Lecture 11: Local smoothing for the wave equation

II

Recall:-

Conjecture 1 (Local smoothing):- For $n \geq 2$, the inequality

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R}^n)}^p \, dt\right)^{1/p} \lesssim \|f\|_{L^p_s(\mathbb{R}^n)}$$

holds for all

$$\begin{cases} s > \bar{s}_p - 1/p & \text{if } \frac{2n}{n-1} \leq p < \infty \\ s > 0 & \text{if } 2 < p \leq \frac{2n}{n-1}. \end{cases}$$

Recall, the fixed time estimate tells us

$$\sup_{t \in [1,2]} \| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p_s(\mathbb{R}^n)}$$

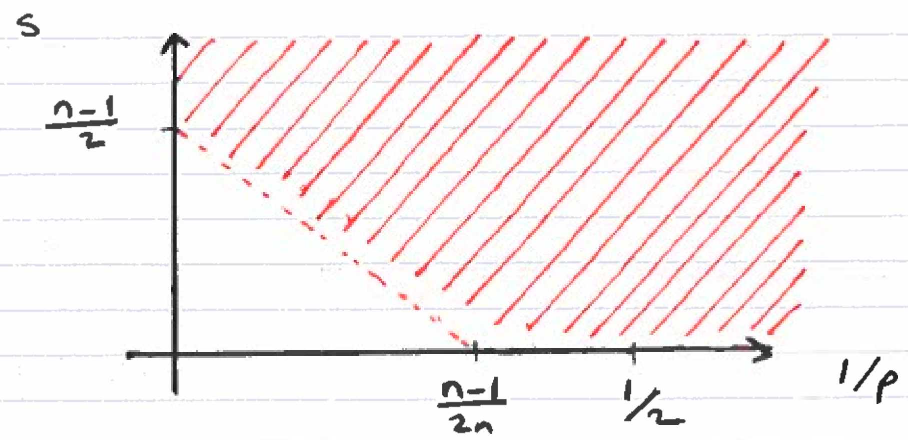
for all $s \geq \bar{s}_p$ if $1 < p < \infty$, so that the problem is to show L^p averaging in t gains $1/p$ - derivatives.

Up until late 2019 this problem was open in all dimensions $n \geq 2$. However a solution to the $n = 2$ case was posted on arXiv in October 2019 by Guth - Wang - Zhang.

The purpose of the next few lectures is to study their work which brings together many of the techniques we've seen so far in the course in new and innovative ways. In particular, we'll be using:-

- Square functions / reverse Littlewood Paley
- Kakeya / Nikodym - type considerations.
- Induction - on - scale techniques.

Preliminary observations:- Since in the 'limiting cases' $p = 2$ and $p = \infty$ the estimates in conjecture 1 agree with the fixed - time range, by interpolation it suffices to consider the $p = \bar{p}_n = \frac{2n}{n-1}$ 'critical' exponent.



Conjectured range for

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R}^n)}^p \, dt \right)^{1/p} \lesssim \|f\|_{L^p_s(\mathbb{R}^n)}.$$

The $p = \bar{p}_n = \frac{2n}{n-1}$ exponent is critical.

Since we are aiming to prove Sobolev estimates with regularity exponent s in an open range, we can frequency localise to some dyadic frequency range.

In particular, let $\eta \in C_c^\infty(\mathbb{R})$ satisfy

$$\begin{cases} \eta(r) = 1 & \text{if } |r| \leq 1 \\ \eta(r) = 0 & \text{if } |r| \geqslant 2 \end{cases}$$

and define $\beta(r) := \eta(r) - \eta(2r)$ so $\operatorname{supp} \beta \subseteq [1/2, 2]$ and

$$\eta(r) + \sum_{j=1}^\infty \beta(2^{-j} r) = 1.$$

Also let $\tilde{\beta}(r) := \eta(r/2) - \eta(4r)$ so that $\operatorname{supp} \tilde{\beta} \subseteq [\frac{1}{4}, 4]$ and $\tilde{\beta}(r) = 1$ if $r \in [1/4, 2]$.

and $\varrho \in \mathcal{S}(\mathbb{R})$ with $\varrho(t) \geq 1$ for $t \in [1,2]$.

Define

$$Tf(x,t) := \varrho(t) \cdot e^{it\sqrt{-\Delta}} f(x)$$

so it suffices to show estimates of the form

$$\|Tf\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^p_s(\mathbb{R}^n)}$$

for the specified range of p and s.

Now let $T^\lambda f := T \circ \beta(\sqrt{-\Delta}/\lambda) f$

$$= \frac{1}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i(\langle x,\xi\rangle + t|\xi|)} \rho(t) \beta(|\xi|/\lambda) \hat{f}(\xi) d\xi$$

so that, by the triangle inequality,

$$\|Tf\|_{L^p(\mathbb{R}^{n+1})} \lesssim \sum_{j=1}^{\infty} \|T^{2^j} f\|_{L^p(\mathbb{R}^{n+1})} + \|f\|_{L^p_s(\mathbb{R}^n)}$$

where the second term comes from estimating the piece of the operator localised to unit frequency scales.

Thus, defining $\alpha(p) := \begin{cases} \bar{s}_p - 1/p & \text{if } \frac{2n}{n-1} \leq p < \infty \\ 0 & \text{if } 2 < p \leq \frac{2n}{n-1} \end{cases}$

it suffices to prove

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_\varepsilon \lambda^{\alpha(p)+\varepsilon} \|f\|_{L^p(\mathbb{R}^n)}. \qquad (1)$$

Indeed, if $s > \alpha(p)$, then taking $\varepsilon = \frac{s-\alpha(p)}{2}$ it follows that

$$\|T^\lambda f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{p,s} \lambda^{s-\varepsilon/2} \|\tilde{\beta}(\sqrt{-\Delta}/\lambda) f\|_{L^p(\mathbb{R}^n)}$$

by applying (1) to the frequency localised piece $\tilde{\beta}(\sqrt{-\Delta}/\lambda) f$. Write $f_j := \tilde{\beta}(\sqrt{-\Delta}/2^j) f$.

Thus, $$\sum_{j=1}^{\infty} \|T^{2^j} f\|_{L^p(\mathbb{R}^{n+1})} \lesssim_{p,s} \sum_{j=1}^{\infty} 2^{j(s-\varepsilon/2)} \|f_j\|_{L^p(\mathbb{R}^n)}$$

(Hölder & rapid decay of $2^{-\varepsilon j/2}$). $$\lesssim_{p,s} \Big(\sum_{j=1}^{\infty} [2^{js} \|f_j\|_{L^p(\mathbb{R}^n)}]^p \Big)^{1/p}$$

$$= \Big\| \Big(\sum_{j=1}^{\infty} [2^{js} |f_j|]^p \Big)^{1/p} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim \Big\| \Big(\sum_{j=1}^{\infty} (2^{js} |f_j|)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim \|f\|_{L^p_s(\mathbb{R}^n)}$$

by the Littlewood-Paley characterisation of Sobolev spaces.

Fourier support Taking the Fourier transform in the t-variable,

$$\mathcal{F}_t[T^\lambda f(x,\cdot)](\tau) =$$
$$\frac{1}{(2\pi)^n}\int_{\hat{\mathbb{R}}^n} e^{i\langle x,\xi\rangle}\int_{\mathbb{R}} e^{-it(\tau-|\xi|)}\rho(t)\,dt\,\beta\left(\tfrac{|\xi|}{\lambda}\right)\hat{f}(\xi)\,d\xi$$

$$= \frac{1}{(2\pi)^n}\int_{\hat{\mathbb{R}}^n} e^{i\langle x,\xi\rangle}\,\hat{\rho}(\tau-|\xi|)\,\beta(|\xi|/\lambda)\,\hat{f}(\xi)\,d\xi.$$

We can choose ρ so that $\operatorname{supp}\hat{\rho} \subseteq [-1,1]$, say. Hence, the space-time Fourier support of $T^\lambda f$ is contained in a neighbourhood of the light-cone

$$\Gamma := \{(\xi,\tau)\in\hat{\mathbb{R}}^{n+1} : \tau = |\xi|\}.$$

More precisely, if for $\lambda \geq 1$ we define

$$\Gamma(\lambda) := \{(\xi,|\xi|) : \xi\in\hat{\mathbb{R}}^n, \ \tfrac{\lambda}{2}\leq|\xi|\leq 2\lambda\},$$

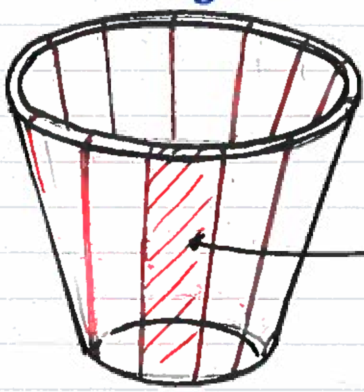
then the space-time Fourier support of $T^\lambda f$ is contained in $N_1\Gamma(\lambda)$.

This looks auspicious for us, since we can adapt many of the techniques / methods used to analyse the parabolic / spherical geometry of the Bochner-Riesz problem to the conical setting.

However, there are difficulties and the techniques do not go over wholesale...

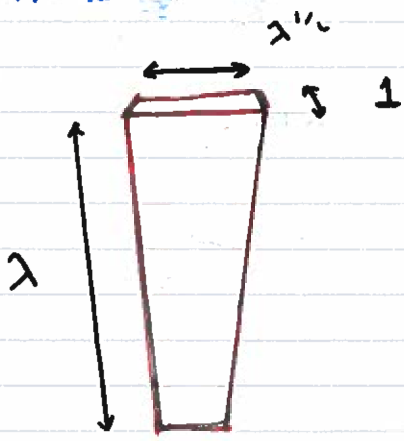
Wave packet decomposition:-

As in the Bochner-Riesz case, the first step is to break up the neighbourhood of the cone into essentially convex regions.



The natural decomposition here is into 'plates' θ of dimensions

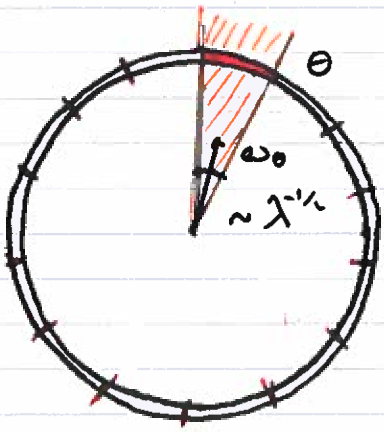
$$\underbrace{\lambda^{1/2} \times \dots \times \lambda^{1/2}}_{(n-1)-\text{fold}} \times 1 \times \lambda$$



The short (length 1) direction lies normal to the cone, the long (length λ) direction lies tangential to the cone in the flat direction and the remaining $\lambda^{1/2}$ sides of the plate lie tangential to the cone in the 'curved' directions.

Each plate θ corresponds to a choice of 'angular direction' $\omega_\theta \in S^{n-1}$ belonging to a $\lambda^{-1/2}$-net in S^{n-1}.

Thus, the total number of plates is $\sim \lambda^{\frac{n-1}{2}}$.



A horizontal cross section of the conic neighbourhood and plate decomposition.

Each plate θ lies over an angular sector in $\mathbb{R}^n$ of aperture $\sim \lambda^{-1/2}$, centred around a vector $\omega_\theta \in S^{n-1}$.

To decompose our operator according to these plates, choose $\tilde{\chi}_\theta \in C^\infty(\mathbb{R}^{n-1} \setminus \{0\})$ homogeneous of degree 0 satisfying

- $$\sum_{\theta : \text{plate}} \tilde{\chi}_\theta \equiv 1$$
- $\tilde{\chi}_\theta(\omega_\theta) = 1$ but $\chi_\theta(\omega) = 0$ if $\omega \in S^{n-1}$, $|\omega - \omega_\theta| \gtrsim \lambda^{-1/2}$
- $|\partial_\omega^\alpha \tilde{\chi}_\theta(\omega)| \lesssim_\alpha \lambda^{|\alpha|/2}$ for all $\alpha \in \mathbb{N}_0^n$, $\omega \in S^{n-1}$.

Now define

$$T_\theta^\lambda f(x,t) := \frac{1}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i(\langle x,\xi\rangle + t|\xi|)} \rho(t) \beta(|\xi|/\lambda) \tilde{\chi}_\theta(\xi) \hat{f}(\xi)\, d\xi .$$

so that

$$T^\lambda f = \sum_{\theta : \text{plate}} T_\theta^\lambda f$$

and each T_θ^λ has space-time Fourier support in θ.

Square Functions

As in the Bochner–Riesz problem, it is crucial to understand the interference patterns between the frequency-localised pieces

$$T_\theta^\lambda f.$$

In particular, one hopes to exploit cross-cancellation via a square function inequality.

In view of the numerology of the local smoothing conjecture, the following estimate appears natural.

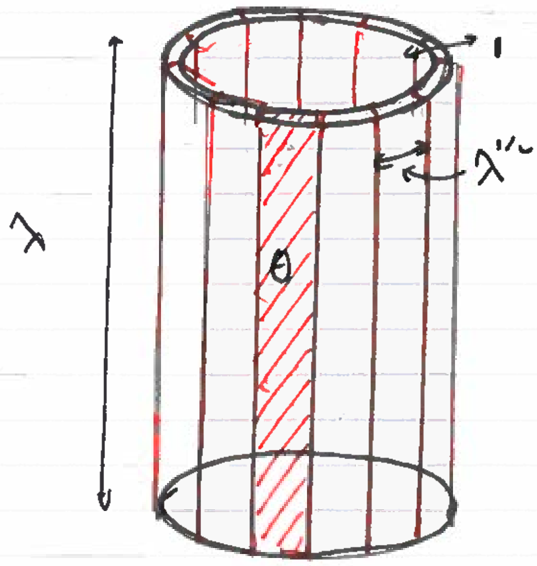
Conjecture 2 (Cone square function):- Suppose $\operatorname{supp} \hat{g} \subseteq N_1 \Gamma(\lambda)$ and $g = \sum_{\theta:\text{plate}} g_\theta$ where $\operatorname{supp} \hat{g}_\theta \subseteq \theta$ for each $\lambda^{1/2} \times \dots \times \lambda^{1/2} \times 1 \times \lambda$ plate θ. Then, for all $\varepsilon > 0$,

$$\Big\| \sum_{\theta:\text{plate}} g_\theta \Big\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^{n+1})} \lesssim_\varepsilon \lambda^\varepsilon \Big\| \Big(\sum_{\theta:\text{plate}} |g_\theta|^2 \Big)^{1/2} \Big\|_{L^{\frac{2n}{n-1}}(\mathbb{R}^{n+1})}.$$

Henceforth, we will focus on the $n=2$ case. Here the exponent $\frac{2n}{n-1}\Big|_{n=2} = 4$ and the above conjecture is a 'cone analogue' of the Córdoba–Fefferman L^4-square function for the parabola / sphere.

Remark (Cylindrical square function):- Consider the analogous problem for the cylinder in $\mathbb{R}^3$ with a "vertical" plate decomposition $\{\theta\}$ as pictured. In this case the critical L^4 estimate follows from the L^4 Córdoba–Fefferman result in $\mathbb{R}^2$ via tensorisation.

Alternatively, one can apply the "biorthogonality" argument of Córdoba–Fefferman directly to the cylinder.



The cylinder and corresponding plate decomposition.

The square function bound

$$\| \sum_\theta g_\theta \|_{L^4(\mathbb{R}^3)} \lesssim \| (\sum_\theta |g_\theta|^2)^{1/2} \|_{L^4(\mathbb{R}^3)}$$

follows from Cordobá–Fefferman (after rescaling and tensorising).

<u>Remark (Failure of "biorthogonality" for the cone)</u>:- One may attempt to carry out the Cordobá–Fefferman "biorthogonality" argument in the conic case. Note that

$$\| \sum_{\theta:\text{plate}} g_\theta \|^2_{L^4(\mathbb{R}^3)} = \| \, | \sum_{\theta:\text{plate}} g_\theta |^2 \|_{L^2(\mathbb{R}^3)}$$

$$= \| \sum_{\theta,\theta':\text{plate}} g_\theta \bar{g}_{\theta'} \|_{L^2(\mathbb{R}^3)}$$

$$= \| \sum_{\theta,\theta':\text{plate}} \hat{g}_\theta * \hat{\bar{g}}_{\theta'} \|_{L^2(\mathbb{R}^3)}$$

where $\operatorname{supp} \hat{g}_\theta * \hat{\bar{g}}_{\theta'} \subseteq \Theta - \Theta'$. Recall, in the Bochner–Riesz case (for the parabolic square function) the key inequality was

$$\sum_{\theta,\theta':\text{slab}} \chi_{\Theta-\Theta'}(\xi) \lesssim 1 \quad \text{for all } \xi \in \hat{\mathbb{R}}^2$$

and we would like something similar to hold in the cone case. However, for 'fairly typical'

$\xi \in \hat{\mathbb{R}}^3$ (meaning a significant proportion of $\bigcup_{\theta,\theta'} \theta - \theta'$) we in fact have

$$\sum_{\theta,\theta' : \text{plate}} \chi_{\theta-\theta'}(\xi) \gtrsim \lambda^{1/2} \quad (2)$$

The bound (2) precludes the use of such simple "biorthogonality" arguments (at least in any direct sense) in proving the cone square function estimate.

Nevertheless, Guth–Wang–Zhang were able to establish the critical L^4 square function bound in $\mathbb{R}^3$.

<u>Theorem 1 (Guth–Wang–Zhang, 2019)</u> For $n+1 = 3$, if

$\operatorname{supp} \hat{g} \subseteq N_1 \Gamma(\lambda)$ and $g = \sum_{\theta : \text{plate}} g_\theta$ where

$\operatorname{supp} \hat{g}_\theta \subseteq \theta$, then, for all $\varepsilon > 0$,

$$\Big\| \sum_{\theta : \text{plate}} g_\theta \Big\|_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon \lambda^\varepsilon \Big\| \Big(\sum_{\theta : \text{plate}} |g_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}$$

The proof of this result will occupy many of the upcoming lectures.

Before turning to the proof of Theorem 1, however, we shall discuss how the result can be used to prove the Local Smoothing Conjecture in 2 spatial dimensions.

<u>Corollary (Local smoothing for $n=2$).</u> For $n=2$ and $\varepsilon > 0$,

$$\| T^\lambda f \|_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon \lambda^\varepsilon \| f \|_{L^4(\mathbb{R}^2)} .$$

<u>Proof</u> :- We apply the Guth–Wang–Zhang square function to $T^\lambda f$ to deduce that

$$\| T^\lambda f \|_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon \lambda^{\varepsilon/2} \Big\| \Big(\sum_{\theta : \text{plate}} |T^\lambda_\theta f|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)} .$$

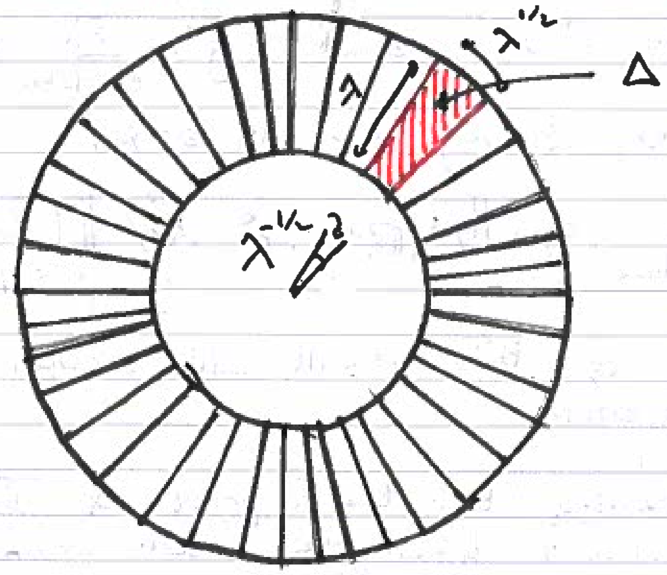
To estimate the square function, we use a duality argument similar to the Bochner-Riesz case. In particular,

$$\Big\|\Big(\sum_{\theta:\text{plate}} |T_\theta^\lambda f|^2\Big)^{1/2}\Big\|^2_{L^4(\mathbb{R}^3)} = \Big\|\sum_{\theta:\text{plate}} |T_\theta^\lambda f|^2\Big\|_{L^2(\mathbb{R}^3)}$$

$$= \int_{\mathbb{R}^3} \sum_{\theta:\text{plate}} |T_\theta^\lambda f|^2 \cdot g$$

for some $g \in L^2(\mathbb{R}^3)$; $\|g\|_{L^2(\mathbb{R}^3)} = 1$.

Now, $T_\theta^\lambda f = T_\theta^\lambda P_\theta f$ where here P_θ is a Fourier projection onto the angular sector of aperture $\lambda^{-1/2}$ of the λ scale annulus which contains $\text{proj}_{\mathbb{R}^2}\theta$:-



The operator P_θ localises a function f in the frequency space to an annular region $\Delta = \Delta(\theta)$ as pictured :-

$$(P_\theta f)^\wedge = \tilde{\chi}_{\Delta(\theta)} \cdot \hat{f}$$

where θ "lies over" $\Delta(\theta)$.

We can also write

$$T_\theta^\lambda f(x,t) = \int_{\mathbb{R}^2} K_\theta^\lambda(x,t;y)\, f(y)\, dy$$

It is, in fact, no more difficult to prove a version of this lemma in arbitrary dimensions.

To simplify notation, write

$$K_\theta^\lambda(x,t) := K_\theta^\lambda(x,t,0) = \frac{1}{(2\pi)^n}\int_{\widehat{\mathbb{R}}^n} e^{i(\langle x,\xi\rangle + t|\xi|)} a_\theta^\lambda(\xi,t)\, d\xi$$

so that it suffices to show

<u>Lemma 2</u>. For all $n \geq 2$, $N \in \mathbb{N}$,

$$|K_\theta^\lambda(x,t)| \lesssim_N \lambda^{\frac{n+1}{2}}\left(1 + \lambda\,|x\cdot\omega_\theta + t| + \lambda^{1/2}\,|\mathrm{proj}_{\omega_\theta^\perp} x|\right)^{-N} \quad (6)$$

where $\mathrm{proj}_{\omega_\theta^\perp}$ is the orthogonal projection onto the space

$$\mathrm{span}\{\omega_\theta\}^\perp.$$

Before proving the lemma, we note that the above bound shows $K_\theta^\lambda(x,t)$ is concentrated on a box

$$T_s := \left\{(x,t)\in\mathbb{R}^2\times[\tfrac{1}{2},2] : \left|\begin{pmatrix}x\\t\end{pmatrix}\cdot\begin{pmatrix}\cos s\\ \sin s\\ 1\end{pmatrix}\right| \lesssim \lambda^{-1} \text{ and } \left|\begin{pmatrix}x\\t\end{pmatrix}\cdot\begin{pmatrix}-\sin s\\ \cos s\\ 0\end{pmatrix}\right| \lesssim \lambda^{-1/2}\right\}.$$

for a suitable choice of s (ie $\begin{pmatrix}\cos s\\ \sin s\end{pmatrix} = \omega_\theta$).

Thus, defining

$$\mathcal{N}^\lambda g(y) := \sup_{s\in[0,2\pi]} \fint_{T_s} |g(x-y,t)|\,dx\,dt,$$

by suitable rescaling and the rapid decay in (6), we see that (5) would follow from a maximal bound

$$\|\mathcal{N}^\lambda g\|_{L^2(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^\varepsilon \|g\|_{L^2(\mathbb{R}^3)}. \quad (5')$$

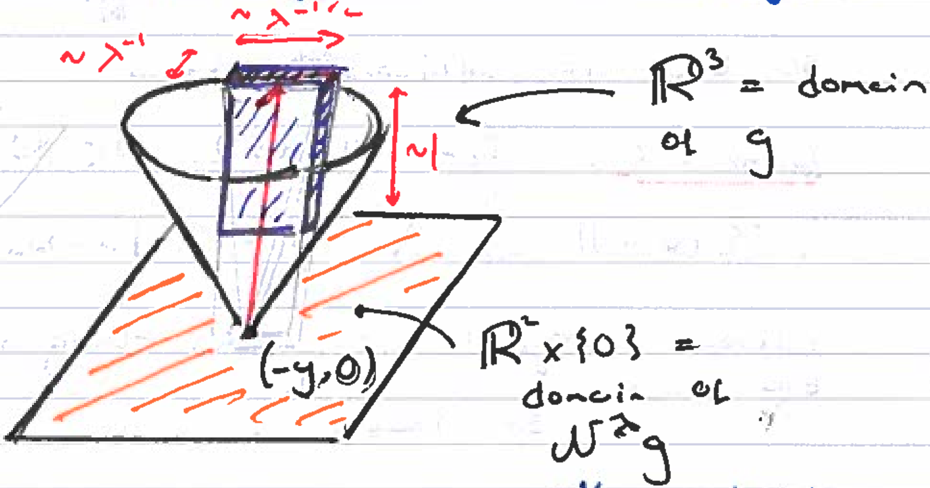
We will prove (5') (along with (4)) in the next lecture.

The maximal function $\mathcal{N}^\lambda$ can be described as follows:-

For $g \in \mathcal{S}(\mathbb{R}^2)$, say, if $y \in \mathbb{R}^2$, which we think of as a point in $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$, then

$\mathcal{W}^\lambda g(y)$ is a maximal average over plates

on the cone with apex centred at $(-y, 0)$:-



These plates have dimensions $\lambda^{-1/2} \times \lambda^{-1} \times 1$ and are translates of the dual plates θ^*.

Proof (of Lemma 2):- Let $\Phi(x, t; \xi) := \langle x, \xi \rangle + t|\xi|$ denote the phase of K_θ^λ. Write

$$\Phi(x, t; \xi) =: \langle \partial_\xi \Phi(x, t; \omega_\theta), \xi \rangle + \mathcal{E}(\xi, t)$$

where $\mathcal{E}(\xi, t) = t|\xi|\left(1 - \langle \omega_\theta, \frac{\xi}{|\xi|} \rangle\right)$ satisfies

$\mathcal{E}(\omega_\theta, t) = 0$ and $\partial_\xi \mathcal{E}(\omega_\theta, t) = 0$.

We will choose coordinate axes for $\mathbb{R}^n$ so that the ξ_1-coordinate is in the ω_θ-direction.

Recall :-

Theorem (Euler's homogeneous function theorem):- Suppose $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is C^1. Then f is homogeneous of degree $k \in \mathbb{Z}$ if and only if

$$\langle x, \partial_x f(x) \rangle = k f(x)$$

Consequently, if f is homogeneous of degree k and in addition belongs to the class $C^j(\mathbb{R}^n \setminus \{0\})$, then any jth-order partial derivative of f is homogeneous of degree $k - j$.

where the kernel K_θ^λ is given by

$$K_\theta^\lambda(x,t;y) := \frac{1}{(2\pi)^2}\int_{\hat{\mathbb{R}}^2} e^{i(\langle x-y,\xi\rangle + t|\xi|)}\, a_\theta^\lambda(\xi,t)\, d\xi$$

for $\quad a_\theta^\lambda(\xi,t) := \rho(t)\cdot\beta(|\xi|/\lambda)\cdot\tilde{\chi}_\theta(\xi).$

It will be shown below that

$$\int_{\mathbb{R}^2} |K_\theta^\lambda(x,t,y)|\, dy \lesssim 1 \qquad (3)$$

and so, by Cauchy – Schwarz,

$$|T_\theta^\lambda f|^2 = |T_\theta^\lambda P_\theta f|^2 \lesssim T_\theta^\lambda |P_\theta f|^2$$

so that

$$\int_{\mathbb{R}^3} \sum_{\theta:\text{plate}} |T_\theta^\lambda f|^2 \cdot g \lesssim \sum_{\theta:\text{plate}} \int_{\mathbb{R}^3} T_\theta^\lambda |P_\theta f|^2 \cdot |g|$$

$$= \sum_{\theta:\text{plate}} \int_{\mathbb{R}^2} |P_\theta f(y)|^2 \Big(\int_{\mathbb{R}^3} |K_\theta^\lambda(x,t;y)|\, |g(x,t)|\, dx\, dt\Big) dy$$

$$\leq \Big\| \sum_{\theta:\text{plate}} |P_\theta f|^2 \Big\|_{L^2(\mathbb{R}^2)} \, \| \tilde{\mathcal{N}}^\lambda g \|_{L^2(\mathbb{R}^2)}$$

where

$$\tilde{\mathcal{N}}^\lambda g(y) := \max_\theta \int_{\mathbb{R}^3} |K_\theta^\lambda(x,t;y)|\, |g(x,t)|\, dx\, dt.$$

Thus, as in the Bochner – Riesz case, the problem is reduced to proving:-

- A <u>forward square function estimate</u>

$$\Big\| \Big(\sum_{\theta:\text{plate}} |P_\theta f|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^\varepsilon \|f\|_{L^4(\mathbb{R}^2)} \qquad (4)$$

where the frequency projections P_θ are now defined with respect to the sectors described above.

- A <u>Nikodym-type maximal estimate</u>

$$\|\widetilde{\mathcal{N}}^\lambda g\|_{L^2(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^\varepsilon \|g\|_{L^2(\mathbb{R}^3)} . \qquad (5)$$

Both (4) and (5) will be treated in the next lecture. It is remarked that

- The square function (4) is significantly more complicated than the corresponding square function used for the Bochner-Riesz problem (recall, this square function corresponded to frequency projection onto even-lengthed strips).

In particular, since the angular sectors corresponding to the P_θ's in (4) are orientated in distinct directions, Kakeya / Nikodym-type considerations play a rôle here.

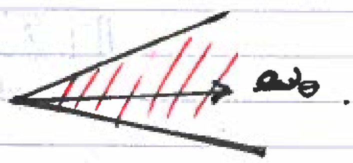
- The Nikodym estimate (5) is also somewhat different than the corresponding bound in the Bochner-Riesz case, most notably because here we are dealing with an operator mapping functions of 3 variables to functions of 2 variables (rather than "2 to 2" as before).

We note the following bound for the kernel K_θ^λ, which will reduce the study of $\widetilde{\mathcal{N}}^\lambda$ to that of a "geometric" maximal function $\mathcal{N}^\lambda$, defined below.

<u>Lemma 1</u> For all $N \in \mathbb{N}$,

$$|K_\theta^\lambda(x,t;y)| \lesssim_N \lambda^{\frac{3}{2}} \left(1 + \lambda|(x-y)\cdot\omega_\theta + t| + \lambda^{1/2}|(x-y)\cdot\omega_\theta^\perp|\right)^{-N}.$$

Here ω_θ is the "direction" of the radial sector defining θ.



Note that the L^1-type bound (3) immediately follows from this lemma.

Using this theorem and our choice of coordinates, $\partial_{\xi_1}^N \mathcal{E}(\xi_1, 0, \dots, 0, t) = 0$ and $\partial_{\xi'} \mathcal{E}(\xi_1, 0, \dots, 0, t) = 0$ and so

$$\partial_{\xi'} \partial_{\xi_1}^N \mathcal{E}(\xi_1, 0, \dots, 0, t) = 0$$

for any $N \geq 1$. Consequently, if one applies Taylor's theorem to the function

$$\xi' \mapsto \partial_{\xi_1}^N \mathcal{E}(\xi_1, \xi', t)$$

about 0, then it follows that

$$|\partial_{\xi_1}^N \mathcal{E}(\xi_1, \xi', t)| \lesssim M(\xi) \cdot |\xi'|^2 \quad \text{where}$$

$$M(\xi) := \max_{|\alpha| = N+2} \sup\{ |\partial_\xi^\alpha \mathcal{E}(\xi_1, r\xi', t)| : 0 < r < 1\}$$

$$\lesssim \max_{|\alpha| = N+2} \sup\{ |\partial_\xi^\alpha \mathcal{E}(\omega, t)| : \omega \in S^{n-1}\} \, |\xi|^{-(N+1)}$$

if $|\xi| \sim |\xi_1|$

$$\lesssim_N |\xi|^{-(N+1)} \quad \text{by homogeneity.}$$

For $\xi \in \operatorname{supp} a_\theta^\lambda$ it follows that

$$|\xi| \sim |\xi_1| \sim \lambda \quad \text{and} \quad |\xi'| \lesssim \lambda^{1/2}$$

and therefore

$$|\partial_{\xi_1}^N \mathcal{E}(\xi; t)| \lesssim_N \lambda^{-N}. \qquad (7)$$

Similarly, since $\partial_{\xi'} \mathcal{E}(\xi_1, 0, \dots, 0, t) = 0$ it follows by applying Taylor's theorem to

$$\xi' \mapsto \partial_{\xi'} \mathcal{E}(\xi_1, \xi', t)$$

that

$$|\partial_{\xi'} \mathcal{E}(\xi_1, \xi', t)| \lesssim \max_{|\alpha|=2} \sup\{ |\partial_\xi^\alpha \mathcal{E}(\xi_1, r\xi', t)| : 0 < r < 1\} \cdot |\xi'|$$

$$\lesssim |\xi|^{-1}|\xi'| \lesssim \lambda^{-1/2} \qquad \text{if } |\xi| \sim |\xi_1| . \qquad (8)$$

On the other hand, by homogeneity,

$$|(\partial_{\xi'})^N \mathcal{E}(\xi, t)| \lesssim_N |\xi|^{1-N} \lesssim \lambda^{-(N-1)} . \qquad (9)$$

and, combining (8) and (9), it follows that

$$|(\partial_{\xi'})^N \mathcal{E}(\xi, t)| \lesssim_N \lambda^{-N/2} \quad \text{for } \xi \in \operatorname{supp} a_\theta^\lambda . \qquad (10)$$

Now express the kernel as

$$K_\theta^\lambda(x,t) = \frac{1}{(2\pi)^n}\int_{\hat{\mathbb{R}}^n} e^{i\langle \partial_\xi \Phi(x,t,\omega_\theta), \xi\rangle} b_\theta^\lambda(\xi,t)\, d\xi$$

where $$b_\theta^\lambda(\xi,t) := e^{i\mathcal{E}(\xi,t)} a_\theta^\lambda(\xi,t) .$$

To estimate this oscillatory integral we apply an integration-by-parts argument.

Consider the self-adjoint differential operator

$$L^N := i^{2N} I + \lambda^{2N} \partial_{\xi_1}^{2N} + \lambda^N \Delta_{\xi'}^N .$$

and the function

$$w^N(x) := (-1)^N \left(1 + \lambda^{2N} |\partial_{\xi_1} \Phi(x,t;\omega_\theta)|^{2N} + \lambda^N |\partial_{\xi'} \Phi(x,t;\omega_\theta)|^{2N}\right)$$

so that the factor

$$e^{i\langle \partial_\xi \Phi(x,t,\omega_\theta), \xi\rangle}$$

is fixed by $w^N(x) \cdot L^N$.

Observe, on the other hand, that the bounds (7) and (10) imply that

$$|L^N e^{i\mathcal{E}(\xi,t)}| \lesssim_N 1$$

and, furthermore,

$$|L^N b_\theta^\lambda(\xi,t)| \lesssim_N 1 . \qquad (11)$$

To see the latter bound, one uses the fact that

$$|\partial_{\xi_1}^N \tilde{\chi}_\theta(\xi)| \lesssim_N |\xi|^{-N} \quad \text{(in our choice of coordinates)}$$

where $\tilde{\chi}_\theta$ is the smooth cut-off introduced above. Indeed, this is because

$$\partial_{\xi_1} = \partial_r + O(\lambda^{-1/2}) \cdot \partial_\xi$$

on supp χ_θ, where ∂_r denotes the radial derivative, and $\partial_r^N \chi_\theta \equiv 0$ by homogeneity.

Combining the above observations,

$$|K_\theta^\lambda(x,t)| \lesssim |\omega^N(x)| \, \Big| \int_{\hat{\mathbb{R}}^n} \big(L^N e^{i\langle \partial_\xi \Phi(x,t,\omega_\theta), \xi \rangle}\big) \cdot b_\theta^\lambda(\xi,t)\, d\xi \Big|$$

$$\lesssim |\omega^N(x)| \int_{\hat{\mathbb{R}}^n} |L^N b_\theta^\lambda(\xi,t)|\, d\xi$$

$$\lesssim \lambda^{\frac{n+1}{2}} |\omega^N(x)| \qquad (12)$$

where the final inequality follows from (11) and the fact that

$$(\xi,t) \mapsto b_\theta^\lambda(\xi,t)$$

is essentially supported in a $\lambda^{1/2} \times \dots \times \lambda^{1/2} \times \lambda \times 1$ rectangle.

Recalling once again our choice of coordinates, it is clear that (12) immediately implies the desired bound (6).