Lecture 12 : Local smoothing for the wave equation III.

In these lectures we will deal with two ingredients in the proof of the local smoothing conjecture for $n=2$, as discussed previously.

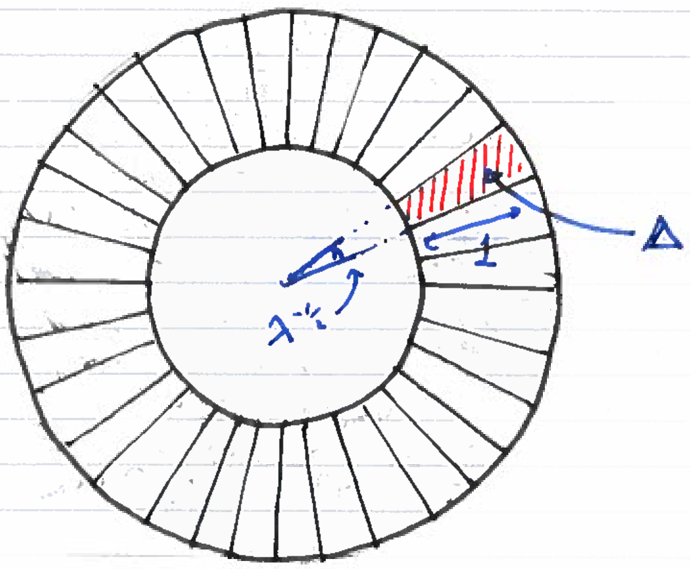
In particular, we will consider :-

- **A forward square function estimate.**

Let $\{\Delta\}$ be a partition of the unit scale annulus

$$\{\xi \in \hat{\mathbb{R}}^2 : 1 \leq |\xi| \leq 2\}$$

into sectors of aperture $\sim \lambda^{-1/2}$, as shown :-



For each region Δ define the projection P_Δ by

$$(P_\Delta f)^\wedge := \tilde{\chi}_\Delta \cdot \hat{f}, \qquad f \in \mathcal{S}(\mathbb{R}^2),$$

where the multiplier is a smooth cutoff adapted to Δ.

Theorem 1 (Cordoba) :- For $2 \leq p \leq 4$, $\varepsilon > 0$

$$\Big\| \Big(\sum_\Delta |P_\Delta f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^\varepsilon \, \|f\|_{L^p(\mathbb{R}^2)}.$$

A Nikodym maximal function estimate :-

For $s \in [0, 2\pi]$ define

$$T_s := \left\{ (x,t) \in \mathbb{R} \times [\tfrac{1}{2}, 2] : \left| \begin{pmatrix} x \\ t \end{pmatrix} \cdot \begin{pmatrix} \cos s \\ \sin s \end{pmatrix} \right| \leq \lambda^{-1} \text{ and } \left| \begin{pmatrix} x \\ t \end{pmatrix} \cdot \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \right| \leq \lambda^{-1/2} \right\}$$

and the maximal operator

$$\mathcal{N}^\lambda g(y) := \sup_{s \in [0, 2\pi]} \fint_{T_s} |g(x-y, t)| \, dx \, dt.$$

Theorem 2 (Mockenhaupt - Seeger - Sogge) :- For $2 \leq p \leq \infty$ and $\varepsilon > 0$,

$$\| \mathcal{N}^\lambda g \|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^\varepsilon \| g \|_{L^p(\mathbb{R}^2)}.$$

Remark :- The arguments will give Theorems 1 and 2 with explicit $(\log \lambda)^\alpha$ dependencies.

The square function

We begin with the proof of Theorem 1. First recall the analogous square function from Lecture 7. In particular, let $\zeta \in C_c^\infty(\mathbb{R}^n)$, $n \geq 1$, satisfy

$$\operatorname{supp} \zeta \subseteq [-1, 1]^n \; ; \qquad \sum_{k \in \mathbb{Z}^n} \zeta(x - k) \equiv 1$$

and define the projection operators P_k, $k \in \mathbb{Z}^n$, by

$$(P_k f)^\wedge := \zeta(\cdot - k) \cdot \hat{f}$$

Theorem 3 For $2 \leq p \leq \infty$,

$$\left\| \left(\sum_{k \in \mathbb{Z}^n} |P_k f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)}.$$

We essentially proved the $n = 1$ case ~~of~~ this theorem in Lecture 7 (strictly speaking, we considered a square function associated to vertical strips in $\mathbb{R}^2$, but the same analysis "ignoring the vertical direction" gives the stated result). The same argument works for all n.

We will use a strengthened version of this result here.

<u>Theorem 4</u> For each $s > 1$,

$$\int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} |P_k f|^2 \cdot \omega \lesssim_s \int_{\mathbb{R}^n} |f|^2 \cdot M_s \omega$$

where $M_s \omega := (M_{HL}|\omega|^s)^{1/s}$ for M_{HL} the Hardy–Littlewood maximal function.

<u>Theorem 4 implies Theorem 3</u> away from the endpoint at ∞ via the duality arguments we have already encountered and the Hardy–Littlewood maximal theorem. Indeed, let $2 \leq p < \infty$ and define $q := (p/2)'$ so that $1 < q \leq \infty$. By duality,

$$\Big\|\Big(\sum_{k \in \mathbb{Z}^n} |P_k f|^2\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} = \Big\|\sum_{k \in \mathbb{Z}^n} |P_k f|^2\Big\|_{L^{p/2}(\mathbb{R}^n)}$$

$$= \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} |P_k f|^2 \cdot \omega$$

for some $\omega \in L^q(\mathbb{R}^n)$ with $\|\omega\|_{L^q(\mathbb{R}^n)} = 1$. Given $s > 1$, Theorem 4 therefore yields

$$\Big\|\Big(\sum_{k \in \mathbb{Z}^n} |P_k f|^2\Big)^{1/2}\Big\|^2_{L^p(\mathbb{R}^n)} \lesssim_s \int_{\mathbb{R}^n} |f|^2 \cdot M_s \omega$$

$$\leq \|f\|^2_{L^p(\mathbb{R}^n)} \cdot \|M_s \omega\|_{L^q(\mathbb{R}^n)}$$

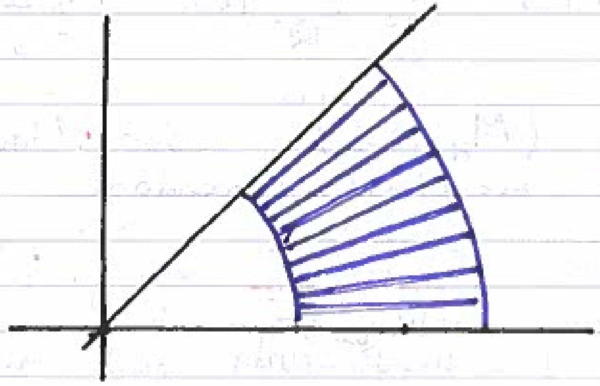
Now choose $1 < s < q$ so that $q/s > 1$ and

$$\|M_s \omega\|_q = \|M_{HL}|\omega|^s\|^{1/s}_{q/s} \lesssim \||\omega|^s\|^{1/s}_{q/s} = \|\omega\|_q = 1$$

by the Hardy–Littlewood theorem. $\square$

We will not provide a proof of Theorem 4. The argument involves combining the simple analysis used to establish Theorem 3 with weighted estimates for singular integrals due to Cordoba – Fefferman.

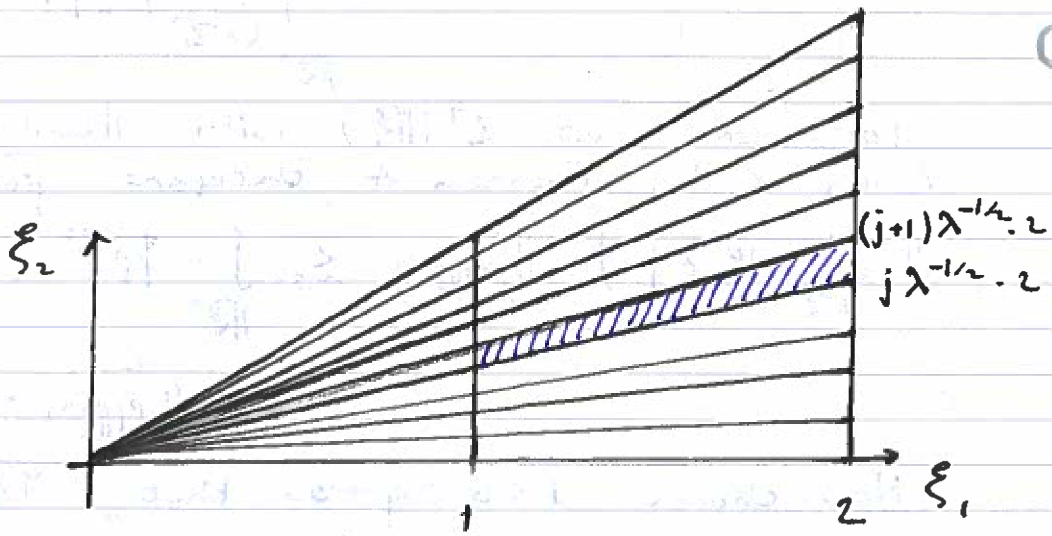
Proof (of Theorem 1):- By the triangle inequality and rotation invariance, it suffices to consider only those sectors which lie between the positive ξ_1-axis and the diagonal



We will simplify the setup a little as follows. Replace the sectors Δ with regions

$$\Delta_j := \left\{ \xi \in \mathbb{R}^2 : 1 \leq \xi_1 \leq 2, \quad \frac{\xi_2}{\xi_1} \in [j\lambda^{-1/2}, (j+1)\lambda^{-1/2}] \right\}$$

$j = 0, 1, \ldots, \lceil \lambda^{-1/2} \rceil$.



These sectors are "essentially" the same as the original sectors, but the definition is easier to work with.

The $p=2$ case of the theorem follows by Plancherel and so it remains to consider $p=4$. We will use a "biorthogonality" argument.

$$\Big\|\Big(\sum_j |P_{\Delta_j} f|^2\Big)^{1/2}\Big\|_{L^4(\mathbb{R}^2)}^4 = \Big\|\sum_j |P_{\Delta_j} f|^2\Big\|_{L^2(\mathbb{R}^2)}^2$$
$$= \sum_{j,k} \int_{\mathbb{R}^2} |P_{\Delta_j} f \cdot P_{\Delta_k} f|^2 .$$

We decompose the latter sum according to the separation between j and k :-

(1a) $$\sum_j \|P_{\Delta_j} f\|_{L^4(\mathbb{R}^2)}^4$$

(1b) $$+ \sum_{\nu=0}^{\lceil \frac{1}{4}\log\lambda\rceil} \; \sum_{2^{-\nu}\lambda^{1/4} \le |j-k| \le 2^{-\nu+1}\lambda^{1/4}} \int_{\mathbb{R}^2} |P_{\Delta_j} f \cdot P_{\Delta_k} f|^2$$

(1c) $$+ \sum_{|j-k| \gtrsim \lambda^{1/4}} \int_{\mathbb{R}^2} |P_{\Delta_j} f \cdot P_{\Delta_k} f|^2$$

The term (1a) corresponds to the diagonal, where $j=k$ and there is no separation. It is easy to see

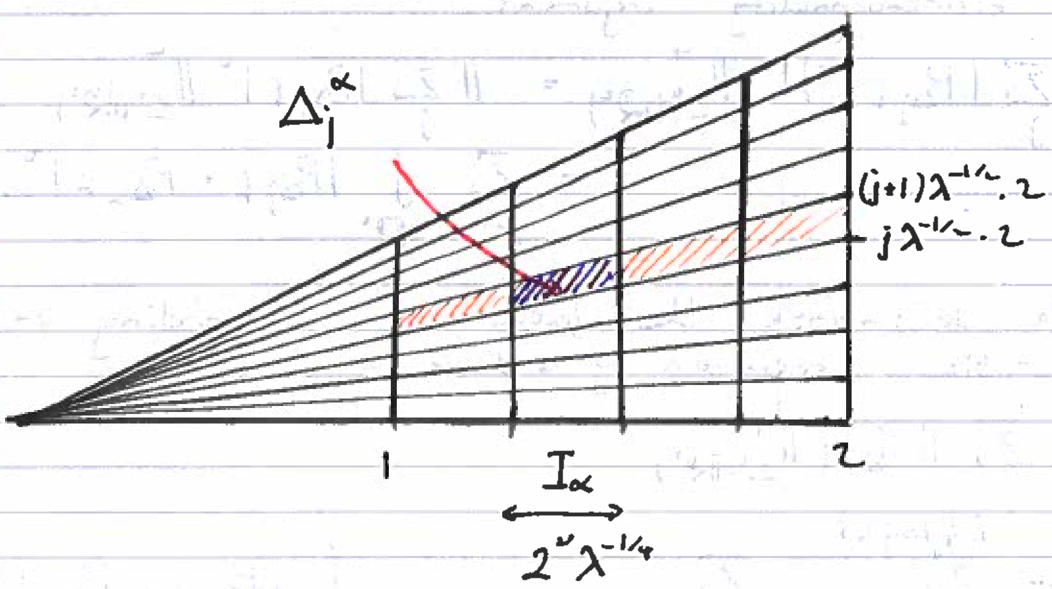
$$\Big(\sum_j \|P_{\Delta_j} f\|_{L^p(\mathbb{R}^2)}^p\Big)^{1/p} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

for $2 \le p \le \infty$, with the left-hand side interpreted in the obvious manner for $p=\infty$. Indeed, the $p=2$ case follows from Plancherel whilst the $p=\infty$ case follows since the kernels of the P_{Δ_j} are uniformly in L^1.

It remains to bound the terms in (1b) and (1c). We will bound the terms in (1b) only; (1c) follows by an identical argument.

Fix $0 \le \nu \le \lceil \frac{1}{4}\log\lambda \rceil$ and decompose the interval $[1,2]$ into essentially disjoint closed sub-intervals I_α of length $2^\nu \lambda^{-1/4}$. Define.

$$\Delta_j^\alpha := \{\xi \in \hat{\mathbb{R}}^2 : \xi \in \Delta_j \text{ and } \xi_1 \in I_\alpha\}.$$



There are two key geometric observations :

1. "Biorthogonality" between Δ_j^α.

If $|j-k| \geq 2^{-\nu}\lambda^{1/4}$, then the sets

$$\Delta_j^\alpha + \Delta_k^\beta, \quad \alpha,\beta = 1,\dots,\lambda^{1/4}$$

are finitely-overlapping:-

$$\sum_{\alpha,\beta=1}^{\lambda^{1/4}} \chi_{\Delta_j^\alpha+\Delta_k^\beta}(\xi) \lesssim 1 \quad \text{for } \xi\in\widehat{\mathbb{R}^2}. \qquad (2)$$

Once we have (2) we may bound

$$\int_{\mathbb{R}^2} |P_{\Delta_j} f \cdot P_{\Delta_k} f|^2 \lesssim \sum_{\alpha,\beta=1}^{\lambda^{1/4}} \int_{\mathbb{R}^2} |P_{\Delta_j^\alpha} f\cdot P_{\Delta_k^\beta} f|^2$$

so that

$$\sum_{|j-k|\geq 2^{-\nu}\lambda^{1/4}} \int_{\mathbb{R}^2} |P_{\Delta_j}f\cdot P_{\Delta_k}f|^2 \lesssim \Big\|\Big(\sum_{j,\alpha}|P_{\Delta_j^\alpha}f|^2\Big)^{1/2}\Big\|^4_{L^4(\mathbb{R}^2)}. \qquad (3)$$

To prove (2), note that each element of Δ_j^α can be expressed as

$$t\begin{pmatrix}1\\ j\cdot\lambda^{-1/2}\end{pmatrix} + \begin{pmatrix}0\\ \eta\end{pmatrix} \quad \text{for some } t\in I_\alpha,\ 0\leq\eta\leq\lambda^{-1/2}.$$

Thus, if $\Delta_j^\alpha + \Delta_k^\beta \cap \Delta_j^{\alpha'} + \Delta_j^{\beta'} \neq \emptyset$, then there must exist

$$t_\alpha \in I_\alpha, \quad t_\beta \in I_\beta, \quad t_{\alpha'} \in I_{\alpha'}, \quad t_{\beta'} \in I_{\beta'}$$

such that

$$t_\alpha \begin{pmatrix} 1 \\ j\lambda^{-1/2} \end{pmatrix} + t_\beta \begin{pmatrix} 1 \\ k\lambda^{-1/2} \end{pmatrix} = t_{\alpha'} \begin{pmatrix} 1 \\ j\lambda^{-1/2} \end{pmatrix} + t_{\beta'} \begin{pmatrix} 1 \\ k\lambda^{-1/2} \end{pmatrix} + O(\lambda^{-1/2}).$$

Writing this in matrix form,

$$\begin{pmatrix} 1 & 1 \\ j\lambda^{-1/2} & k\lambda^{-1/2} \end{pmatrix} \begin{pmatrix} t_\alpha \\ t_\beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ j\lambda^{-1/2} & k\lambda^{-1/2} \end{pmatrix} \begin{pmatrix} t_{\alpha'} \\ t_{\beta'} \end{pmatrix} + O(\lambda^{-1/2})$$

Now, the determinant of $\begin{pmatrix} 1 & 1 \\ j\lambda^{-1/2} & k\lambda^{-1/2} \end{pmatrix}$ is $(k-j)\cdot\lambda^{-1/2}$ and so, left multiplying by the inverse matrix,

$$\begin{pmatrix} t_\alpha \\ t_\beta \end{pmatrix} = \begin{pmatrix} t_{\alpha'} \\ t_{\beta'} \end{pmatrix} + O\left(\lambda^{-1/2} \frac{1}{|k-j|\lambda^{-1/2}}\right)$$

$$= \begin{pmatrix} t_{\alpha'} \\ t_{\beta'} \end{pmatrix} + O(2^{\nu}\lambda^{-1/4})$$

under the separation hypothesis. In particular, if we think of $t_{\alpha'}$ and $t_{\beta'}$ as fixed, then t_α and t_β must lie in a pair of intervals of length $O(2^{\nu}\lambda^{-1/4})$ around these points. Since the I_α have length $2^{\nu}\lambda^{-1/4}$, this means there are only $O(1)$ choices of α, β for α', β' fixed, as desired. $\square$

2. "Essentially parallel" property. If $|j-k| \leq 2^{-\nu}\lambda^{\frac{1}{4}}$, then $\Delta_j^\alpha, \Delta_k^\alpha$ are essentially parallel, in the sense that

$$\Delta_j^\alpha \subseteq 100\cdot\Delta_k^\alpha + x_0$$

for some translate $x_0 \in \mathbb{R}^2$, say.

Indeed, the "base lines" ℓ_j^α and ℓ_k^α where

$$\ell_j^\alpha := \left\{ t \begin{pmatrix} 1 \\ j\lambda^{-1/2} \end{pmatrix} : t \in \mathbb{R} \right\}, \text{ etc,}$$

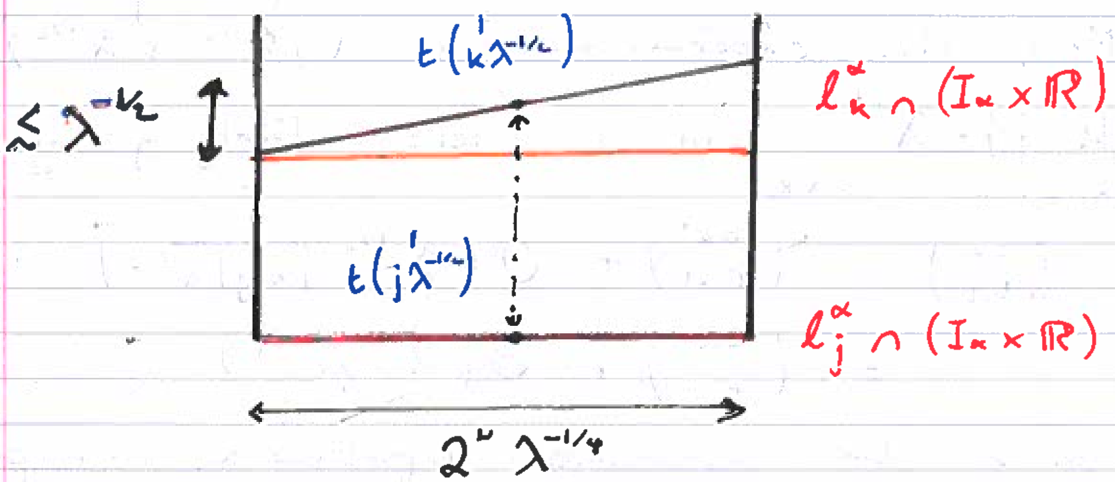
differ by an angle of at most $2^{-\nu}\lambda^{-1/4}$. Consequently, over the interval I_α the vertical displacement

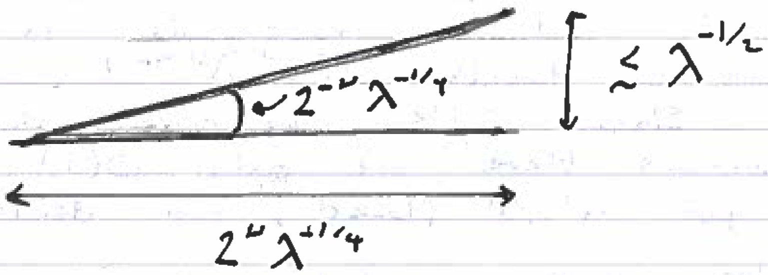
$$t\left[({}_j'\lambda^{-1/2}) - ({}_k'\lambda^{-1/2})\right], \qquad t \in I_\alpha,$$

varies over an interval of length

$$O(\underbrace{2^{\nu}\lambda^{-1/4}}_{\text{length of } I_\alpha} \cdot \underbrace{2^{-\nu}\lambda^{-1/4}}_{\text{difference in angle}}) = O(\lambda^{-1/2}).$$

Since the "height" of the Δ_j^α, Δ_k^α is $\lesssim \lambda^{-1/2}$, this gives the desired "essentially parallel" property.





By duality, we can write the (1/2 power of the) hand side of (3) as

$$\Big\|\Big(\sum_{j,\alpha} |P_{\Delta_j^\alpha} f|^2\Big)^{1/2}\Big\|_{L^4(\mathbb{R}^2)}^2 = \Big\|\sum_{j,\alpha} |P_{\Delta_j^\alpha} f|^2\Big\|_{L^2(\mathbb{R}^2)}$$

$$= \int_{\mathbb{R}^2} \Big(\sum_{j,\alpha} |P_{\Delta_j^\alpha} f|^2\Big) \cdot \omega$$

for some $\omega \in L^2(\mathbb{R}^2)$ with $\|\omega\|_{L^2(\mathbb{R}^2)} = 1$.

We now collect the Δ_j^α into essentially parallel families:-

$$\sum_{j,\alpha} \int_{\mathbb{R}^2} |P_{\Delta_j^\alpha} f|^2 \cdot \omega \qquad (4)$$

$$= \sum_{\ell=1}^{\sim 2^\nu \lambda^{1/4}} \sum_{\alpha=1}^{\lambda^{1/4}} \sum_{j=\ell 2^{-\nu}\lambda^{1/4}}^{(\ell+1)2^{-\nu}\lambda^{1/4}} \int_{\mathbb{R}^2} |P_{\Delta_j^\alpha} Q_\ell^\alpha f|^2 \cdot \omega$$

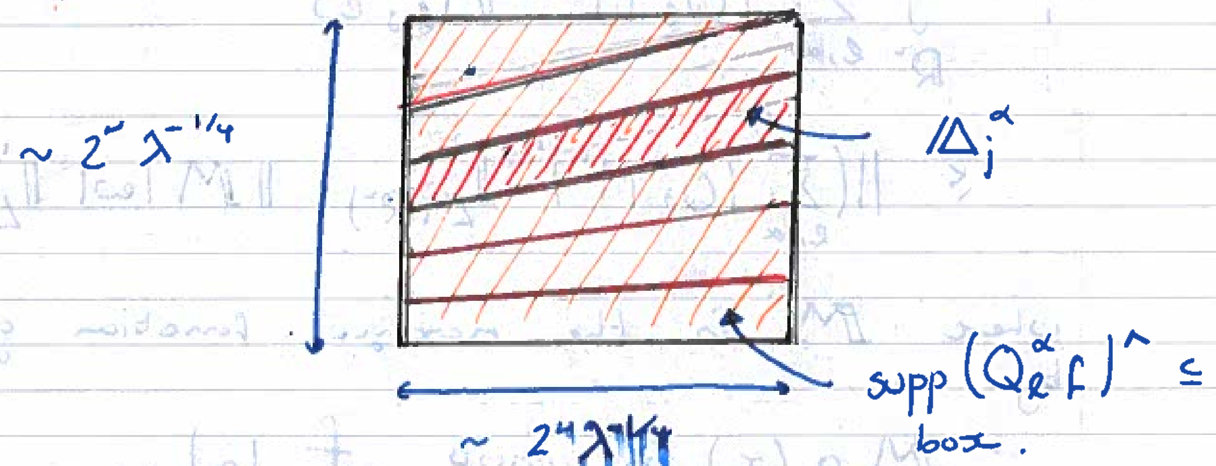
where $(Q_\ell^\alpha f)^\wedge(\xi) = \zeta(2^{-\nu}\lambda^{1/4}(\xi - c_\ell^\alpha))\, \hat{f}(\xi)$

is a frequency projection to a square of side-length

$$O(2^\nu \lambda^{-1/4})$$

containing all the parallel regions

$$\Delta_j^\alpha, \qquad \ell 2^{-\nu}\lambda^{1/4} \leq j \leq (\ell+1) 2^{-\nu}\lambda^{1/4}. \qquad (5)$$



Since the regions in (5) are essentially parallel, we can apply a suitably transformed version of Theorem 4 to bound the inner most sum. In particular, for $s > 1$ we have

$$\int_{\mathbb{R}^2} \sum_{j=\ell 2^{-\nu}\lambda^{1/4}}^{(\ell+1)2^{-\nu}\lambda^{1/4}} |P_{\Delta_j^\alpha} Q_\ell^\alpha f|^2 \cdot \omega$$

$$\lesssim \int_{\mathbb{R}^2} |Q_\ell^\alpha f|^2 \cdot \mathcal{M}_{\ell,s}\, \omega$$

where $\mathcal{M}_{\ell,s}\, \omega := (M_\ell |\omega|^s)^{1/s}$ for M_ℓ

the maximal function taking maximal averages over dilates of the dual rectangle to Δ_j^α (for $j = \ell \cdot 2^{-2\nu}\lambda^{1/4}$, say).

Note that this dual rectangle has dimensions

$$2^{-\nu}\cdot\lambda^{+1/4} \times \lambda^{1/2}$$

with the short $2^{-\nu}\lambda^{1/4}$ side pointing in the direction of $\begin{pmatrix} 1 \\ j\lambda^{-1/2} \end{pmatrix}$.

Plugging this into (4),

$$\sum_{j,\alpha}\int_{\mathbb{R}^2} |P_{\Delta_j^\alpha} f|^2 \cdot \omega \lesssim$$

$$\int_{\mathbb{R}^2} \sum_{\ell,\alpha} |Q_\ell^\alpha f|^2 \; \mathcal{M}_{\ell,s}\,\omega$$

$$\lesssim \Big\|\Big(\sum_{\ell,\alpha} |Q_\ell^\alpha f|^2\Big)^{1/2}\Big\|^2_{L^4(\mathbb{R}^2)} \; \|\mathcal{M}|\omega|^s\|^{1/s}_{L^{2/s}(\mathbb{R}^2)}$$

where $\mathcal{M}$ is the maximal function given by

$$\mathcal{M}g(x) := \sup_{\substack{x\in T \\ T \text{ has eccentricity } 2^\nu\lambda^{1/4}}} \fint_T |g| .$$

That is, $\mathcal{M}$ takes maximal averages over all rectangles centred at a point with fixed eccentricity $2^\nu\lambda^{1/4}$.

(The eccentricity of T is defined to be

$$\frac{\text{length of long side of } T}{\text{length of short side of } T}).$$

This is a larger version of the Nikodym maximal function encountered in the Bochner–Riesz case which took maximal averages over rectangles through a point x with fixed side-lengths.

By Theorem 3 we have:

$$\Big\| \Big(\sum_{\alpha, \ell} |Q_\ell^\alpha f|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^2)} \lesssim \|f\|_{L^4(\mathbb{R}^2)}$$

and so all that remains is to show that

$$\|\mathcal{M} g\|_{L^2(\mathbb{R}^2)} \lesssim \log \lambda \cdot \|g\|_{L^2(\mathbb{R}^2)}, \quad (6).$$

which is a direct strengthening of our previous Nikodym bound.

We will not give a proof of (6) here. To prove the weaker Nikodym estimate in Lecture 7 we appealed to duality with the Kakeya maximal function and then used the L^2 Kakeya bound from Lecture 4. In this case it is convenient to argue directly, using a 'dual version' of the L^2 Cordoba argument from Lecture 4, coupled with a Vitali-style selection process on the rectangles with respect to the side length.