Modern Developments in Fourier Analysis.

Lecture 1

Fourier Analysis : Background.

We begin with a review of basic definitions from Fourier analysis. This is not intended to be comprehensive.

Def^{n}: Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$. Thus,

$$f \in \mathcal{S}(\mathbb{R}^n) \quad \text{iff} \quad \|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\beta (\partial_x^\alpha f)(x)| < \infty$$

$$\text{for all } \alpha, \beta \in \mathbb{N}_0^n.$$

Here

$$x^\beta := x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n} \quad \text{and} \quad \partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

Def^{n}: Given $f \in \mathcal{S}(\mathbb{R}^n)$ define its Fourier transform

$$\hat{f}(\xi) := \int_{\hat{\mathbb{R}}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)\,dx \qquad \xi \in \hat{\mathbb{R}}^n := \mathbb{R}^n \quad (1)$$

The map $\mathcal{F} : f \longmapsto \hat{f}$ is :

- linear between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$
- continuous with respect to the topology induced by $\|\cdot\|_{\alpha,\beta}$.
- a homeomorphism, with inverse $\mathcal{F} : f \longmapsto \check{f}$ where

$$\check{f}(x) := \hat{f}(-x).$$

In particular, we have the

$$\boxed{\begin{array}{l}\underline{\text{Inversion formula:}}\\[1em] f(x) = \displaystyle\int_{\hat{\mathbb{R}}^n} e^{2\pi i\langle x,\xi\rangle} \hat{f}(\xi)\,d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n)\end{array}}$$

Problem: We would like to make sense of $\hat{f}$ for more general (ie less regular) f.

- If $f \in L^1(\mathbb{R}^n)$, then the formula (1) still makes sense and we have the Riemann–Lebesgue bound

$$\|\hat{f}\|_{L^\infty(\hat{\mathbb{R}}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \qquad (2)$$

- If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\|\hat{f}\|_{L^2(\hat{\mathbb{R}}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \qquad (3)$$

Since $\mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ is a dense subspace, we can combine (3) with some functional analysis to conclude that $\mathcal{F}$ extends to a bounded linear operator on L^2 which satisfies

Plancherel's identity

$$\|\hat{f}\|_{L^2(\hat{\mathbb{R}}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } f \in L^2(\mathbb{R}^n)$$

This is supernice because now we have defined $\mathcal{F}$ as an operator on a (very useful!) Hilbert space $L^2(\mathbb{R}^n)$, rather than the (highly restrictive) locally convex topological vector space $\mathcal{S}(\mathbb{R}^n)$.

Note, to prove the Planchetel identity for $f \in L^2(\mathbb{R}^n)$ it sufficed to prove the a priori estimate, i.e. the same bound for f belonging to the dense subclass $\mathcal{S}(\mathbb{R}^n)$. Functional analysis then does the rest of the work. This is a general theme in the kind of problems we are interested in.

- Interpolating (2) and (3) (using Riesz–Thorin), one may extend $\mathcal{F}$ to a bounded linear operator $\mathcal{F}: L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ for $1 \leq p \leq 2$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we have

$\boxed{\begin{array}{l}\underline{\text{Hausdorff – Young inequality}} \quad \text{For } 1 \le p \le 2 \\ \|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in L^p(\mathbb{R}^n).\end{array}}$

<u>Problem</u>:- We would like to make sense of the inversion formula

$$f(x) = \int_{\hat{\mathbb{R}}^n} e^{2\pi i \langle x, \xi \rangle} \hat{f}(\xi)\, d\xi \qquad (4)$$

for more general (i.e. less regular) f.

For (4) to make sense we need $\hat{f} \in L^1(\mathbb{R}^n)$ – this is a tall order!

- $\hat{f} \in L^1(\mathbb{R}^n)$ holds if f is sufficiently smooth with integrable derivatives (by applying integration-by-parts to the formula for $\hat{f}$).
- For $1 \le p \le 2$, if $f \in L^p(\mathbb{R}^n)$, then $\hat{f} \in L^{p'}(\mathbb{R}^n)$ where $2 \le p' \le \infty$ is far from 1!

$\underline{n=1}$ The idea here is to use a summation method. If $\hat{f} \in L^{p'}(\hat{\mathbb{R}}^1)$, then $\hat{f} \in L^1_{loc}(\hat{\mathbb{R}}^1)$ so the partial sums

$$S_R f(x) := \int_{-R}^{R} e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi$$

make sense. We can reformulate our problem more precisely as:-

<u>Problem'</u> Under what hypotheses does

$$S_R f \longrightarrow f \quad \text{as} \quad R \to \infty \ ?$$

Since we are dealing with sequences of functions, there are various different modes of convergence. We focus on 2:-

1. Almost everywhere convergence.

One of the most famous theorems in Fourier analysis (or, indeed, analysis in general) is :-

Theorem (Carleson-Hunt) If $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, then

$$S_R f \to f \quad \text{a.e.} \quad \text{as } R \to \infty.$$

2. L^p convergence. This is much easier than the Carleson-Hunt theorem

Theorem (M. Riesz) If $1 < p < \infty$, then

$$\| S_R f - f \|_p \to 0 \quad \text{as } R \to \infty$$

whenever $f \in L^p(\mathbb{R}^n)$.

If $f \in \mathcal{S}(\mathbb{R}^1)$, then $\| S_R f - f \|_p \to 0$ as $R \to \infty$ trivially holds by the inversion formula. Since $\mathcal{S}(\mathbb{R})$ is dense in L^p, it suffices to show

$$\sup_{R \geq 1} \| S_R f \|_p \lesssim C \| f \|_p$$

ie. the S_R have uniformly bounded L^p operator norms

(Proof :- Let $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$. There exists $g \in \mathcal{S}(\mathbb{R})$ such that

$$\| f - g \|_p < \varepsilon / 2(C+1)$$

and $R_0 \geq 1$ such that

$$\| S_R g - g \|_p < \varepsilon/2 \quad \text{for all } R \geq R_0$$

Thus, if $R \geq R_0$, then

$$\begin{aligned} \| S_R f - f \|_p &\leq \| S_R (f-g) \|_p + \| S_R g - g \|_p + \| g - f \|_p \\ &\leq C \cdot \| f - g \|_p + \| S_R g - g \| + \| f - g \|_p \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad). \end{aligned}$$

By rescaling,

$$\|S_R\|_{p\to p} = \|S_1\|_{p\to p}$$

and so Riesz's theorem follows from (and is in fact equivalent to):-

$$\|S_1\|_{p\to p} < \infty \quad \text{for all} \quad 1 < p < \infty.$$

This is an immediate consequence of the L^p boundedness of the Hilbert transform, since S_1 can be written as a simple superposition of (suitably affine transformed) copies of H.

Higher dimensions? In higher dimensions, there are many different choices of summation method:

Square sums

Define

$$S_R f(x) := \int_{[-R,R]^n} e^{2\pi i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi.$$

In this case everything tensorizes and reduces to the $n=1$ case. In particular:

Carleson-Hunt: If $1 < p \leq 2$ and $f \in L^p(\mathbb{R}^n)$, then $S_R f \to f$ a.e. as $R \to \infty$

M. Riesz: If $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$\|S_R f - f\|_p \to 0 \quad \text{as} \quad R \to \infty.$$

Spherical sums:-

Define

$$S_R f(x) := \int_{B(0,R)} e^{2\pi i\langle x,\xi\rangle} \hat{f}(\xi)\, d\xi$$

Here we cannot tensorize and this is a genuinely higher dimensional problem.

Almost everywhere convergence? MAJOR OPEN PROBLEM!

L^p convergence? Trivially L^2 convergence holds by Plancherel:

$$\|S_R f\|_{L^2(\mathbb{R}^n)} = \|(S_R f)^\wedge\|_{L^2(\hat{\mathbb{R}}^n)}$$
$$= \|\chi_{B(0,R)} \hat{f}\|_{L^2(\hat{\mathbb{R}}^n)}$$
$$\leq \|\hat{f}\|_{L^2(\hat{\mathbb{R}}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$$

What about $p \neq 2$?

Theorem (C. Fefferman) For $p \neq 2$ and $n \geq 2$,

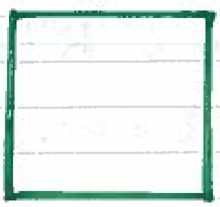
$$\|S_R\|_{p \to p} = \infty.$$

Consequently, there exist $f \in L^p(\mathbb{R}^n)$ such that

$$S_R f \not\to f \text{ in } L^p(\mathbb{R}^n) \quad \text{as } R \to \infty.$$

Remark:- To deduce the failure of L^p convergence from the unboundedness of the operator we have to reverse the reduction described above. This is slightly non-trivial and uses the principle of uniform boundedness.

Moral:



Good

(Very) bad!

Fefferman's theorem revealed some deep underlying connections between higher dimensional Fourier analysis and difficult geometric problems (such as the Kakeya conjecture). These connections are a highly active area of contemporary research, with numerous applications to harmonic analysis, GMT, PDE, analytic number theory and beyond.