Lecture 10: Local smoothing for the wave equation.

In the following lectures we will investigate a problem from hyperbolic PDE closely related to the Bochner-Riesz conjecture, but which turns out to be substantially more challenging.

Let $n \geq 2$ and consider the Cauchy problem for the wave equation in n spatial variables:-

$$\begin{cases} (\Delta_x - \partial_t^2) u = 0 \\ \partial_t^i u(\cdot, 0) = f_i \qquad i = 0, 1 \end{cases} \qquad (W)$$

If the f_i are sufficiently regular, then the solution to (W) can be expressed in terms of the half-wave semigroup

$$e^{it\sqrt{-\Delta}} f(x) := \frac{1}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i(\langle x, \xi \rangle + t|\xi|)} \hat{f}(\xi)\, d\xi .$$

In particular,

$$u(x,t) = e^{it\sqrt{-\Delta}} \phi_-(x) + e^{-it\sqrt{-\Delta}} \phi_+(x) \qquad (1)$$

where $\widehat{\phi_{\pm}}(\xi) := \frac{1}{2}\left(\hat{f}_0(\xi) \pm i \frac{\hat{f}_1(\xi)}{|\xi|} \right)$.

<u>Basic question</u>:- How much regularity must one impose on f_0, f_1 to ensure the solution u lies in L^p?

Using (1), we can recast this question in terms of the propagator $e^{it\sqrt{-\Delta}}$.

<u>Theorem 1 (Fixed-time estimate)</u>:- For $1 < p < \infty$,

$$\| e^{it\sqrt{-\Delta}} f \|_{L^p(\mathbb{R}^n)} \lesssim \| f \|_{L^p_s(\mathbb{R}^n)}$$

for $\quad s \geqslant \bar{s}_p := (n-1) \cdot \left| \frac{1}{p} - \frac{1}{2} \right|.$

Here $L^p_s(\mathbb{R}^n)$ denotes the standard Sobolev (or Bessel potential) space defined with respect to the multiplier

$$(1 + |\xi|^2)^{s/2};$$

ie.

$$L^p_s(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n) : (1 - \Delta_x)^{s/2} f \in L^p(\mathbb{R}^n) \}$$

with

$$\| f \|_{L^p_s(\mathbb{R}^n)} := \| (1 - \Delta_x)^{s/2} f \|_{L^p(\mathbb{R}^n)}.$$

Such 'fixed time' estimates appear, for instance, in the work of Peral and were extended to general Fourier integral operators by Seeger-Sogge-Stein.

Note, the $p = 2$ case of the result is trivial owing to the energy conservation identity

$$\| e^{it\sqrt{-\Delta}} f \|_{L^2(\mathbb{R}^n)} = \| f \|_{L^2(\mathbb{R}^n)}.$$

Theorem 1 then follows via interpolation and duality from an appropriate $H^1 \to L^1$ bound:-

$$\| e^{it\sqrt{-\Delta}} (1 - \Delta_x)^{-\frac{n-1}{4}} f \|_{L^1(\mathbb{R}^n)} \lesssim \| f \|_{H^1(\mathbb{R}^n)}.$$

See the references for details.

Sharpness:- Theorem 1 is sharp in the sense that one cannot replace $\bar{s}_p$ with some smaller exponent. To see this, note

the inverse Fourier transform of the distribution

$$e^{-i|\xi|} (1 + |\xi|^2)^{-\alpha/2}$$

agrees with a function f_α. Moreover:-

- f_α is rapidly decreasing for $|x| > 2$
- f_α satisfies

$$|f_\alpha(x)| \sim |1 - |x||^{-\frac{n+1}{2} + \alpha} \quad \text{for } |x| \leq 2 \qquad (2)$$

To see where these numbers come from, if we write formally

$$f_\alpha(x) = \frac{1}{(2\pi)^n} \int_{\hat{\mathbb{R}}^n} e^{i(\langle x, \xi\rangle - |\xi|)} (1+|\xi|^2)^{-\alpha/2} \, d\xi$$

then the stationary points of the phase occur at

$$x - \frac{\xi}{|\xi|} = 0.$$

A rough sketch of f_α: the function is singular on S^{n-1}.

Hence on S^{n-1} there is no oscillation to help us to integrate $(1+|\xi|^2)^{-\alpha/2}$ and so the function is singular on this set (at least for small α values).

To see where the $-\frac{n+1}{2}+\alpha$ power comes from, we use polar co-ordinates

$$f_\alpha(x) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{S^{n-1}} e^{i\langle x, \omega\rangle} \, d\sigma(\omega) \, e^{-ir} (1+r^2)^{-\frac{\alpha}{2}} r^{n-1} \, dr$$

Recall,

$$\int_{S^{n-1}} e^{i\langle rx, \omega\rangle} \, d\sigma(\omega) = (d\sigma)^\vee(rx) = \sum_{\pm} \frac{e^{\pm ir|x|}}{(1+r|x|)^{\frac{n-1}{2}}} a_\pm(rx)$$

where $a_\pm \in S^0$. Thus, for $|x| \sim 1$, concentrating on the large r regime, we essentially have

$$f_\alpha(x) \sim \int_1^\infty e^{ir(|x|-1)} r^{\frac{n-1}{2}-\alpha} \, dr$$

(N.B. the contribution with phase $e^{-ir(|x|+1)}$ will have rapid decay by non-stationary phase).

Thus, $f_\alpha(x)$ is comparable to the Fourier transform of the homogeneous distribution

$$r \longmapsto r^{\frac{n-1}{2}-\alpha} \qquad (3)$$

evaluated at $|x|-1$. By basic distribution theory, the Fourier transform of (3) is homogeneous of order $-\left(\frac{n-1}{2}-\alpha\right)-1 = -\frac{n+1}{2}+\alpha$, which motivates (2).

A rigorous presentation of this computation can be found in Stein's Harmonic Analysis, Chapter IX, § 6.13.

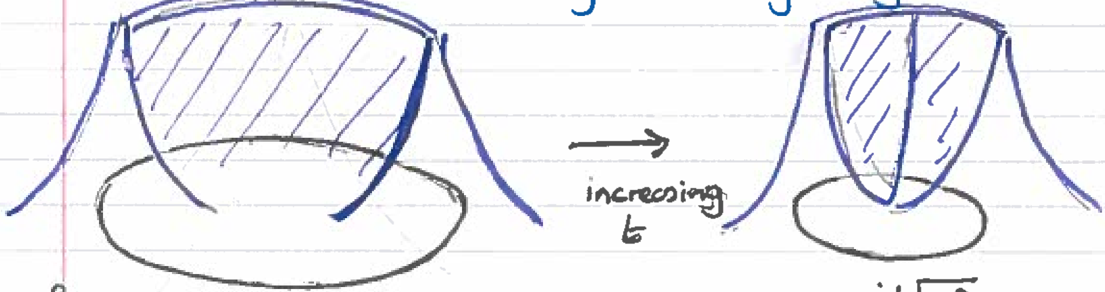
Now consider $e^{i\sqrt{-\Delta}}$ – the unit time propagator – acting on f_α. Formally,

$$e^{i\sqrt{-\Delta}} f_\alpha(x) = \frac{1}{(2\pi)^n}\int_{\hat{\mathbb{R}}^n} e^{i\langle x,\xi\rangle}\left(1+|\xi|^2\right)^{-\alpha/2} d\xi$$

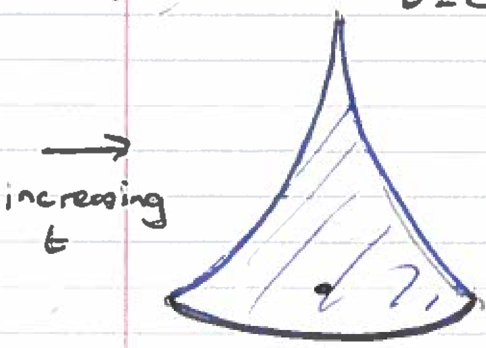
and so

$$\left|e^{i\sqrt{-\Delta}} f_\alpha(x)\right| \gtrsim |x|^{-n+\alpha} \text{ for } |x| \lesssim 1 \qquad (4)$$

(this computation can again be justified at a heuristic level by homogeneity considerations).



f_α $\quad t=0$ $\qquad\qquad e^{it\sqrt{-\Delta}} f_\alpha$



$t=1$ $\qquad e^{i\sqrt{-\Delta}} f_\alpha$

As t increases from 0 to 1 we can think of the initial waves concentrated around S^{n-1} coalescing into a single singularity at the point 0.

The inequality $\|e^{i\sqrt{-\Delta}} f_\alpha\|_{L^p(\mathbb{R}^n)} \lesssim \|f_\alpha\|_{L^p_s(\mathbb{R}^n)}$

can be rewritten $\|e^{i\sqrt{-\Delta}} f_\alpha\|_{L^p(\mathbb{R}^n)} \lesssim \|f_{\alpha-s}\|_{L^p(\mathbb{R}^n)}$,

(5)

Observe, by (2) we have :-

if $\left(-\frac{n+1}{2} + \alpha - s\right)p > -1$, then $f_{\alpha-s} \in L^p(\mathbb{R}^n)$. (5a)

By (4) we have

if $(-n+\alpha)p \leq -n$, then $e^{i\sqrt{-\Delta}} f_\alpha \notin L^p(\mathbb{R}^n)$

(6b)

Combining (5a) and (6b), we see (5) cannot hold if

$$s < (n-1)\cdot\left(\tfrac{1}{2} - 1/p\right)$$

which shows the sharpness of Theorem 1 for $2 \leq p < \infty$. The remaining range can be treated by duality.

Remark :- One may also treat the $1 < p \leq 2$ range via an explicit construction (rather than appeal to duality) by 'dualising' the example given above. In particular, choose g_α so that

- The initial condition g_α is concentrated at the origin with a singularity at this point.
- $e^{i\sqrt{-\Delta}} g_\alpha$ is concentrated around S^{n-1} with a singularity along this surface.

In the above example the specific time $t = 1$ plays an important rôle as it is precisely the instant when the waves coalesce at the origin.

For general t, one may expect $e^{it\sqrt{-\Delta}} f_\alpha$ to be much better behaved.

Example :- For f_α as above, one may show that

$$\left|e^{it\sqrt{-\Delta}} f_\alpha(x)\right| \gtrsim |x|^{-\frac{n-1}{2}} \big|t-1-|x|\big|^{-\frac{n+1}{2}+\alpha}$$

if $t \geq 2|x|+1$

for $|x| \lesssim 1$.

- As before, if $\left(-\frac{n+1}{2} + \alpha - s\right)p > -1$, then $f_{\alpha - s} \in L^p(\mathbb{R}^n)$ (7a)

- If $\alpha \leq n - \frac{n+1}{p}$, then (7b)

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_\alpha \|^p_{L^p(\mathbb{R}^n)} \, dt\right)^{1/p} = \infty .$$

Comparing (7a) and (7b) we see that we can hope for the "averaged"

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f_\alpha \|^p_{L^p(\mathbb{R}^n)} \, dt\right)^{1/p}$$

to be bounded under the weaker regularity hypothesis

$$\| f_\alpha \|_{L^p_s(\mathbb{R}^n)} < \infty \quad \text{for } s \geq \bar{s}_p - 1/p .$$

Conjecture (Local Smoothing):- For $n \geq 2$, the inequality

$$\left(\int_1^2 \| e^{it\sqrt{-\Delta}} f \|^p_{L^p(\mathbb{R}^n)} \, dt\right)^{1/p} \lesssim \| f \|_{L^p_s(\mathbb{R}^n)} \qquad (8)$$

holds for all

$$\begin{cases} s > \bar{s}_p - 1/p & \text{if } \frac{2n}{n-1} \leq p < \infty \\ s > 0 & \text{if } 2 < p \leq \frac{2n}{n-1}. \end{cases}$$

Remark:- • The exponent $\bar{p}_n := \frac{2n}{n-1}$ corresponds to the value where $\bar{s}_p - 1/p = 0$.

One can show using Fefferman-type counter-examples that (8) cannot hold with $s = 0$ for $p = \bar{p}_n$.

- The exponent cannot be improved beyond $s \geq 0$ for $p = 2$ by conservation of energy.

- The exponent cannot be improved beyond $s \geq \bar{s}_p$ for $1 < p \leq 2$ because of the g_α example above.