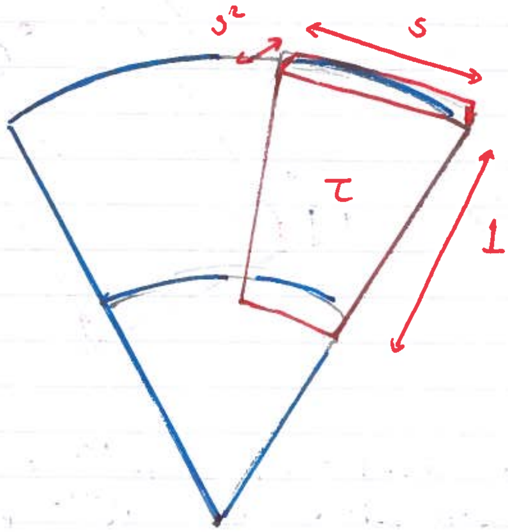
Lecture 14: Cone square function II

A key innovation in the Guth – Wang – Zhang argument is to work with an alternative form of the square function estimate which allows one to take full advantage of the favourable scaling and L^2-orthogonality properties.

The setup in Guth – Wang – Zhang involves additional frequency and spatial localisation, which we describe presently.

Localising the square function

We first localise the square function in frequency. We will work with planks on the cone of intermediate widths $R^{-1/2} \le s \le 1$.



$d(\tau) = s$

Let τ be a plank of dimensions $s \times s^2 \times 1$. We write $d(\tau) = s$ in this case.

We consider the frequency localised square function

$$\Big(\sum_{\substack{d(\theta) = R^{-1/2} \\ \theta \subseteq \tau}} |f_\theta|^2 \Big)^{1/2}$$

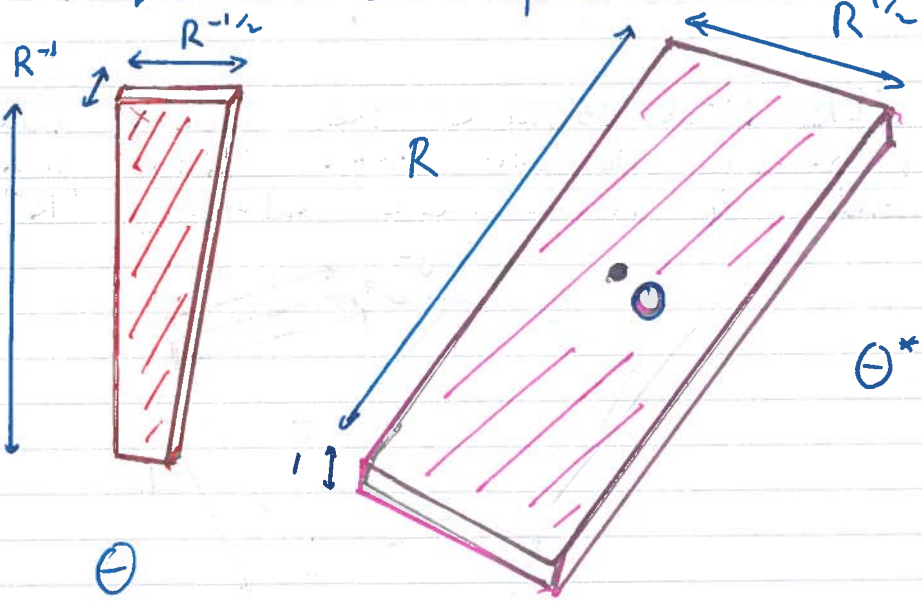
consisting only of contributions from those

$R^{-1/2}$-plates θ lying in τ.

- When $s = R^{-1/2}$, this corresponds to a single plate $|f_\theta|$
- When $s = 1$, we obtain the entire square function
$$\Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2}.$$

We next localise the square function in frequency. This is carried out in a manner which respects the uncertainty principle.

Given a $R^{-1/2}$-plank θ, since $\operatorname{supp} \hat{f}_\theta \subseteq \theta$ we expect f_θ to be locally constant on translates of the dual plank θ^*



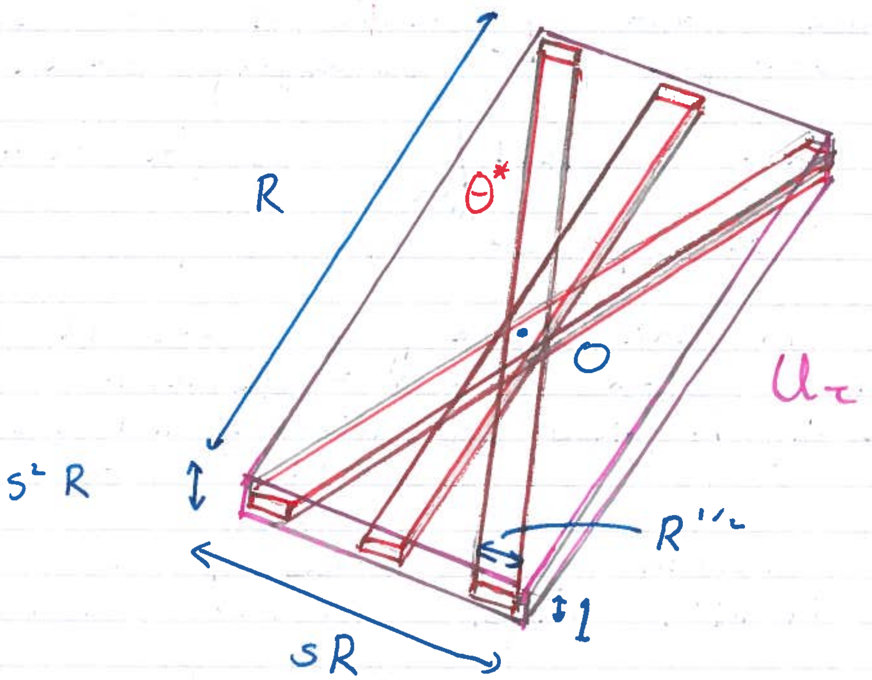
Fix τ with $d(\tau) = s$, $R^{-1/2} \leq s \leq 1$ and consider the collection of $R^{-1/2}$ planks θ contained in τ. We want to understand the corresponding collection of dual planks θ^* and, in particular, the set
$$\bigcup_{\theta \subseteq \tau} \theta^* .$$

To do this, first consider the dual τ^* of τ. This is a plank of dimension
$$s^{-1} \times s^{-2} \times 1$$

and thus smaller than the θ^*. We blow this plank up by a factor of $s^2 R$: to obtain

$$U_\tau = U_{\tau,R} := s^2 R \cdot \tau^*$$

so U_τ is a plank of dimension $sR \times R \times s^2 R$. In particular, we have chosen the scaling factor $s^2 R$ so that the length of the longest side of U_τ — that is, R — agrees with the common length of the long sides of the θ^*.



It follows that $\bigcup_{\theta \subseteq \tau} \theta^* \subseteq U_\tau$ (essentially); this observation relies on the choice of scaling factor above and the fact that the long direction of U_τ, which is dictated by the normal direction to the cone at the centre of τ, will differ from the long direction of θ^* for any $\theta \subseteq \tau$ by at most $\lesssim s$.

We tessellate $\mathbb{R}^3$ with translates U of U_τ. We write $U \parallel U_\tau$ to indicate that U belongs to this tessellation.

The fully localised square function is then defined to be

$$S_{\mathcal{U}} f := \Big(\sum_{\substack{d(\theta)=R^{-1/2} \\ \theta \subseteq \tau}} |f_\theta|^2 \Big)^{1/2} \Big|_{\mathcal{U}} .$$

for each $\mathcal{U} \parallel \mathcal{U}_\tau$.

Examples:- Let's look at two extreme cases:-

- $s \approx R^{-1/2}$. In this case the sets $\mathcal{U}$ have dimension $sR \times s^2 R \times R = R^{1/2} \times 1 \times R$ and agree with translates of the (essentially) unique dual plank θ^* satisfying $\theta \subseteq \tau$.

Hence $S_{\mathcal{U}} f = |f_\theta| \big|_{\mathcal{U}}$ where $\mathcal{U} \parallel \theta^*$

is a "wave packet" of f.

- $s \sim 1$. In this case the sets $\mathcal{U}$ have dimension $sR \times s^2 R \times R = R \times R \times R$ and are boxes. In view of this special case, we let

$\mathcal{Q}_R$ denote a lattice of axis-parallel R-cubes in $\mathbb{R}^3$. Thus, whenever we have

$S_{Q_R} f$ for some $Q_R \in \mathcal{Q}_R$

this is understood to correspond to taking $d(\tau) = 1$ so that

$$S_{Q_R} f = \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big|_{Q_R} .$$

From the point of view of our definitions, here we are considering the union of θ^* over all $R^{-1/2}$-plates across the whole cone. Since these point in many different directions they essentially fill out the whole of a $R \times R \times R$ box.

The main estimate.

Fix an intermediate scale s and cover $N_{1/R}\Gamma$ with a collection of s-planks τ.

We can then write

$$\Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2\Big)^{1/2} \sim \Big(\sum_{d(\tau)=s} \sum_{\substack{d(\theta)=R^{-1/2}\\ \theta\subseteq\tau}} |f_\theta|^2\Big)^{1/2}.$$

Let $p \geq 2$; we wish to bound the L^p-norm of the left-hand side of the above display from below.

Clearly,

$$\int_{\mathbb{R}^3} \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2\Big)^{p/2} \gtrsim \sum_{d(\tau)=s} \int_{\mathbb{R}^3} \Big(\sum_{\substack{d(\theta)=R^{-1/2}\\ \theta\subseteq\tau}} |f_\theta|^2\Big)^{p/2}$$

and by decomposing the spatial domain $\mathbb{R}^3 = \bigcup_{U \| U_\tau} U$, it follows that

$$\Big\| \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2\Big)^{1/2} \Big\|^p_{L^p(\mathbb{R}^3)}$$

$$\gtrsim \sum_{d(\tau)=s} \sum_{U\|U_\tau} \| S_U f \|^p_{L^p(\mathbb{R}^3)} \qquad (1)$$

so we now wish to prove a bound involving the right-hand side of (1).

Theorem (Guth–Wang–Zhang, version 1):–
For all $\varepsilon > 0$,

$$\|f\|^4_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon R^\varepsilon \sum_{\substack{R^{-1/2}\leq s\leq 1\\ s\in 2^{\mathbb{Z}}}} \sum_{d(\tau)=s} \sum_{U\|U_\tau} \|S_U f\|^4_{L^4(\mathbb{R}^3)}$$

$\operatorname{supp}\hat{f} \subseteq N_{1/R}\Gamma$

From the above discussion, this immediately implies the square function bound

$$\|f\|_{L^4(\mathbb{R}^3)}^4 \lesssim_\varepsilon R^\varepsilon \|(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2)^{1/2}\|_{L^4(\mathbb{R}^3)}^4$$

noting that the sum in s is only over $O(\log R)$ dyadic scales.

A key innovation in the Guth-Wang-Zhang argument is to shoot for a stronger estimate :-

Theorem 2 (Guth-Wang-Zhang, version 2) For all $\varepsilon > 0$,

$$\|f\|_{L^4(\mathbb{R}^3)}^4 \lesssim_\varepsilon R^\varepsilon \cdot \sum_{\substack{R^{-1/2} \le s \le 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau)=s} \sum_{U \| U_\tau} |U|^{-1} \|S_U f\|_{L^2(U)}^4$$

whenever $\operatorname{supp} \hat{f} \subseteq N_{1/R}\Gamma$.

Theorem 2 $\Rightarrow$ Theorem 1 :- This is an immediate consequence of Cauchy-Schwarz, since

$$\|S_U f\|_{L^2(U)}^2 = \int_U |S_U f|^2$$
$$\le |U|^{1/2} \left(\int_{\mathbb{R}^3} |S_U f|^4\right)^{1/2} \qquad (2)$$

so that

$$|U|^{-1} \|S_U f\|_{L^2(U)}^4 \le \|S_U f\|_{L^4(\mathbb{R}^3)}^4 . \qquad \square$$

The fact that we can shoot for a stronger estimate here is bound up with certain local-constancy properties of the frequency localised square functions

$$\Big(\sum_{\substack{d(\theta)=R^{-1/2} \\ \theta \subseteq \tau}} |f_\theta|^2 \Big)^{1/2}$$

In particular, were the above function

to be constant on $\mathcal{U}$, then the reverse inequality in (2) also holds.

We make one final reformulation of the problem. If $\operatorname{supp}\hat{f} \subseteq N_{1/R}\Gamma \subseteq B(0,10)$, say, then f should be locally constant at unit scale by the uncertainty principle. We can therefore write

$$\|f\|_{L^4(\mathbb{R}^3)}^4 = \sum_{Q_1 \in \mathcal{Q}_1} \|f\|_{L^4(Q_1)}^4$$

$$\sim \sum_{Q_1 \in \mathcal{Q}_1} |Q_1|^{-1} \|f\|_{L^2(Q_1)}^4, \qquad (3)$$

where the second step is by local constancy. On the other hand,

$$S_{Q_1} f = \Big(\sum_{d(\sigma)=1} |f_\sigma|^2\Big)^{1/2}\Big|_{Q_1} = |f|\,\Big|_{Q_1}$$

since there is only a single unit scale plank which contains the entire Fourier support of f.

Thus, (3) can be written

$$\|f\|_{L^4(\mathbb{R}^3)}^4 \sim \sum_{Q_1 \in \mathcal{Q}_1} |Q_1|^{-1} \|S_{Q_1} f\|_{L^2(Q_1)}^4$$

and it therefore suffices to show

Theorem 3 (Guth-Wang-Zhang, version 3) For all $\varepsilon > 0$, $1 \leq r \leq R$

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \|S_{Q_r} f\|_{L^2(Q_r)}^4 \lesssim_\varepsilon (R/r)^\varepsilon \cdot \qquad (4)$$

$$\sum_{\substack{R^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau)=s} \sum_{U \| U_\tau} |U|^{-1} \|S_U f\|_{L^2(U)}^4,$$

$\operatorname{supp}\hat{f} \subseteq N_{1/R}\Gamma$.

Indeed, the $r=1$ case of Theorem 3 $\Rightarrow$ Theorem 2 by the preceding discussion.

Key features of the reformulation :-

What have we gained by passing to version 3 of the theorem ?

There are two nice features of the inequality (4), which relate to the two nice features of square functions discussed in the previous lecture :-

- 'L^2-based' inequality Both sides of (4) involve L^2, rather than L^4, norms which is useful in view of L^2 orthogonality.

- Self-similarity The left and right-hand sides of (4) both have a similar form, albeit at different scales (r and R, respectively). More precisely, if we consider the term of the sum in s corresponding to $s=1$ on the right-hand side, we obtain

$$\sum_{d(\tau)=1} \sum_{U \| U_\tau} |U|^{-1} \| S_U f \|^4_{L^2(U)}$$

$$= \sum_{Q_R \in \mathcal{Q}_R} |Q_R|^{-1} \| S_{Q_R} f \|^4_{L^2(Q_R)}$$

which corresponds precisely with the left-hand side but with r replaced by R.

Incidentally, this shows that (4) trivially holds when $r = R$.

We will exploit the self-similarity property in an induction-on-scale argument. In view of this, we make the following definition.

Def^n Let $S(r,R)$ denote the infemum over all $C \geqslant 1$ for which the inequality

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)}$$

$$\leq C \cdot \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau)=s} \sum_{U \| U_\tau} |U|^{-1} \| S_U f \|^4_{L^2(U)}$$

holds whenever $\operatorname{supp}\hat{f} \subseteq N_{1/R}\Gamma$.

With this definition, Theorem 3 can be succinctly expressed as: For all $1 \le r \le R$, $\varepsilon > 0$,

$$S(r, R) \lesssim_\varepsilon (R/r)^\varepsilon . \qquad (5)$$

The basic idea is to prove (5) by inducting on the ratio R/r. When $R/r = 1$, we have $R = r$ and the inequality is trivial as noted above. When $R/r = R$, we have $r = 1$ and this is the case of interest.

To facilitate the induction argument, it is necessary to compare the $S(r, R)$ between different choices of scales. This is achieved by the following lemma.

<u>Lemma 1 (Lorentz rescaling, revisited)</u> Let $1 \le r_1 \le r_2 \le r_3$. Then

$$S(r_1, r_3) \lesssim (\log r_2) \cdot S(r_1, r_2) \cdot \max_{\substack{r_2^{-1/2} \le s \le 1 \\ s \in 2^{\mathbb{Z}}}} S(s^2 r_2, s^2 r_3). \qquad (6)$$

To understand the statement of this lemma, it is useful to consider a 'moral' version of the above inequality:-

$$S(r_1, r_3) \le S(r_1, r_2) \cdot S(r_2, r_3). \qquad (6')$$

This is obtained by ignoring the $\log r_2$ factor in Lemma 1 and assuming the maximum occurs when $s = 1$. For practical purposes, it makes little difference whether (6) or (6') is used and so we will typically work with (6').

We can see from (6') that to "pass" from scale r_1 to scale r_3, it suffices to pass through an intermediate scale r_2 and then take the product of the resulting constants:-

$$\begin{tikzcd} r_1 \arrow[r, "S(r_1,r_2)"'] \arrow[rr, bend left, "S(r_1,r_3)"] & r_2 \arrow[r, "S(r_2,r_3)"'] & r_3 \end{tikzcd}$$

Lemma 1 is the raison d'être for the ostensibly convoluted setup in Theorem 3. The ability to factor through intermediate scales is crucial in the upcoming induction arguments.

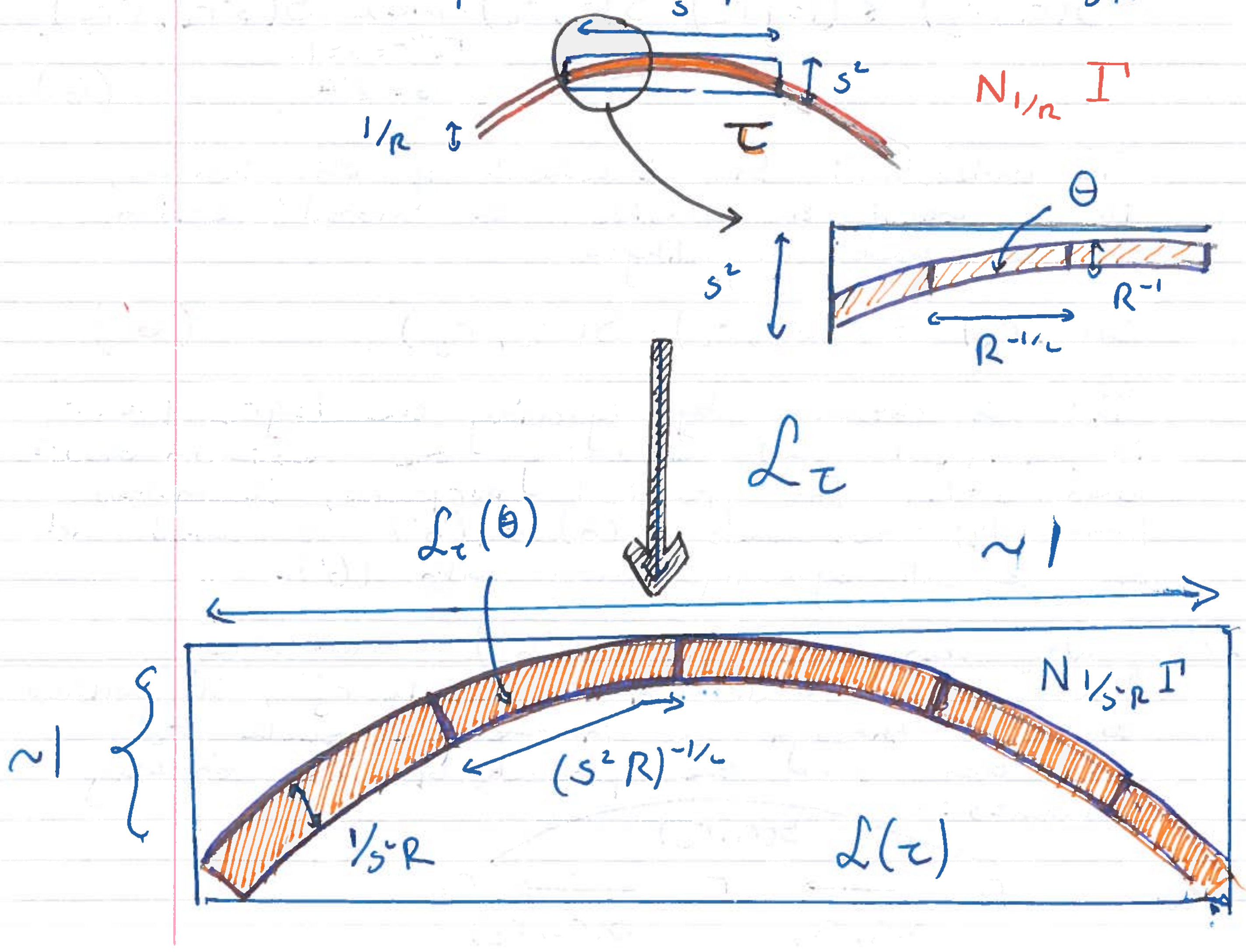
Proof (of Lemma 1) We begin by recalling some facts about Lorentz rescaling from the previous lecture.

Fix an s-plank τ and let $\mathcal{L} = \mathcal{L}_\tau$ denote the Lorentz rescaling taking τ to the unit-scale plank.

Thus, $\mathcal{L}_\tau$ maps each $\theta \leq \tau$ to a plank $\sigma = \mathcal{L}_\tau(\theta)$ with $d(\sigma) = (s^2 R)^{-1/2}$ and, moreover,

$$\{ \sigma = \mathcal{L}_\tau(\theta) : \theta \leq \tau \}$$

forms a plank decomposition of $N_{1/s^2R}\Gamma$.



In particular, given $(f_\theta)_\theta$ with $\operatorname{supp} \hat{f}_\theta \subseteq \theta$ define

$$h_{\mathcal{L}(\theta)} := f_\theta \circ \mathcal{L}^{T}.$$

It follows that $(h_{\mathcal{L}(\theta)})^{\wedge} = \hat{f}_\theta \circ \mathcal{L}^{-1}$ is supported in $\mathcal{L}(\theta)$ and so

$$\Big\| \Big(\sum_{\substack{d(\theta) = R^{-1/2} \\ \theta \subseteq \tau}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^3)} = |\det \mathcal{L}|^{1/p} \cdot \Big\| \Big(\sum_{\substack{d(\theta) = R^{-1/2} \\ \theta \subseteq \tau}} |h_{\mathcal{L}(\theta)}|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^3)}$$

$$= |\det \mathcal{L}|^{1/p} \Big\| \Big(\sum_{d(\sigma) = (s^2 R)^{-1/2}} |h_\sigma|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^3)}.$$

We also wish to study the effect on the transformation on the spatial localisation. In particular,

$\mathcal{L}$ maps τ to a unit scale plank
$\mathcal{L}^{-T}$ maps τ^* to a unit scale plank

since the mapping $\mathcal{L}^{-T}$ will "preserve duality".
Thus, if $U_\tau = s^2 R \cdot \tau^*$, then

$$\mathcal{L}^{-T}(U_\tau) \text{ is an } s^2R\text{-cube}.$$

In particular,

$$\| S_U f \|_{L^p(\mathbb{R}^3)} = |\det \mathcal{L}|^{1/p} \Big\| \Big(\sum_{d(\sigma) = (s^2 R)^{-1/2}} |h_\sigma|^2 \Big)^{1/2} \Big\|_{L^p(\mathcal{L}^{-T}(U))}$$

$$= |\det \mathcal{L}|^{1/p} \Big\| \Big(\sum_{d(\sigma) = (s^2 R)^{-1/2}} |h_\sigma|^2 \Big)^{1/2} \Big\|_{L^p(Q_{s^2R})}$$

Having made these preliminary observations, we are now ready to turn to the proof of the lemma.

By definition,

$$\sum_{Q_{r_1} \in \mathcal{Q}_{r_1}} |Q_{r_1}|^{-1} \| S_{Q_{r_1}} f \|_{L^2(Q_{r_1})}^4 \leq S(r_1, r_2) \cdot \qquad (7)$$

$$\sum_{\substack{r_2^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau) = s} \sum_{U \| \mathcal{U}_{\tau, r_2}} |U|^{-1} \| S_U f \|_{L^2(U)}^4 .$$

Fix $r_2^{-1/2} \leq s \leq 1$ and τ with $d(\tau) = s$. By the above, if $\mathcal{L}$ is the Lorentz transformation sending τ to the unit-scale plank, then

$$\sum_{U \| \mathcal{U}_{\tau, r_2}} |U|^{-1} \| S_U f \|_{L^2(U)}^4 = |\det \mathcal{L}|^3 . \qquad (8)$$

$$\sum_{Q_{s^2 r_2} \in \mathcal{Q}_{s^2 r_2}} |Q_{s^2 r_2}|^{-1} \| S_{Q_{s^2 r_2}} h \|_{L^2(Q_{s^2 r_2})}^4$$

for $h := f \circ \mathcal{L}^{-1}$.

By definition,

$$\sum_{Q_{s^2 r_2} \in \mathcal{Q}_{s^2 r_2}} |Q_{s^2 r_2}|^{-1} \| S_{Q_{s^2 r_2}} h \|_{L^2(Q_{s^2 r_2})}^4 \leq \qquad (9)$$

$$\max_{r_2^{-1/2} \leq s \leq 1} S(s^2 r_2, s^2 r_3) \cdot$$

$$\sum_{\substack{(s^2 r_3)^{-1/2} \leq s' \leq 1 \\ s' \in 2^{\mathbb{Z}}}} \sum_{d(\tau') = s'} \sum_{U' \| \mathcal{U}_{\tau', s^2 r_3}} |U'|^{-1} \| S_{U'} h \|_{L^2(U')}^4 .$$

We now undo all the changes of variable. In particular, each τ', $d(\tau') = s'$, is mapped to a ss'-plate $\mathcal{L}^{-1}(\tau')$ under $\mathcal{L}^{-1}$ and it follows that the final line of the above display is

$$|\det \mathcal{L}|^{-3} \cdot \sum_{(s^2 r_3)^{-1/2} \leq s' \leq 1} \sum_{d(\tau') = s \cdot s'} \sum_{U' \| \mathcal{U}_{\tau', r_3}} |U'|^{-1} \| S_{U'} f \|_{L^2(U')}^4 . \qquad (10)$$

$$= |\det \mathcal{L}|^{-3} \sum_{r_3^{-1/2} \le s' \le s} \sum_{d(\tau')=s'} \sum_{\mathcal{U}' \| \mathcal{U}_{\tau', r_3}} |\mathcal{U}'|^{-1} \| S_{\mathcal{U}'} f \|^4_{L^2(\mathcal{U}')} .$$

Finally, combining (7), (8), (9) and (10),

$$\sum_{Q_{r_1} \in \mathcal{Q}_{r_1}} |Q_{r_1}|^{-1} \| S_{Q_{r_1}} f \|^4_{L^2(Q_{r_1})} \lesssim S(r_1, r_2) \cdot \max_{r_2^{-1/2} \le s \le 1} S(s^2 r_2, s^2 r_3)$$

$$\cdot \sum_{r_2^{-1/2} \le s \le 1} \sum_{r_3^{-1/2} \le s' \le s} \sum_{d(\tau')=s'} \sum_{\mathcal{U}' \| \mathcal{U}_{\tau', r_3}} |\mathcal{U}'|^{-1} \| S_{\mathcal{U}'} f \|^4_{L^2(\mathcal{U}')} \qquad (11)$$

which implies, by definition,

$$S(r_1, r_3) \lesssim (\log r_2) \cdot S(r_1, r_2) \max_{r_2^{-1/2} \le s \le 1} S(s^2 r_2, s^2 r_3),$$

as required. Here we have relaxed the summation in s' in (11) to $r_3^{-1/2} \le s' \le 1$ and used the fact that there are $\lesssim \log r_2$ summands in the sum in s.

$\square$.