Informal course:

Paracontrolled distributions and singular stochastic PDEs

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(Notes taken by Justin Forlano)

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Contents:

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Stochastic Quantization equation (SQE) on $\mathbb{T}^d$: (dynamical Φ^4_d model).

Real-valued
$$\begin{cases} u|_{t=0} = u_0, & \partial_t u + (m-\Delta)u = -u^3 + \xi \end{cases}$$
ξ ↳ spacetime white noise

- $d=2$: Da Prato–Debussche '03 ⇝ u^{2k+1}, odd power nonlinearities
- $d=3$:
  - Hairer '14 (Theory of regularity structures)
  - → Gubinelli–Imkeller–Perkowski '15 (Paracontrolled distributions)
  - Kupiainen '16 (Renormalization group method).

Focus on this approach here

$\xi(t,x)$ = Gaussian space-time white noise.
$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$
$$\Downarrow$$
$$\xi(t,x) = \sum_{n\in\mathbb{Z}^d} e_n(x)\, d\beta_n(t), \quad e_n(x) = e^{2\pi i n\cdot x}, \; x\in\mathbb{T}^d,\; n\in\mathbb{Z}^d.$$

$\{\beta_n\}_{n\in\mathbb{Z}^d}$ = independent, standard complex-valued(*) Brownian motions conditioned that $\beta_n = \overline{\beta_{-n}}$.

($\overline{\beta_n} = \beta_{-n}$) so that ξ real)

β_n:
- $\beta_n(0) = 0$ a.s.
- $\beta_n(t_2) - \beta_n(t_1) \sim \mathcal{N}_{\mathbb{C}}(0, 2(t_2-t_1))$
- Independent increments over disjoint intervals.

Wiener Integral: $I(f) = \int_a^b f(t)\, d\beta(t)$, f deterministic. , $\omega\in\Omega$.

For simple f, i.e. $f(t) = \sum_{j=1}^N c_j \chi_{I_j}(t)$, $I_j = [a_j, b_j]$ disjoint.

Then define: $\int_a^b f(t)\,d\beta(t) := \sum_{j=1}^N f(a_j)[\beta(b_j;\omega) - \beta(a_j;\omega)]$.

(*): $\mathrm{Re}\,\beta_n$, $\mathrm{Im}\,\beta_n$ are indep $\mathbb{R}$-valued Brownian motions.

Can then show that I extends to a mapping

$$I : L^2([a,b]) \to L^2(\Omega)$$

which is "isometric" onto its image (β_n complex valued so have a factor of 2) i.e. $\|I(f)\|_{L^2(\Omega)} = 2\|f\|_{L^2([a,b])}$.

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SQE: We say u is a solution to (SQE) if u satisfies the mild formulation (= Duhamel formulation).

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-t')\Delta}(u^3 - mu)(t')\,dt' + \int_0^t e^{(t-t')\Delta}\,dW(t'),$$

$W(t) = \sum_{n\in\mathbb{Z}^d} e_n \beta_n(t)$,

$\Downarrow$

"$dW(t) = \xi(t)$"

$S(t) := e^{t\Delta}$

Here

$$\Psi(t) := \int_0^t e^{(t-t')\Delta}\,dW(t')$$

$$= \text{"}\sum_{n\in\mathbb{Z}^d} e_n \int_0^t S(t-t')\,d\beta_n(t')\text{."}$$

$$= \sum_{n\in\mathbb{Z}^d} e_n(x) \int_0^t e^{-(t-t')|n|^2}\,d\beta_n(t').$$

is called the stochastic convolution.

W – also called the L^2-cylindrical Wiener process.

Regularity of W $\Rightarrow$ $\left[C_t^{\frac12-}H_x^{-\frac d2-}\right]$ $\to$ β_n essentially $\{g_n\}$

b/c BM (or $W_{t,loc}^{\frac12,\infty}$).

$\{g_n\}$ indep std $\mathbb{C}$-val Gaussians

$\Rightarrow$ Spatial regularity like that of the Fourier series $\sum e_n g_n(\omega)$

Why? Set $\eta(x) = \sum_n e_n(x) g_n(\omega)$. Then

$$\mathbb{E}[\|\eta\|^2_{H^s}] = \mathbb{E}\Big[\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |g_n|^2\Big] = 2\sum_n \langle n\rangle^{2s} < \infty \qquad \langle n\rangle := (1+|n|^2)^{1/2}$$

$$\Longleftrightarrow$$

$$s < -d/2.$$

Also

$$\big\| \|\eta\|_{W^{s,p}_x} \big\|_{L^p(\Omega)} = \Big\| \Big\| \sum_n \langle n\rangle^s e_n g_n \Big\|_{L^p_x} \Big\|_{L^p(\Omega)}$$

$$= \Big\| \Big\| \sum_n \langle n\rangle^s e_n(x) g_n(\omega) \Big\|_{L^p(\Omega)} \Big\|_{L^p_x(\mathbb{T}^d)}.$$

$$\lesssim \sqrt{p}\, \Big\| \Big\| \quad \text{---} \quad \Big\|_{L^2(\Omega)} \Big\|_{L^p_x(\mathbb{T}^d)} \quad \to \mathbb{E}[g_n g_m] = 2\delta_{n+m=0}.$$

$$\sim \sqrt{p}\, \cancel{\Big\|} \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \cancel{\Big\|_{L^p_x}}$$

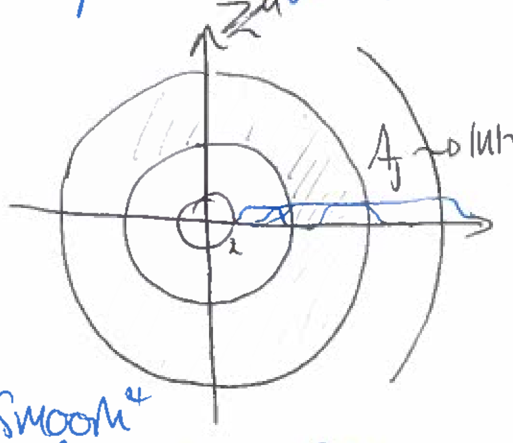
$$< \infty \iff s < -d/2.$$

Hölder Regularity $C^s = B^s_{\infty,\infty}$ = Besov space. $(0<s<1)$.

(Hölder semi-norm + L^∞ norm)

In order to define Besov spaces, we recall the Littlewood-Paley decomposition of a function (dyadic decomposition).

Dyadic numbers $N \in 2^{\mathbb{N}\cup\{0\}}$



- $\varphi_0(\xi) = \begin{cases} 1, & |\xi| \le 5/8 \\ 0, & |\xi| > 8/5 \end{cases}$ $(N \in 2^{\mathbb{Z}})$.
- $\varphi_j(\xi) := \varphi_0(\xi/2^j) - \varphi_0(\xi/2^{j-1})$, $j \ge 1$.
- $\sum_{j\ge 0} \varphi_j(\xi) \equiv 1 \quad \forall \xi.$

"Smooth" Localization in frequency space on annuli of width $\sim 2^j$"

Define (LP projector) $\widehat{P_j f}(n) := \varphi_j(n)\hat{f}(n)$
"Smooth localization around $\{|n| \sim 2^j\}$."

$$\Rightarrow f = \sum_{j\geq 0} P_j f.$$

LP-Th^m: For $1<p<\infty$,
$$\boxed{\|f\|_{L^p} \sim \|S(f)\|_{L^p} := \Big\| \underbrace{\Big(\sum_j |P_j f(x)|^2\Big)^{1/2}}_{\text{"Square function"}} \Big\|_{L^p}}$$

Besov space: $B^s_{p,q}$,
$$\|f\|_{B^s_{p,q}} := \big\| 2^{js} \|P_j f\|_{L^p_x} \big\|_{\ell^q_j}. \quad (*)$$

$$\|f\|_{W^{s,p}} = \|\langle\nabla\rangle^s f\|_{L^p} \sim \Big\| \Big(\sum_j |P_j \langle\nabla\rangle^s f(x)|^2\Big)^{1/2} \Big\|_{L^p_x}$$
$$\sim \Big\| \Big(\sum_j 2^{2js} |P_j f(x)|^2\Big)^{1/2} \Big\|_{L^p_x}.$$

$\Rightarrow B^s_{2,2} = H^s$

- $B^0_{p,\infty} \supset L^p \supset B^0_{p,1}$, $\quad \|\cdot\|_{C^s} \sim \sup_j 2^{js}\|P_j f\|_{L^\infty}$
$= \|f\|_{B^s_{\infty,\infty}}$. $\quad \Big(0<s<1$, $s\notin(0,1)$, $C^s \underset{\text{defn}}{\equiv} B^s_{\infty,\infty} \Big)$.

$$\not\!\!=$$

Recall $\|\eta\|_{W^{s,p}} < \infty$ a.s. $\forall p<\infty$, $s<-d/2$

Then by Sobolev inequality, $\|\eta\|_{W^{s,\infty}} \lesssim \|\eta\|_{W^{s+\varepsilon,q}} < \infty$ a.s. s.t. $s+\varepsilon < -d/2$ (i.e. q large).

$$\|\eta\|_{W^{s+\varepsilon,q}} \gtrsim \sup_j \|P_j \eta\|_{W^{s,\infty}} = \|\eta\|_{B^s_{\infty,\infty}} = C^s.$$

Sobolev Thm
Cheap Littlewood Paley
i.e. $\|P_j \eta\|_{L^p} \lesssim \|\eta\|_{L^p}$.

$$\Big\Downarrow$$

(*): $L^p_x \ell^q_j$ – $F^s_{p,q}$ Triebel-Lizorkin space, $\ell^q_j L^p_x$ – Besov space.

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$$\mathrm{Reg}(W) \sim C_t^{1/2-} C_x^{-d/2-}.$$

d=2:

lect → semigroup, gain 1 derivative.

$\Psi \in C_t C_x^{0-}$ → Slight problem then

$$u(t) = S(t)u_0 - \int_0^t S(t-t')\,u^3(t')\,dt' + \Psi(t)$$
$$= \Gamma_{\xi,u_0}(u)$$

u sol^n $\Longleftrightarrow$ $u = \Gamma(u)$

Say $u_0 \equiv 0$. At First Picard iteration, we have a term "Ψ^3" but this is not defined as Ψ has strictly negative spatial regularity.

$\Rightarrow$ Need to renormalize the non-linearity u^3.

Main issue: To make sense of the product of two distributions.

Bony's paraproduct decomposition

Alternative notation:
- $f \prec g$ · $\pi_f(g)$
- $f \odot g$ · $T_f(g)$
- $R(f,g)$

$$fg = \sum_{j\geq 0}\sum_{k\geq 0} P_j f \cdot P_k g.$$

$$=: f \circledless g + f \circleddash g + f \circledgtr g.$$

$$=: \sum_{k\geq 0}\underbrace{\Big(\sum_{j\leq k-2} P_j f\Big)}_{\text{“}S_{k-2}(f)\text{”}} P_k g + \sum_{|j-k|<2} P_j f\cdot P_k g$$

(2 here unimportant: just "well separated")

$$+ \sum_{j\geq 0} P_j f \cdot S_{j-2}(g).$$

$f \circledless g$ = "Paraproduct of g by f".

→ To remember order: high frequency $f^{=}$ more important

$f \odot g$ = resonant product (or remainder)

(for more see Bahouri-Chemin-Danchin, Ch2)

Key: The paraproducts $f \olessthan g$, $f \ogreaterthan g$ ALWAYS make sense!
$\Rightarrow$ Problems arise because of the resonant product.

Suppose $Reg(f) = \alpha$, $Reg(g) = \beta$ $\Rightarrow$

- If $\alpha \geq 0$, $Reg(f \olessthan g) \sim \beta$
- If $\alpha < 0$, $Reg(f \olessthan g) \sim \alpha + \beta < \beta$.

$f \olessthan g \Rightarrow \langle n\rangle^s \widehat{fg}(n) = \sum_{n=n_1+n_2} \hat{f}(n_1)\hat{g}(n_2)$

$\overset{?}{\lesssim} \langle n_2\rangle^s$

$\langle n\rangle^{\alpha+\beta} \sim \langle n_1\rangle^{\alpha} \hat{f}(n_1)\langle n_2\rangle^{\beta}\hat{g}(n_2)$

$\downarrow$ if $\alpha < 0$

Prop^n: $\alpha, \beta \in \mathbb{R}$, $p, q \in [1,\infty]$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

→ Product so recall Hölder inequality for the integrability p_1, p_2.

① If $\alpha \geq 0$,
$$\|f \olessthan g\|_{B^{\beta}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{\beta}_{p_2,q}}$$

② If $\alpha < 0$,
$$\|f \olessthan g\|_{B^{\beta}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q}} \|g\|_{B^{\beta}_{p_2,q}}$$

Resonant product: $|n_1| \sim |n_2|$ → $n = n_1 + n_2$
$\Rightarrow |n| \lesssim |n_1|, |n_2|$ as $|n_1| \sim |n_2|$.

Need $\langle n_1\rangle^{\alpha}\hat{f}(n_1)\langle n_2\rangle^{\beta}\hat{g}(n_2)$ → $\langle n\rangle^{\alpha+\beta}$ → If $\alpha+\beta \geq 0$, then of course
$$\langle n\rangle^{\alpha+\beta} \lesssim \langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta}$$
but $\not\lesssim$ if $\alpha+\beta < 0$.

Prop^n: $\alpha, \beta \in \mathbb{R}$, $p, q \in [1,\infty]$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.
Then for $\underline{\alpha+\beta > 0}$,
$$\|f \odot g\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q}} \|g\|_{B^{\beta}_{p_2,q}}.$$

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Sps $\alpha < 0 < \beta$, $\alpha+\beta > 0$,

$$f \circledless g \sim \alpha+\beta$$
$$f \circleddash g \sim \alpha+\beta$$
$$f \circledgreater g \sim \alpha \longrightarrow \text{Worst regularity}$$

Moral: • When the product makes sense (i.e. $\alpha+\beta>0$), then the paraproduct is responsible for the worst regularity.
• The product not making sense (because $\alpha+\beta<0$) is because of the resonant product.

Lemma (Heat semigroup properties):

See appendix of Mourrat–Weber '17, CMP (3-d SQE Global)
Also: M–W '17 Annals of Prob. (2-d SQE on $\mathbb{R}^2$)
B–C–D as before

$\alpha, \beta \in \mathbb{R}$, $p, q \in [1, \infty]$,

$$\alpha \geq \beta: \quad \|e^{t\Delta} f\|_{B^{\alpha}_{p,q}} \leq C\, t^{\frac{\beta-\alpha}{2}} \|f\|_{B^{\beta}_{p,q}}$$

($\frac{\beta-\alpha}{2}$: Negative)

2-d SQE: Write $u = v + \Psi$ (Da-Prato-Debussche trick)

v: Reg $= 2-$. Ψ: Reg $= 0-$

• Study fixed pt problem for v.

Step 1: $\xi \longmapsto :\Psi^{\ell}: =$ Renormalized Ψ^{ℓ} (Wick powers)

(Stochastic analysis)

$$\Psi \in C_t C^{-\varepsilon} \rightarrow :\Psi^{\ell}: \in C_t C^{-\ell\varepsilon} \quad \text{a.s.}$$

Logarithmic so no issue with ℓ.) In 3-D, $\varepsilon \sim \frac{1}{2}$ so $\ell\varepsilon \gg 0$!)

$$:(v+\Psi)^k: = :u^k: = \sum_{\ell=0}^{k} \binom{k}{\ell} :\Psi^{\ell}: v^{k-\ell}$$

Suppose $u_0 \in C^{2-2\varepsilon}$, v satisfies

$$v(t) = S(t)u_0 - \sum_{l=0}^{k} \binom{k}{l} \int_0^t S(t-t') :\Psi^l: v^{k-l}(t')\,dt'$$

Step 2:

$$- m \int_0^t S(t-t')(v+\Psi)(t')\,dt'.$$

Set $s = 2-2\varepsilon$. Then by Minkowski & Heat semigroup property

$$\|v\|_{GC^s} \lesssim \|u_0\|_{C^s} + \sum_{l=0}^{k}\binom{k}{l}\int_0^t (t-t')^{-1+\varepsilon/2} \|:\Psi^l: v^{k-l}\|_{GC_x^{\varepsilon}}\,dt' \quad \left(:\Psi^l: \in G_t C^{-\varepsilon},\ l=0,1,\dots,k.\right)$$

$$+ \int_0^T (t-t')^{-1+\varepsilon/2}\|v+\Psi\|_{GC_x^{-\varepsilon}}\,dt'. \quad \left(\begin{matrix}\beta=-\varepsilon \\ \alpha = 2-2\varepsilon\end{matrix}\right).$$

Paraproduct estimate: $\|:\Psi^l: v^{k-l}\|_{C^{-\varepsilon}} \lesssim \underbrace{\|:\Psi^l:\|_{C^{-\varepsilon}}}_{\lesssim 1}\|v^{k-l}\|_{C^{2\varepsilon}}$

$$\lesssim \|v\|_{C^s}^{k-l}.$$

$$\Rightarrow \|v\|_{GC^s} \lesssim \|u_0\|_{C^s} + C(\omega)T^{\theta}\sum_{l=0}^{k}\|v\|_{GC^s}^{k-l}.$$

+ (Difference estimate)

$\Downarrow$ Contraction mapping principle

Local well-posedness

Summary:

$$\begin{tikzcd}
(u_0, \xi) \arrow[r, "\times"] \arrow[d, "\text{Step 1: Stochastic analysis}"'] & u & v \arrow[l, bend right, "u=v+\Psi"'] \\
(u_0, \Psi, :\Psi^2:, \dots, :\Psi^k:) \in C^s \times (C_t C^{-\varepsilon})^k \arrow[urr, "\text{Step 2: Deterministic}"'] & &
\end{tikzcd}$$

This map is continuous.

Rmk: We dropped the linear 'm' term for 2d.
In 3-d though, $:h^3: = h^3 - 3\infty h$ ← Choose m to kill this.

From last time:

Lemma: $\alpha, \beta \in \mathbb{R}$, $p, q \in [1,\infty]$, $\alpha \geq \beta$.

$$\boxed{\|e^{t\Delta} f\|_{B^{\alpha}_{p,q}} \lesssim t^{\underbrace{\frac{\beta-\alpha}{2}}_{\leq 0.}} \|f\|_{B^{\beta}_{p,q}}}$$ (Smoothing estimate)

2-d SQE

$$(\partial_t - \Delta) u = -u^k + \xi$$

$k=3$: $\xi_\varepsilon = \eta_\varepsilon * \xi$, $\eta_\varepsilon(x) = \eta(\frac{x}{\varepsilon})$ ← Mollification kernel.

(SQE$_\varepsilon$) $$(\partial_t - \Delta) u_\varepsilon = -u_\varepsilon^3 + m_\varepsilon u_\varepsilon + \xi_\varepsilon.$$

NB: Mollification ≈ frequency truncation at $\sim 1/\varepsilon$.

Write $u_\varepsilon = \underbrace{\Psi_\varepsilon}_{\text{Stochastic convolution.}} + v_\varepsilon$, $(\partial_t - \Delta) v_\varepsilon = -(v_\varepsilon + \Psi_\varepsilon)^3 + m_\varepsilon(v_\varepsilon + \Psi_\varepsilon)$

But $\Psi_\varepsilon^3 \not\to$ (does not converge), as $\Psi \in C_t W^{-\varepsilon,\infty}$ so product Ψ^3 does not make sense.

Need to renormalize. → $:\Psi_\varepsilon^3: := \Psi_\varepsilon^3 - 3\sigma_\varepsilon \Psi_\varepsilon$, $\sigma_\varepsilon \sim \log(1/\varepsilon)$.

Then $\exists\, m_\varepsilon \to \infty$ s.t. $v_\varepsilon \to v \Rightarrow u_\varepsilon \to u$ in $C_t W^{-\varepsilon,\infty}$.

Last time: We studied

$$\Gamma v(t) = S(t)u_0 - \sum_{l=0}^{k} \binom{k}{l} \int_0^t S(t-t') :\Psi^l: v^{k-l}(t')\,dt'$$ (SQE$_v$)

where $:\Psi^l: \in C_t C^{-\varepsilon} \subseteq B^{-\varepsilon}_{\infty,\infty}$.

and constructed a fixed pt $v \in C_t C^s$, $s = 2-2\varepsilon$. assuming $u_0 \in C^s$.

Q: What about rough initial data?

Want to understand the Gibbs measure

$$e^{-H}du = e^{-\frac{1}{k+1}\int \underbrace{:u^{k+1}:}_{\text{Renormalized}}} e^{-\frac{1}{2}\int |\nabla u|^{2} dx} du \quad , k\in 2\mathbb{N}+1$$

Since $\partial_t u = \frac{\partial H}{\partial u} + \xi$

but the Gaussian measure is supported on $W^{-\varepsilon,p}\ \forall p\le\infty$

i.e. on data of the form $\sum_{n\in\mathbb{Z}^2} \frac{g_n}{\langle n\rangle} e^{inx} \in W^{-\varepsilon,\infty}$ a-s.

We proved LWP for data in $C^{s=2-2\varepsilon} \rightarrow$ 2-ish derivatives! Way too much.

Ans 1: Write $u_0^{\omega} = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{inx} \in W^{-\varepsilon,\infty} \setminus W^{0,\infty}$ a-s.

Set $z(t)=S(t)u_0^{\omega}$ and write $u = z + \Psi + v$,

and redefine Ψ to solve

$$\begin{cases} (\partial_t-\Delta)\Psi = \xi \\ \Psi|_{t=0} = u_0^{\omega} \end{cases}$$

What about deterministic rough u_0?

Ans 2: Study (SQE$_v$) with $u_0\in C^{\sigma}$, $\sigma<0$. From (SQE$_v$) & Heat estimate

$$t^{\theta}\|v(t)\|_{C^s} \lesssim t^{\theta} t^{\underbrace{\frac{\sigma-s}{2}}_{<0}} \|u_0\|_{C^{\sigma}} \qquad \left(\begin{matrix}\theta = \frac{s-\sigma}{2} > 0 \\ s = 2\varepsilon\end{matrix}\right.$$

$$+\, t^{\theta} \sum_{l=0}^{k} \binom{k}{l} \int_0^t (t-t')^{-\frac{3}{2}\varepsilon} \underbrace{\|:\Psi^{l}:\, v^{k-l}(t')\|_{C^{-\varepsilon}}} dt'$$

$$\lesssim \|:\Psi^{l}(t'):\|_{C^{-\varepsilon}} \|v^{k-l}(t')\|_{C^{2\varepsilon}}$$

(Paraproduct estimate see Lecture 1).

$$\lesssim \|v(t')\|_{C^{2\varepsilon}}^{k-l}.$$

$$t^\theta \|v(t)\|_{C^s} \lesssim \|u_0\|_{C^\sigma} + \sum_{l=0}^{k} \binom{k}{l} t^\theta \int_0^t (t-t')^{-\frac{3}{2}\varepsilon}$$
$$\times \|:\Psi^l(t'):\|_{C^{-\varepsilon}} (t')^{-(k-l)\theta} (t')^{(k-l)\theta} \|v(t')\|_{C^{2\varepsilon}}^{k-l} dt'$$

Using $\|:\Psi^l(t):\|_{C([0,T];C^{-\varepsilon})} \lesssim C(\omega)$,

have

$$t^\theta \|v(t)\|_{C^s} \lesssim \|u_0\|_{C^\sigma} + C(\omega) \sum_{l=0}^{k} \binom{k}{l} t^\theta \sup_{t'\in[0,T]} \left((t')^\theta \|v(t')\|_{C^{2\varepsilon}}\right)^{k-l}$$
$$\times \underbrace{\int_0^t (t-t')^{-\frac{3}{2}\varepsilon} (t')^{-(k-l)\theta} dt'}_{}\,!$$

Is this finite?

Let's check

FACT: $t^{\alpha_1} \int_0^t (t-t')^{\alpha_2} (t')^{\alpha_3} dt'$
$= B(\alpha_2+1, \alpha_3+1) < \infty$
when $\alpha_1+\alpha_2+\alpha_3 = -1$, $\alpha_2, \alpha_3 > -1$
& $B(x,y) := \int_0^1 (1-t)^{x-1} t^{y-1} dt$, $\operatorname{Re} x, \operatorname{Re} y > 0$
is the Beta function.

So need

$$\theta + \left(-\frac{3}{2}\varepsilon\right) + (-k\theta) \geq -1. \quad , \theta = \varepsilon - \frac{\sigma}{2}.$$

$$-(k-1)\varepsilon + \frac{k-1}{2}\sigma - \frac{3}{2}\varepsilon \geq -1.$$

So need to impose $\frac{k-1}{2}\sigma > -1 \Rightarrow \sigma > \frac{-2}{k-1}$

$(\varepsilon \ll 1)$.

~~Notice: Scaling critical Sobolev index is $s_c = \frac{d}{r} - \frac{2}{k-1}$ (for $\dot{W}^{s,r}$)~~

~~We have $\sigma > \frac{d}{r} - \frac{2}{k-1}$, with $d=2, r=\infty$.~~

$$\left[\alpha_1 = \theta,\ \alpha_2 = -\frac{3}{2}\varepsilon,\ \alpha_3 = -(k-\ell)\theta. \quad \ell = 0,\dots,k.\right]$$

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In order to bound the time integral we need:

$$\begin{cases} \text{i)}\ \alpha_1 + \alpha_2 + \alpha_3 = -1 \\ \text{ii)}\ \alpha_2, \alpha_3 > -1. \end{cases}$$

At worst, these have to be satisfied even when $\ell = 0$, in which case we have:

$$\text{i)}\ \alpha_1 + \alpha_2 + \alpha_3 = \theta + \left(-\frac{3}{2}\varepsilon\right) + (-k\theta) \quad , \theta = \varepsilon - \frac{\sigma}{2},$$

$$= -(k-1)\varepsilon + \frac{k-1}{2}\sigma - \frac{3}{2}\varepsilon$$

$$\geqslant -1.$$

$$\left[\Rightarrow \sigma > -\frac{2}{k-1}.\right] \quad \text{---}(1)$$

while

ii) $\alpha_2 > -1 \Rightarrow$ True since $\varepsilon \ll 1$.

$\underline{\alpha_3 > -1}$: $\quad -k\theta > -1 \Rightarrow \theta < \frac{1}{k}$.

$$\Downarrow$$

$$\left[\sigma > -\frac{2}{k}.\right] \quad \text{--}(2)$$

Clearly (2) is the more limiting restriction of (1) and (2).

$\Rightarrow$ We can close the fixed point argument only with initial data in $C_x^\sigma(\mathbb{T}^2)$, $\underline{\sigma > -\frac{2}{k} > -\frac{2}{k-1}}$.

(We <u>miss</u> the scaling critical exponent) i.e. Suppose u satisfies $(\partial_t - \Delta)u = -u^k$ on $\mathbb{R}^d$,

Then $u^\lambda(t,x) = \frac{1}{\lambda^{2/(k-1)}}\, u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$ $(\lambda>0)$

is also a solution.

s_c defined because of the scaling invariance of the $\dot{W}^{s_c,r}$ norm: $\|u^\lambda\|_{\dot{W}^{s_c,r}} = \|u\|_{\dot{W}^{s_c,r}}$.

$$s_c = \frac{d}{r} - \frac{2}{k-1}.$$

e.g. 4d-cubic

$s_c(2) = 1$

$s_c(\infty) = -1$.

Rmk: For Schrödinger like equations (dispersive PDE) we take $r=2$. For parabolic, $r=\infty$ natural.

Taking a supremum in time, and defining the norm

$$\|v\|_{X_T} = \sup_{t\in[0,T]} t^\theta \|v(t)\|_{\mathcal{C}^{s=2\varepsilon}}, \quad \theta = \frac{s-\sigma}{2}.$$

$$\Downarrow$$

(SQE_v) is LWP in $C^{\cancel{-\frac{2}{k-1}}+}(\mathbb{T}^2)$ $\boxed{-\frac{2}{k}}$

(a.s. wrt noise).

($\alpha_3 > -1$ conditions restricts the $\alpha_1+\alpha_2+\alpha_3=-1$ one)

$$\Downarrow$$

(SQE).

Proof of Smoothing estimate Lemma

Firstly, consider the following lemma.

Lemma': $A = \{|k|\sim 1\}$ annulus. Then $\exists\, c, C>0$ s.t. $\forall p\in[1,\infty]$, $\forall t,\lambda>0$, $\operatorname{supp}\hat{f} \subset \lambda A$, we have

$$\|e^{t\Delta} f\|_{L^p} \le C e^{-ct\lambda^2} \|f\|_{L^p}.$$

(see B-C-D Lemma 2.4)

Proof of Lemma: We will prove for $\mathbb{R}^d$. ⑤

By scaling, we can assume $\lambda = 1$. $\to$ Set $f_\lambda(x) = f(\frac{x}{\lambda})$, $\lambda > 0$.

Let ϕ be a smooth function s.t. $\phi \equiv 1$ on A.

Then we can write

$$e^{t\Delta} f = \mathcal{F}^{-1}\{\phi(\xi) e^{-t|\xi|^2} \hat{f}(\xi)\} \quad \to \text{as } \hat{f} \text{ supported on } A.$$

$$= g(t,\cdot) * f,$$

where $$g(t,x) = \int e^{ix\cdot\xi} \phi(\xi) e^{-t|\xi|^2} d\xi$$

By Young's inequality, it suffices to show

$$\|g(t)\|_{L^1_x} \le Ce^{-ct}. \quad \text{---} \; (*).$$

Write

$$g(t,x) = (1+|x|^2)^{-d} \int \underbrace{(1+|x|^2)^d e^{ix\cdot\xi}}_{(Id-\Delta_\xi)^d(e^{ix\cdot\xi})} \phi(\xi) e^{-t|\xi|^2} d\xi$$

$$= (1+|x|^2)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \underbrace{(Id-\Delta_\xi)^d(\phi(\xi)e^{-t|\xi|^2})}_{} d\xi$$

(Leibniz) $$= \sum_{\substack{\beta\le\alpha \\ |\alpha|\le 2d}} C_{\alpha,\beta} (\partial^{\alpha-\beta}\phi(\xi)) \underbrace{(\partial^\beta e^{-t|\xi|^2})}_{}$$

$$\to \left| (\partial^{\alpha-\beta}\phi(\xi))(\partial^\beta e^{-t|\xi|^2}) \right| \le C(1+t)^{|\beta|} e^{-t|\xi|^2}.$$

$|\xi_j| \lesssim 1$ as ϕ supported on say $2A$.

From lower bound on A. $\to (1+t)^{|\beta|} e^{-t|\xi|^2} \lesssim e^{-ct}$.

$$\Rightarrow |g(t,x)| \lesssim (1+|x|^2)^{-d} \int_{|\xi|\sim 1} \sum_{\substack{\beta\le\alpha \\ |\alpha|\le 2d}} C_{\alpha\beta} e^{-ct} d\xi.$$

$$\in L^1_x(\mathbb{R}^d)$$

$\square$

Lemma' $\iff$ $\|T_t f\|_{L^p} \le C\|f\|_{L^p}$ $(\mathbb{T}^d)$.

$\quad$ ↳ indep of p, t.

$$\widehat{T_t}(n) = \frac{e^{-t|n|^2}}{e^{-ct\lambda^2}}\,\theta\left(\frac{n}{\lambda}\right), \quad \theta \equiv 1 \text{ on } \{|n| \sim \lambda\}$$

Direct proof: Use Poisson summation formula.

Proof of Lemma: Using Lemma', hence

$$\|e^{t\Delta} P_j f\|_{L^p} \lesssim e^{-ct2^{2j}} \|P_j f\|_{L^p}.$$

$$\Rightarrow \quad 2^{\alpha j}\|e^{t\Delta}P_j f\|_{L^p} \lesssim t^{\frac{\beta-\alpha}{2}} 2^{\beta j}\Big[\underbrace{(t2^{2j})^{\overbrace{\frac{\alpha-\beta}{2}}^{}_{\ge 0}} e^{-ct2^{2j}}}_{x^{\theta} e^{-cx},\ \theta \ge 0,\ \lesssim_\theta 1.}\Big]\|P_j f\|_{L^p}.$$

$\Rightarrow$ Sum in ℓ^q_j.

$\square$.

Renormalization via Wick powers

- Definition of Wick powers : $:\Psi^\ell:$
- Showing : $:\Psi^\ell: \in C_t C^{-\varepsilon}$.

Hermite polynomials: $H_k(x;\sigma)$, defined by the generating function

$$F(t,x;\sigma) = e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma).$$

e.g. $H_0(x;\sigma) = 1$

$H_1(x;\sigma) = x$

$H_2(x;\sigma) = x^2 - \sigma.$

$H_3(x;\sigma) = x^3 - 3\sigma x.$

• $L^2(\mu_\sigma)$ – Gram-Schmidt $\to c_k H_k$.

$\quad \mathcal{N}(0,\sigma)$

Orthogonality: $\int_{\mathbb{R}} H_k(x)\, H_m(x)\, d\gamma(x) = \delta_{km}\, k!$

$\mathbb{R}^d$:

$$H_{\vec{k}}(\vec{x}) := \prod_{j=1}^{d} H_{k_j}(x_j), \qquad k = |\vec{k}| = \sum_{j=1}^{d} k_j.$$

$$\mathcal{H}_k := \overline{\{ H_{\vec{k}}(\vec{x}) \ ; \ |\vec{k}| = k \}}^{\|\cdot\|_{L^2(\mu)}}$$

$= k^{th}$ homogeneous Wiener chaoses.

Itô-Wiener decomposition

$$L^2(\mu) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$$

- $L = \Delta - x\cdot\nabla$ → Ornstein-Uhlenbeck operator (Hartree-Fock op).

↳ $F \in \mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$.

Hypercontractivity of O-U Semigroup (Nelson '65)

$$p \geq q > 1,$$

$$\boxed{\|e^{tL} f\|_{L^p(\mu)} \leq \|f\|_{L^q(\mu)}, \ \forall\, t \geq \tfrac{1}{2}\log\left(\tfrac{p-1}{q-1}\right).}$$

(Gaussian integrability after some time).

Corollary: $F \in \mathcal{H}_k$, $\forall\, p \geq 2$,

$$\boxed{\|F\|_{L^p(\mu)} \leq (p-1)^{k/2} \|F\|_{L^2(\mu)}.} \quad (*)$$

Proof: $e^{tL}F = e^{-tk}F$, Taking $q=2$ & $t=\frac{1}{2}\log(p-1)$.

↪ eigenfunction.

$$\hookrightarrow \quad \|F\|_{L^p} \geq \underset{\underset{(p-1)^{k/2}}{\shortparallel}}{e^{tk}} \|F\|_{L^2}.$$

$\boxtimes$

Corollary (i.e. ⊛) also holds for $F \in \bigoplus_{j=0}^{k} \mathcal{H}_j$

(see Barry Simon, "$P(\varphi)_2$ Euclidean Quantum Field Theory," Ch 1) Th$^{\underline{m}}$ 1-22).

Proof of Lemma′ on the Torus:

The aim of this note is to describe how the estimate

$$\|g(t,\cdot)\|_{L^1_x(\mathbb{R}^d)} \le Ce^{-ct} \quad \forall\, t>0, \tag{$*$}$$

which was the key estimate in the proof of lemma′, along with the Poisson Summation Formula can be used to prove lemma′ on the torus.

First we describe how $(*)$ can be upgraded to the stronger estimate

$$\|g_\lambda(t,\cdot)\|_{L^1_x(\mathbb{R}^d)} \le Ce^{-ct\lambda^2} \quad \forall\, t,\lambda>0 \tag{$**$}$$

Where $g_\lambda(t,x) = \int_{\mathbb{R}^d_\xi} e^{ix\cdot\xi}\phi(\xi/\lambda)e^{-t|\xi|^2}\,d\xi$. By a change of variables we have

$$\begin{aligned} g_\lambda(t,x) &= \int_{\mathbb{R}^d_\xi} e^{ix\cdot\xi}\phi(\xi/\lambda)e^{-t|\xi|^2}\,d\xi \\ &= \lambda^d\int_{\mathbb{R}^d_\xi} e^{i\lambda x\cdot\xi}\phi(\xi)e^{-t\lambda^2|\xi|^2}\,d\xi \\ &= \lambda^d g(t\lambda^2,\lambda x), \end{aligned} \tag{$***$}$$

As $\|\cdot\|_{L^1(\mathbb{R}^d)}$ is invariant under $f(\cdot)\mapsto\lambda^d f(\lambda\cdot)$ we have,

$$\begin{aligned} \|g_\lambda(t,\cdot)\|_{L^1(\mathbb{R}^d)} &= \|\lambda^d g(t\lambda^2,\lambda\cdot)\|_{L^1(\mathbb{R}^d)} \\ &= \|g(t\lambda^2,\cdot)\|_{L^1} \\ &\le Ce^{-ct\lambda^2} \quad (\text{by } (*)). \end{aligned}$$

This proves (**).

We will soon need the Poisson Summation Formula. We state it here for convienence. For a proof of this theorem see "Classical Fourier Analysis" by Grafakos

<u>(Poisson Summation Formula):</u> Suppose that $f, \hat{f} \in L^1(\mathbb{R}^d)$ satisfy

$$|f(x)| + |\hat{f}(x)| \leq C(1+|x|)^{-d-\delta}$$

for some $C, \delta > 0$. Then f and $\hat{f}$ are both continuous and for all $x \in \mathbb{R}^d$ we have

$$\sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x} = \sum_{n \in \mathbb{Z}^d} f(x+n).$$

We now have all the tools we need to prove lemma' on the torus

Let $\hat{f}_n$ denote the n-th Fourier coefficent of a function on $\mathbb{T}^d$. Analogous to the proof of lemma' on $\mathbb{R}^d$, if $\operatorname{supp} \hat{f}_n \subset \lambda A$ we have,

$$e^{t\Delta} f = \mathcal{F}^{-1}\left(\phi(n/\lambda) e^{-t|n|^2} \hat{f}_n\right)$$

$$= g_\lambda^{per} * f$$

where $g_\lambda^{per} = \sum_{n \in \mathbb{Z}^d} \phi(n/\lambda) e^{-t|n|^2} e^{in\cdot x}$. As in the proof of lemma' on $\mathbb{R}^d$, by Young's inequality it suffices to show

$$\| g_\lambda^{per} \|_{L^1(\mathbb{T}^d)} \leq C e^{-ct\lambda^2} \quad \forall\, t, \lambda > 0.$$

If ϕ is nice enough (say Schwartz) then $\widehat{g_\lambda} = \phi(\cdot/\lambda)e^{-t|\cdot|^2}$ is also Schwartz. By properties of the Fourier transform g_λ is also Schwartz. Hence the hypothesis of Poisson's Summation Formula are satisfied and so we have,

$$\begin{aligned}
\|g_\lambda^{Per}\|_{L^1(\mathbb{T}^d)} &= \Big\| \sum_{n\in\mathbb{Z}^d} \phi(n/\lambda)\, e^{-t|n|^2} e^{in\cdot x} \Big\|_{L^1(\mathbb{T}^d)} \\
&= \Big\| \sum_{n\in\mathbb{Z}^d} \widehat{g_\lambda}(n)\, e^{in\cdot x} \Big\|_{L^1(\mathbb{T}^d)} \\
&= \Big\| \sum_{n\in\mathbb{Z}^d} g_\lambda(x+n) \Big\|_{L^1(\mathbb{T}^d)} \qquad \text{(Poisson)} \\
&= \|g_\lambda\|_{L^1(\mathbb{R}^d)} \\
&\leq C e^{-ct\lambda^2} \qquad \text{(by (**))}.
\end{aligned}$$

This completes the proof.

Lecture 3: 7/02/2018

Renormalisation continued

Hyperconstructive

Recall from last lecture the Wiener Chaos estimate /

Corollary: Let $F \in \mathcal{H}_k$. Then for all $p \geq 2$,

$$\|F\|_{L^p(\mu)} \leq (p-1)^{k/2} \|F\|_{L^2(\mu)}$$

Recall the stochastic convolution

$$\Psi(t) = \int_0^t S(t-t')\,dW(t')$$

$$= \sum_{n \in \mathbb{Z}^2} e_n(x) \int_0^t e^{-(t-t')|n|^2} d\beta_n(t').$$

Given $N \in \mathbb{N}$, set

$$\Psi_N := P_{\leq N} \Psi.$$

Ψ_N is smooth i.e. a "nice" function.

For fixed $(t,x) \in \mathbb{R} \times \mathbb{T}^2$, $\Psi_N(x,t)$ is a mean-zero Gaussian random variable with variance

$$\sigma_N(t) = \mathbb{E}[\Psi_N^2(t,x)] = \mathbb{E}\Big[\sum_{n,m} e_n(x) e_m(x) \int_0^t e^{-(t-t')|n|^2} d\beta_n(t') \int_0^t e^{-(t-t'')|m|^2} d\beta_m(t'')\Big]$$

$$= \sum_{|n| \leq N} \mathbb{E}\Big[\int_0^t e^{-(t-t')|n|^2} d\beta_n(t') \cdot \overline{\int_0^t e^{-(t-t'')|n|^2} d\beta_n(t'')}\Big].$$

$$= \mathbb{E}\Big[\Big(\int_0^t d\beta_0\Big)^2\Big] + \sum_{\substack{n \neq 0 \\ |n| \leq N}} \mathbb{E}\big[\ \_\_\_\ \big]$$

$$= \int_0^t dt' + \sum_{0 < |n| \leq N} 2 \int_0^t e^{-2(t-t')|n|^2} dt'.$$

$$= t + \sum_{0 < |n| \leq N} \frac{1 - e^{-2t|n|^2}}{|n|^2} \sim_t \log N. \quad (\text{as } d = 2).$$

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In the third equality, we used the independence of the Brownian motions which implied that the only non-zero contribution occurs when

$$n+m=0$$
$$\Rightarrow \quad m=-n.$$

Notice also that this implies $e_n(x)e_m(x) = e_n(x)e_{-n}(x) = 1$.

Now

- $\sigma_N(t) \sim_t \log N \to \infty$ as $N \to \infty$
- $\sigma_N(t)$ is independent of $x \in \mathbb{T}^2$.

The blow-up in the variance indicates the need for a renormalization.

We define

$$\boxed{:\Psi_N^{\ell}(t,x): \stackrel{\text{def}^n}{=} H_\ell\left(\Psi_N(t,x)\,;\,\sigma_N(t)\right)}$$

Remark: The Hermite polynomials are appropriate for the real-valued setting. In the complex-valued setting, the (generalised) Laguerre polynomials are used.

Lemma: Let f and g be Gaussian random variables (mean-zero) with variances σ_f and σ_g, respectively. Then

$$\mathbb{E}\left[H_k(f;\sigma_f)H_\ell(g;\sigma_g)\right] = \delta_{k\ell}\, k!\left(\mathbb{E}[fg]\right)^k.$$

In the rest of this lecture, we discuss the proof of the following proposition.

<u>Proposition:</u> Let $\ell \in \mathbb{N}$, $T>0$ and $p \geq 1$. Then $\{:\Psi_N^\ell:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in

$$L^p(\Omega; C([0,T]; C^{-\varepsilon}(\mathbb{T}^2))) \quad , \forall \varepsilon>0.$$

In particular, denoting the limit by $:\Psi^\ell:$, we have

$$:\Psi^\ell: \in C([0,T]; C^{-\varepsilon}(\mathbb{T}^2)) \quad a\text{-}s.$$

<u>Remark:</u> For simplicity, we will show the sequence $\{:\Psi_N^\ell:\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$, $\forall \varepsilon>0$, and hence that

$$:\Psi^\ell: \in C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)) \quad a\text{-}s.$$

This will be equivalent to that in the proposition as the spatial regularity is not sharp and hence we can accept the ε-loss in using the embeddings

$$\circledast \ldots \quad C^{s+\varepsilon} \hookrightarrow W^{s,\infty} \hookrightarrow C^s, \quad \forall s\in\mathbb{R},\ \varepsilon>0.$$

To see where $\circledast$ comes from, we essentially write down the definitions of the norms. We have

$$\|f\|_{C^s} = \sup_j 2^{js}\|P_j f\|_{L^\infty_x} \sim \sup_j \|P_j(\langle\nabla\rangle^s f)\|_{L^\infty_x}$$
$$\lesssim \sup_j \|\langle\nabla\rangle^s f\|_{L^\infty_x}$$
$$= \|f\|_{W^{s,\infty}},$$

and

$$\|f\|_{W^{s,\infty}} = \Big\|\sum_{j=0}^\infty 2^{js} P_j f\Big\|_{L^\infty_x} = \Big\|\sum_{j=0}^\infty 2^{-j\varepsilon} 2^{j(s+\varepsilon)} P_j f\Big\|_{L^\infty_x}$$
$$\leq \Big(\sum_j 2^{-j\varepsilon}\Big) \sup_j 2^{j(s+\varepsilon)} \|P_j f\|_{L^\infty_x}$$
$$\lesssim \|f\|_{C^{s+\varepsilon}}.$$

Proof of Proposition: It suffices to prove the claim for all $p \geq 1$ large enough (since $L^{p_1}(\Omega) \subset L^{p_2}(\Omega)$, $p_2 \leq p_1$).
Let $t_1 \geq t_2$. We have

$$
\begin{aligned}
\mathbb{E}[\Psi_N(t_1,x)\Psi_N(t_2,y)] &= \sum_{|n|\leq N} \mathbb{E}\Big[\int_0^{t_2} e^{-(t_1-t')|n|^2} d\beta_n(t') \\
&\qquad \times \int_0^{t_2} e^{-(t_2-t'')|n|^2} d\bar{\beta}_n(t'')\Big] e_n(x-y) \\
&\quad + \sum_{|n|\leq N} \cancel{\mathbb{E}\Big[\int_{t_2}^{t_1} e^{-(t_1-t')|n|^2} d\beta_n(t') \cdot \int_0^{t_2} e^{-(t_2-t'')|n|^2} d\bar{\beta}_n(t'')\Big]} \longrightarrow 0 \\
&\qquad \cdot e_n(x-y)
\end{aligned}
$$

(by independent increments property of BM.)

$$
\begin{aligned}
&= \sum_{0<|n|\leq N} e_n(x-y) \cdot 2\int_0^{t_2} e^{2t'|n|^2} dt' \cdot e^{-(t_1+t_2)|n|^2} \\
&\quad + \int_0^{t_2} 1^2 \, dt' \\
&= t_2 + \sum_{0<|n|\leq N}{}' e_n(x-y) \underbrace{e^{-t_1|n|^2} \frac{e^{t_2|n|^2} - e^{-t_2|n|^2}}{|n|^2}}_{=:\gamma(n,t_1,t_2)} \quad \ldots (1)
\end{aligned}
$$

We now apply the Bessel potentials $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ to both sides of (1). For the right hand side of (1) we have*

$$
\langle\nabla_x\rangle^{-\varepsilon}\langle\nabla_y\rangle^{-\varepsilon}\Big(\sum_{n\in\mathbb{Z}^2}\big(\gamma(n,t_1,t_2)[0<|n|\leq N] + t_2[n=0]\big)e_n(x-y)\Big)
$$

$$
= \sum_{0<|n|\leq N} \frac{e_n(x-y)}{\langle n\rangle^{2\varepsilon}} \gamma(n,t_1,t_2) + t_2.
$$

(*) (I have used the notation:

$$
[P] = \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{if } P \text{ is false.} \end{cases}
$$
)

For the left hand side we expand the Ψ_N as Fourier series and interchange the finite sums with the expectation, i.e.

$$\langle\nabla_x\rangle^{-\varepsilon}\langle\nabla_y\rangle^{-\varepsilon}\mathbb{E}[\Psi_N(t_1,x)\,\Psi_N(t_2,y)]$$

$$=\langle\nabla_x\rangle^{-\varepsilon}\langle\nabla_y\rangle^{-\varepsilon}\Big(\sum_{|n_2|\le N}e_{n_2}(y)\Big(\sum_{|n_1|\le N}e_{n_1}(x)\,\mathbb{E}[\widehat{\Psi_N}(t_1,n_1)\widehat{\Psi_N}(t_2,n_2)]\Big)\Big)$$

$$=\langle\nabla_x\rangle^{-\varepsilon}\Big(\sum_{|n_1|\le N}e_{n_1}(x)\Big(\sum_{|n_2|\le N}\frac{e_{n_2}(y)}{\langle n_2\rangle^{\varepsilon}}\,\mathbb{E}[\widehat{\Psi_N}(t_1,n_1)\widehat{\Psi_N}(t_2,n_2)]\Big)\Big)$$

$$=\sum_{\substack{|n_1|\le N\\|n_2|\le N}}\frac{e_{n_1}(x)}{\langle n_1\rangle^{\varepsilon}}\frac{e_{n_2}(y)}{\langle n_2\rangle^{\varepsilon}}\,\mathbb{E}[\widehat{\Psi_N}(t_1,n_1)\,\widehat{\Psi_N}(t_2,n_2)]$$

$$=\mathbb{E}\big[\langle\nabla_x\rangle^{-\varepsilon}\Psi_N(t_1,x)\cdot\langle\nabla_y\rangle^{-\varepsilon}\Psi_N(t_2,y)\big].$$

Setting $x=y$ now yields (and $t_1=t_2=t$)

$$\mathbb{E}[(\langle\nabla\rangle^{-\varepsilon}\Psi_N(t,x))^2]=t+\sum_{0<|n|\le N}\frac{1}{\langle n\rangle^{2\varepsilon}}\frac{1-e^{-2t|n|^2}}{4|n|^2}$$

$$\lesssim t+\sum_{n\neq0}\frac{1}{\langle n\rangle^{2+2\varepsilon}}$$

$$\lesssim_t 1<\infty,$$

uniformly in $N\in\mathbb{N}$ and $x\in\mathbb{T}^2$.
Now since $\langle\nabla\rangle^{-\varepsilon}\Psi_N(t,x)\in\mathcal{H}_1$, we use the Wiener Chaos estimate to get for all $p\ge2$,

$$\mathbb{E}[|\langle\nabla\rangle^{-\varepsilon}\Psi_N(t,x)|^p]\lesssim_{t,p}1,$$

and thus by Sobolev embedding (and choosing p large enough) we get

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$$\mathbb{E}\left[\|\Psi_N(t,\cdot)\|^p_{W^{-\varepsilon,\infty}}\right] \lesssim \mathbb{E}\left[\|\Psi_N(t,\cdot)\|^p_{W^{-\varepsilon',p}}\right]$$

$$= \int_{\mathbb{T}^2} \mathbb{E}\left[|\langle\nabla\rangle^{-\varepsilon'}\Psi_N(t,x)|^p\right] dx$$

$$\lesssim_{t,p} 1$$

for any $\varepsilon>0$, $t>0$ and $p\geq 1$, uniformly in $N\in\mathbb{N}$.
Hence for fixed $t>0$,

$$\Psi_N(t,\cdot) \in W_x^{-\varepsilon,\infty}(\mathbb{T}^2) \text{ a.s.}$$

We now show that, for fixed $t>0$, the Wick powers $:\Psi_N^{\ell}(t,\cdot):$ carry the same spatial regularity almost surely.

(When $t_1=t_2=t$, we write $\gamma(n,t_1,t_2)=\gamma(n,t)$).
Using the Lemma, we have

$$\mathbb{E}\left[:\Psi_N^{\ell}(t,x)::\Psi_N^{\ell}(t,y):\right] = \ell!\left\{\mathbb{E}\left[\Psi_N(t,x)\Psi_N(t,y)\right]\right)^{\ell}.$$

$$= \Big(t+\sum_{0<|n|\leq N} e_n(x-y)\,\gamma(n,t)\Big)^{\ell}\cdot \ell!$$

$$= \ell!\sum_{k=0}^{\ell}\binom{\ell}{k} t^{k-\ell}\sum_{\substack{n_1,\dots,n_k\\ 0<|n_j|\leq N}} e_{n_1+\dots+n_k}(x-y)\prod_{j=1}^{k}\gamma(n_j,t).$$

$$= \ell!\sum_{k=0}^{\ell}\binom{\ell}{k} t^{k-\ell}\sum_{\substack{|n|\leq N}} e_n(x-y)\sum_{\substack{n_1,\dots,n_k\\ n=n_1+\dots+n_k\\ 0<|n_j|\leq N}}\prod_{j=1}^{k}\gamma(n_j,t) \quad \text{--- (2)}$$

($n=n_1+\dots+n_k \Rightarrow |n|\lesssim N.$)

Now we apply the same approach as before: act with $\langle\nabla_x\rangle^{-\varepsilon}$ and $\langle\nabla_y\rangle^{-\varepsilon}$ and then set $x=y$.

Remark: The whole point of evaluating the 'covariance' with general x and y, then applying $\langle\nabla_x\rangle^{-\varepsilon}$ & $\langle\nabla_y\rangle^{-\varepsilon}$ followed by setting $x=y$ is so that we can make appear the act of the smoothing by $\langle\nabla_0\rangle^{-\varepsilon}$. That is, we get the factors $\langle n_1+\dots+n_k\rangle^{-2\varepsilon}$ in the sums which are necessary for convergence. Directly estimating $\mathbb{E}[(\langle\nabla\rangle^{-\varepsilon} :\Psi_N^l(t,x):)]$ will not make appear the sums, and furthermore, it is not even clear how one applies Lemma in this situation anyway.

For (LHS) of (2):

$$\langle\nabla_x\rangle^{-\varepsilon}\langle\nabla_y\rangle^{-\varepsilon}\,\mathbb{E}[:\Psi_N^l(t,x):\,:\Psi_N^l(t,y):]$$

$$= \mathbb{E}[\langle\nabla_x\rangle^{-\varepsilon}:\Psi_N^l(t,x):\cdot\langle\nabla_y\rangle^{-\varepsilon}:\Psi_N^l(t,y):], \quad \dots(2a)$$

because $:\Psi_N^l(t,x):$ is a polynomial in Ψ_N and since Ψ_N is finitely frequency restricted, so too is any product of Ψ_N's $\Rightarrow$ $:\Psi_N^l(t,x):$ is frequency restricted (i.e. $\operatorname{supp}\{\widehat{:\Psi_N^l(t):}(n)\} \subset \{|n|\le N\}$).

For (RHS) of (2):

$$\langle\nabla_x\rangle^{-\varepsilon}\langle\nabla_y\rangle^{-\varepsilon}\left(l!\sum_{k=0}^{l}\binom{l}{k}t^{k-l}\sum_{|n|\le N}e_n(x-y)\sum_{\substack{n_1,\dots,n_k\\ n=n_1+\dots+n_k\\ 0<|n_j|\le N}}\prod_{j=1}^{k}\gamma(n_j,t)\right)$$

$$= l!\sum_{k=0}^{l}\binom{l}{k}t^{k-l}\sum_{|n|\le N}\frac{e_n(x-y)}{\langle n\rangle^{2\varepsilon}}\sum_{\substack{n_1,\dots,n_k\\ n=n_1+\dots+n_k\\ 0<|n_j|\le N}}\prod_{j=1}^{k}\gamma(n_j,t). \quad \dots(2b)$$

Equating (2a) & (2b), then setting $x=y$ yields

$$\mathbb{E}\left[\left|(\langle\nabla\rangle^{-\varepsilon} : \Psi_N^{\ell}(\cdot,t):)(x)\right|^2\right]$$

$$= \ell! \sum_{k=0}^{\ell} \binom{\ell}{k} t^{k-\ell} \sum_{\substack{n_1,\dots,n_k \\ 0<|n_j|\leq N}} \frac{1}{\langle n_1+\dots+n_k\rangle^{2\varepsilon}} \prod_{j=0}^{k} \gamma(n_j,t).$$

$$\lesssim \ell! \sum_{k=0}^{\ell} \binom{\ell}{k} t^{k-\ell} \underbrace{\sum_{n_1,\dots,n_k\in\mathbb{Z}^2} \frac{1}{\langle n_1+\dots+n_k\rangle^{2\varepsilon}\langle n_1\rangle^2\cdots\langle n_k\rangle^2}}_{\text{Converges for any } k.}$$

Consider $k=2$: $\displaystyle\sum_{n_1,n_2} \frac{1}{\langle n_1+n_2\rangle^{2\varepsilon}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2} = \sum_{n_1} \frac{1}{\langle n_1\rangle^2} \sum_{n_2} \frac{1}{\langle n_1+n_2\rangle^{2\varepsilon}} \frac{1}{\langle n_2\rangle^2}$

Since $n_1+n_2-n_2=n_1$, $\Rightarrow \langle n_1\rangle \leq \max(\langle n_1+n_2\rangle, \langle n_2\rangle)$, thus

$$\sum_{n_2} \frac{1}{\langle n_1+n_2\rangle^{2\varepsilon}} \frac{1}{\langle n_2\rangle^2} = \sum_{\substack{\max(\cdot,\cdot) \\ =\langle n_1+n_2\rangle}} (\cdot) + \sum_{\substack{\max(\cdot,\cdot) \\ =\langle n_2\rangle}}' (\cdot)$$

$$\lesssim \frac{1}{\langle n_1\rangle^{\varepsilon}} \sum_{n_2} \frac{1}{\langle n_1+n_2\rangle^{\varepsilon}} \frac{1}{\langle n_2\rangle^2} + \frac{1}{\langle n_1\rangle^{\varepsilon}} \sum_{n_2} \frac{1}{\langle n_1+n_2\rangle^{2\varepsilon}} \frac{1}{\langle n_2\rangle^{2-\varepsilon}}$$

$$\lesssim \frac{1}{\langle n_1\rangle^{\varepsilon}}.$$

$$\Rightarrow \sum_{n_1,n_2} \frac{1}{\langle n_1+n_2\rangle^{2\varepsilon}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2} \lesssim \sum_{n_1} \frac{1}{\langle n_1\rangle^{2+\varepsilon}} < \infty.$$

Apply this iteratively!

$$\Rightarrow \mathbb{E}\left[\left|(\langle\nabla\rangle^{-\varepsilon} : \Psi_N^{\ell}(\cdot,t):)(x)\right|^2\right] \lesssim_{t,p,\ell} \lesssim 1$$

uniformly in $N\in\mathbb{N}$ & $x\in\mathbb{T}^2$.

So by the Wiener chaos estimate and Sobolev embedding as before,

$$\| \langle \nabla \rangle^{-\varepsilon} :\Psi_N^{\ell}(t,x): \|_{L^p(\Omega)} \lesssim_{p,t,\ell} 1,$$

$$\Downarrow$$

$$\| \| :\Psi_N^{\ell}(t,x): \|_{W_x^{-\varepsilon,\infty}} \|_{L^p(\Omega)} \lesssim_{p,t,\ell} 1 \quad \underline{\text{unif in } N}$$

$$\Downarrow$$

$$:\Psi_N^{\ell}(t,\cdot): \in W_x^{-\varepsilon,\infty}(\mathbb{T}^2) \quad a\text{-}s.$$

- Temporal regularity for $:\Psi_N^{\ell}(t,x):$

Define the difference operator δ_h by

$$\delta_h :\Psi_N^{\ell}(t,x): := :\Psi_N^{\ell}(t+h,x): - :\Psi_N^{\ell}(t,x):, \quad |h| \leq 1.$$

As before, we expand and use Lemma to obtain

$$\mathbb{E}\left[(\delta_h :\Psi_N^{\ell}(t,x):)(\delta_h :\Psi_N^{\ell}(t,y):) \right]$$

$$= \mathbb{E}[:\Psi_N^{\ell}(t+h,x): :\Psi_N^{\ell}(t+h,y):] - \mathbb{E}[:\Psi_N^{\ell}(t+h,x): :\Psi_N^{\ell}(t,y):]$$

$$- \mathbb{E}[:\Psi_N^{\ell}(t,x): :\Psi_N^{\ell}(t+h,y):] + \mathbb{E}[:\Psi_N^{\ell}(t,x): :\Psi_N^{\ell}(t,y):]$$

$$= \ell! \Big[\big(\mathbb{E}[\Psi_N(t+h,x)\Psi_N(t+h,y)] \big)^{\ell} - \big(\mathbb{E}[\Psi_N(t+h,x)\Psi_N(t,y)] \big)^{\ell}$$
$$- \big(\mathbb{E}[\Psi_N(t,x)\Psi_N(t+h,y)] \big)^{\ell} + \big(\mathbb{E}[\Psi_N(t,x)\Psi_N(t,y)] \big)^{\ell} \Big].$$

Joining up the first and third terms and the second and fourth terms and writing them as telescoping sums gives

$$= l!\ \mathbb{E}[\delta_h \Phi_N(x,t)\ \Phi_N(t+h,y)]$$
$$\times \sum_{j=0}^{l-1} \left(\mathbb{E}[\Phi_N(t+h,x)\Phi_N(t+h,y)]\right)^{l-j-1}\left(\mathbb{E}[\Phi_N(t,x)\Phi_N(t+h,y)]\right)^{j}$$
$$- l!\ \mathbb{E}[\delta_h \Phi_N(t,x)\ \Phi_N(t,y)]$$
$$\times \sum_{j=1}^{l-1} \left(\mathbb{E}[\Phi_N(t,x)\Phi_N(t+h,y)]\right)^{l-j-1}\left(\mathbb{E}[\Phi_N(t,x)\Phi_N(t,y)]\right)^{j}$$

$$=: (A) - (B).$$

Recall from (1) that for $t_1 \geq t_2$, we have

$$\mathbb{E}[\Phi_N(t_1,x)\Phi_N(t_2,x)] \sim_t \sum_{0<|n|\leq N} e_n(x-y)\underbrace{e^{-t_1|n|^2}\frac{e^{t_2|n|^2}-e^{-t_2|n|^2}}{|n|^2}}_{\gamma(n,t_1,t_2)},$$

where we will now neglect the zeroth frequency contribution as the core of the argument requires estimation of the sums over non-zero frequencies.

Inserting the above into (A) and (B), then applying $\langle \nabla_x \rangle^{-\varepsilon}$ and $\langle \nabla_y \rangle^{-\varepsilon}$ and then setting $x=y$ as before, we are led to bounding sums of the form

$$(*)\quad \sum_{\substack{n_1,\dots,n_j \\ 0<|n_j|\leq N}} \frac{1}{\langle n_1+\dots+n_j\rangle^{2\varepsilon}} G_1(n_1,t,h)\, G_2(n_2,t,h) \cdots G_j(n_j,t,h),$$
$$(j=2,\dots,l)$$

where $G_1(n,t,h) = \mathbb{E}[\delta_h \widehat{\Phi_N}(t,n)\ \widehat{\Phi_N}(t_1,n)]$

$$= \gamma(n,t+h,t_1) - \gamma(n,t,t_1),$$

$$G_j(n_j, t_1, t_2) = \mathbb{E}[\widehat{\Phi_N}(t_1, n_j)\widehat{\Phi_N}(t_2, n_j)], \quad j = 2, \dots, \ell.$$
$$= \gamma(n_j, t_1, t_2) \quad (t_1 \geq t_2),$$

and $t_1, t_2 \in \{t, t+h\}$.

From the definition of γ, we see that

$$|G_j(n_j, t_1, t_2)| \lesssim_t \frac{1}{\langle n_j \rangle^2}, \quad j = 2, \dots, \ell$$

$$|G_1(n_1, t, h)| \underset{\text{(M.V.T.) (Mean value Theorem)}}{\lesssim_t} \min\left(h, \frac{1}{\langle n_1 \rangle^2}\right).$$

By interpolation, we have

$$|G_1(n_1 t, h)| \lesssim_t \frac{h^\alpha}{\langle n \rangle^{2-2\alpha}}, \quad \alpha \in [0,1].$$

$$\Rightarrow (*) \lesssim \sum_{\substack{n_1, \dots, n_j \\ |n_i| \leq N}} \frac{1}{\langle n_1 + \dots + n_j \rangle^{2\varepsilon}} \frac{h^\alpha}{\langle n_1 \rangle^{2-2\alpha}} \frac{1}{\langle n_2 \rangle^2} \cdots \frac{1}{\langle n_j \rangle^2}.$$

$$\lesssim h^\alpha,$$ provided that $2\varepsilon - 2\alpha > 0$.

Summarising, we have shown

$$\mathbb{E}[(\delta_h \langle \nabla \rangle^{-\varepsilon} :\Phi_N^\ell(t,x):)^2] \lesssim h^\alpha, \quad 2\varepsilon - 2\alpha > 0, \; |h| \ll 1.$$

$\Downarrow$ Wiener Chaos estimate / Hypercontractivity

$$\mathbb{E}[\| \delta_h :\Phi_N^\ell(t,\cdot): \|^p_{W^{-\varepsilon,p}}] \lesssim_{\ell,p,t} h^{\frac{\alpha p}{2}}.$$

Thus by Sobolev embedding, given $\varepsilon>0$,

$$\mathbb{E}\left[\|S_h(:\Psi_N^{\ell}(t,\cdot):)\|^p_{W^{-\varepsilon,\infty}}\right] \leq \mathbb{E}\left[\|S_h(:\Psi_N^{\ell}(t,\cdot):)\|^p_{W^{-\frac{\varepsilon}{2},p}}\right]$$

$$\lesssim_{p,t,\varepsilon} |h|^{\frac{p}{2}\alpha}, \quad \alpha\in[0,1].$$

for p large enough so that $\varepsilon p>4$.

For p large enough so that, when $\alpha\in(0,\varepsilon)$, we have

$$\frac{p\alpha}{2}>1,$$

we can apply Kolmogorov's Continuity Criterion (see Bass, "Stochastic Processes", Cor 8.4 & Ex 8.2):

Kolmogorov Cty Criterion:

Suppose $\exists$ c_1, η, and $p>0$ s.t.

$$\mathbb{E}[d(X_s,X_t)^p] \leq c_1|t-s|^{1+\eta}.$$

Then $\exists$ $C_2=C_2(c_1,\varepsilon,p)$ s.t.

$$\mathbb{P}\left(\sup_{s\neq t}\frac{d(X_s,X_t)}{|t-s|^{\frac{\eta}{p}-\gamma}}\geq M\right) < C_2M^{-p},$$

$\gamma<\eta/p$.

For us, choose: $\eta=\frac{p\alpha}{2}-1$, γ very close to zero,

$$\frac{\eta}{p}=\frac{\alpha}{2}-\frac{1}{p} \to \frac{\alpha}{2}- \text{ as } p\to\infty.$$

$$\alpha\in(0,\varepsilon).$$

$$\Rightarrow :\Psi_N^{\ell}: \in C([0,T];W^{-\varepsilon,\infty}(\mathbb{T}^2)) \text{ a.s.}$$

with a uniform in N bound.

Remark: By giving up spatial regularity, we can gain Hölder regularity in time, i.e.

$$:\Psi_N^\ell: \in C^\delta([0,T]; W^{-2\delta-,\infty}(\mathbb{T}^2)) \text{ a.s. } \forall\, 0 \le \delta < \tfrac{1}{2}.$$

Taking $\varepsilon = 2\delta +$, γ very small, then

$$\delta = \frac{\varepsilon}{p} - \gamma \sim \frac{\alpha}{2} - < \frac{\varepsilon}{2} - -$$

$$\hookrightarrow \quad \delta = \frac{\alpha}{2} - \text{ but } \alpha \in (0,1) \Rightarrow \underline{\delta \in [0, 1/2)}.$$

It remains to show $\{:\Psi_N^\ell:\}_{N\in\mathbb{N}}$ is Cauchy in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$.

Letting $N \ge M \ge 1$, $\beta > 0$, $\varepsilon > 0$, $\alpha \in [0,1]$, such that $2\varepsilon - 2\alpha - 2\beta > 0$, we can show, in a similar manner as above,

$$\mathbb{E}\left[\left| \delta_h \left(\langle\nabla\rangle^{-\varepsilon} (:\Psi_N^\ell(t,\cdot): - :\Psi_M^\ell(t,\cdot):) \right) \right|^2 \right] \lesssim_{t,\ell} \frac{h^\alpha}{M^{2\beta}}.$$

We extract the small negative power of M from the sums since we have the lower restriction

$$M \le |n_j| \le N, \quad j = 1, \dots, \ell.$$

By hypercontractivity, we get

$$\mathbb{E}\left[\| \delta_h (:\Psi_N^\ell(t,\cdot): - :\Psi_M^\ell(t,\cdot):) \|^p_{W^{-\varepsilon,p}} \right] \lesssim_{p,t,\ell} \frac{h^{\frac{\alpha p}{2}}}{M^{\beta p}},$$

provided $2\varepsilon > \alpha + 2\beta$.

By Sobolev embedding and Kolmogorov's Continuity criterion, we get that $\{:\Psi_N^2:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in

$$L^p(\Omega; C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))), \quad T>0, \; \varepsilon>0,$$

and hence it converges to a limit, denoted by $:\Psi^2:$, which is in $C([0,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ a.s.

$\boxtimes$.

Lecture 4: 7/04/18

Recall from Lecture 1 Bony's paraproduct decomposition

$$fg = f \circledless g + f \circleddash g + f \circledgtr g.$$

Notations:

$$f \circledleq g := f \circledless g + f \circleddash g$$
$$f \circledgeq g := f \circledless g + f \circledgtr g.$$

Suppose $Reg(f) = \alpha$, $Reg(g) = \beta$

$\alpha < 0 < \beta$, $\alpha + \beta > 0$

Then

$$\begin{cases} f \circledless g \sim \alpha + \beta \\ f \circleddash g \sim \alpha + \beta \\ f \circledgtr g \sim \alpha. \end{cases}$$

→ To make sense of the product, we need $\underline{\alpha + \beta > 0}$.

Here, '$\sim$' will be used to denote the spatial regularity of a given function, e.g. $f \sim 1 \Leftrightarrow f$ has spatial regularity 1.

In this lecture we work in $C^\alpha_x = B^\alpha_{\infty,\infty,x}$.

Dynamical Φ^4_3-model on the torus

The following discussion follows that in Section 2 of Mourrat-Weber "The dynamic Φ^4_3 model comes down from infinity '17, CMP.)

$$(\Phi^4_3) \begin{cases} \partial_t u = \Delta u - u^3 + mu + \xi, \text{ on } \mathbb{R}_+ \times \mathbb{T}^3 \\ u|_{t=0} = u_0. \end{cases}$$

Stochastic Convolution

$$\Psi := (\partial_t - \Delta)^{-1} \xi = \int_0^t S(t-t') dW(t') \in C_t C_x^{-\frac{d}{2}+1-}.$$

(Recall $\xi \sim -\frac{d}{2}-$, the heat Duhamel integral $(\partial_t - \Delta)^{-1}$ gives 2 derivatives but we lose one because of dW
$\Rightarrow$ Total gain for Ψ over ξ is $+1$ derivative. ②

$$d=3: \quad \Psi \sim -\tfrac{1}{2}-.$$

(We will be interested only in spatial regularities.).

First Attempt (at Φ^4_3):

- Study at the level of u

(Even in 2-d, this failed).

By the Duhamel formulation (mild), we have

$$u \sim \Psi \sim -\tfrac{1}{2}- \quad (\text{or } 0- \text{ in 2-d})$$

$$\Rightarrow u^3 \text{ does } \underline{\text{not}} \text{ make sense!}$$

Second attempt: Da-Prato - Debussche Trick

Write $\quad u = v + \Psi = v + ⫯$

(It will be notationally convenient to write

$$⫯ := L^{-1}\xi := \Psi, \quad (\partial_t - \Delta) ⫯ = \xi \; \text{---}(1)$$

for the stochastic convolution;

"$\bullet = \xi$"
"$| = L^{-1} = (\partial_t - \Delta)^{-1}$" $\Big\}$ "$⫯ = \overset{\bullet}{|} = L^{-1}\xi$".)

Then v solves

$$\left[(\partial_t - \Delta) v = -v^3 - 3v^2 ⫯ - 3v \vee - \Psi \right. \\ \left. + m(⫯ + v). \right] \quad (\dagger)$$

We have renormalised the following products:
(i.e. defined new stochastic objects)

$$\begin{cases} \overset{\bullet}{|}^{\,2} \rightsquigarrow \overset{\bullet\ \bullet}{\vee} \\ \overset{\bullet}{|}^{\,3} \rightsquigarrow \overset{\bullet\bullet\bullet}{\Psi} \end{cases}$$

($\overset{\bullet}{|}^{\,2}$ makes no sense since $\overset{\bullet}{|} \sim -\tfrac{1}{2}-$ & $(-\tfrac{1}{2}-)+(-\tfrac{1}{2}-)<0$; Same for $\overset{\bullet}{|}^{\,3}$).

These renormalised objects are rigorously defined as limits of smoothed/mollified stochastic objects:

i.e. Let $\eta_\delta(x) = \eta\left(\frac{x}{\delta}\right)$ be a smooth mollifier and set

$$\overset{\bullet}{|}_\delta := \overset{\bullet}{|} * \eta_\delta .$$

Since $\overset{\bullet}{|}^{\,2}$ & $\overset{\bullet}{|}^{\,3}$ do not make sense, the objects $\overset{\bullet}{|}_\delta^{\,2}$ & $\overset{\bullet}{|}_\delta^{\,3}$ fail to converge as $\delta \to 0$. However, for a suitable constant $C_\delta^{(1)}$, we have convergence for the mildly modified terms to some objects we label $\overset{\bullet\ \bullet}{\vee}$ and $\overset{\bullet\bullet\bullet}{\Psi}$, resp;

$$\begin{cases} \overset{\bullet\ \bullet}{\vee}_\delta := (\overset{\bullet}{|}_\delta)^2 - C_\delta^{(1)} \xrightarrow{\delta \to 0} \overset{\bullet\ \bullet}{\vee} \in C_t C_x^{-d+2-} \\ \overset{\bullet\bullet\bullet}{\Psi}_\delta := (\overset{\bullet}{|}_\delta)^3 - 3C_\delta^{(1)} \overset{\bullet}{|}_\delta \xrightarrow{\delta \to 0} \overset{\bullet\bullet\bullet}{\Psi} \in C_t C_x^{-\frac{3}{2}d+3-} . \end{cases}$$

$C_\delta^{(1)}$ is "related" to the variance of $\overset{\bullet}{|}_\delta$. Recall that for sharp frequency truncation $P_{\leq N} \overset{\bullet}{|}$, the variance behaves like

$$\sum_{\substack{|n| \leq N \\ n \in \mathbb{Z}^d}} \frac{1}{|n|^2} \sim \begin{cases} \log N & \text{in } 2\text{-}d \\ N & \text{in } 3\text{-}d , \end{cases}$$

and we think of $N \sim \delta^{-1}$, so

$$C_\delta^{(1)} \sim \begin{cases} \log(1/\delta) , & 2\text{-}d \\ \delta^{-1} , & 3\text{-}d \end{cases}$$

$$\longrightarrow \infty \text{ as } \delta \to 0 .$$

In 2-D: $\mathbf{V} \sim 3(0-) = 0-$

(Expect) $\Rightarrow v \sim 2-$ (Heat gain of 2-dims)

$\Rightarrow$ All the products in (+) make sense

$\Rightarrow$ Can proceed with a fixed point argument.

Note: $\Psi, \mathbf{V}, \mathbf{V}$ depend on the choice of mollifier η, but the solution "u" can be shown to not depend on η.

In 3-D: $\mathbf{V} \sim -\frac{3}{2}- \Rightarrow v \sim \left(-\frac{3}{2}-\right)+2 = \frac{1}{2}-.$

but now "$v^2\Psi$" and "$v\mathbf{V}$" do not make sense in (+)

$$\underbrace{v^2 \Psi}_{\frac{1}{2}-,\ -\frac{1}{2}-} \qquad \underbrace{v \mathbf{V}}_{\frac{1}{2}-,\ -1-}$$

$$\left(\tfrac{1}{2}-\right)+\left(-\tfrac{1}{2}-\right)<0 \qquad \left(\tfrac{1}{2}-\right)+(-1-)<0.$$

So Da-Prato - Debussche trick fails for 3-D.

Third Attempt:

Da-Prato-Debussche trick expands about the first Picard iterate. We now expand about the second Picard iterate. The worst term in (+) is $\mathbf{V}$ so we "subtract it off."

Write

$$u = \underbrace{v}_{\substack{\text{Should be smoother}\\ \text{than } \Psi \,\&\, \mathbf{Y}}} + \underbrace{\Psi}_{1^{st}\text{ Picard iterate}} - \underbrace{\mathbf{Y}}_{2^{nd}\text{ Picard iterate}}$$

$$\mathbf{Y} := \underbrace{\mathcal{L}^{-1}(\mathbf{V})}_{\text{“}\int_0^t S(t-t')\mathbf{V}\,dt'\text{”}} \sim \left(-\frac{3}{2}-\right)+2 = \frac{1}{2}-.$$

$\downarrow$ In itself is a replacement of "$\int_0^t S(t-t')\Psi^3 dt'$".

$$(\partial_t-\Delta)\mathsf{Y} = \mathsf{W} \quad \cdots (2)$$

Note: The mu term is written into v since $m\mathcal{L}^{-1}\mathsf{I} \sim \frac{3}{2}-$ which is smoother than these other terms in the expansion of u.

From (Φ^4_3), (1) and (2) we have

$$(\partial_t-\Delta)v = \text{“}\,\mathsf{W} - (v+\mathsf{I}-\mathsf{Y})^3 + m(v+\mathsf{I}-\mathsf{Y})\,\text{”}$$

Expand and renormalise $\mathsf{I}^2 \rightsquigarrow \mathsf{V}$, $\mathsf{I}^3 \rightsquigarrow \mathsf{W}$ $\longrightarrow$

$$= \cancelto{0}{\mathsf{W} - \mathsf{W}} - (v-\mathsf{Y})^3 - 3(v-\mathsf{Y})^2\,\mathsf{I}$$
$$-3(v-\mathsf{Y})\mathsf{V} + m(v+\mathsf{I}-\mathsf{Y}).$$

Worst term cancels!

Cleaning up, we have v satisfies

$$(\partial_t-\Delta)v = -v^3 - 3(v-\underline{\underline{\mathsf{Y})\mathsf{V}}} + Q(v), \quad \text{---} \circledast$$

Where

$$Q(v) := b_0 + b_1 v + b_2 v^2,$$

$$\begin{cases} b_0 := m(\mathsf{I}-\mathsf{Y}) + (\mathsf{Y})^3 - 3\,\underline{\underline{\mathsf{I}(\mathsf{Y})^2}} \\ b_1 := m + 6\,\underline{\underline{\mathsf{I}\mathsf{Y}}} - 3(\mathsf{Y})^2 \\ b_2 := -3\mathsf{I} + 3\mathsf{Y} \end{cases}$$

All these 3 coefficients have regularity $-\frac{1}{2}-$, however, the terms $\mathsf{I}\mathsf{Y}$, and $\mathsf{I}(\mathsf{Y})^2$ are ill-defined since the sum of their regularities is $0-$.

We will be able to make sense of the product of explicit stochastic objects by renormalisation.

A worse issue is the term $v\mathsf{V}$.

$\mathsf{V} \sim -1-$, and the worst term in $\circledast$ is $\mathsf{Y}\mathsf{V} \sim -1-$

Hence, expect $v \sim (-1-)+2 = 1-$

$\Rightarrow v V \sim 0- \Rightarrow$ Ill-defined!

Rmk: 2nd attempt: Bad term $vV \sim -\frac{1}{2}-$
3rd attempt: $vV \sim 0-$ } Improved by $\frac{1}{2}$.

Idea: Use paraproducts.

- $v \prec V \sim -1-$ (always makes sense)
- $\underbrace{v}_{1-} \succ \underbrace{V}_{-1-} \sim 1+(-1-) = 0-$ ("——")

Write: $v = X+Y$ with

$$\begin{cases} (\partial_t - \Delta) X = -3(X+Y-\Psi) \prec V \\ (\partial_t - \Delta) Y = -(X+Y)^3 - 3(X+Y-\Psi) \succcurlyeq V + Q(X+Y) \end{cases}$$

Idea: X carries the rough regularity of v while Y is smoother.

Indeed,

- $X \sim (-1-)+2 = 1- \sim v$ (from $\Psi \prec V$)
- $Y \sim (-\frac{1}{2}-)+2 = \frac{3}{2}-$ (from $\Psi \succcurlyeq V \sim (-1-)+(\frac{1}{2}-) = -\frac{1}{2}-$ & Q).

However we still need to make sense of the resonant product

$$-3(X+Y-\Psi) \circledcirc V$$

Notice: $Y \circleddash V \sim \left(\tfrac{3}{2}-\right) + (-1-) = \tfrac{1}{2}-$ (7)

So this resonant product is well-defined.

Need to make sense of

$$\Psi \circleddash V \quad \& \quad X \circleddash V.$$

$\Psi \circleddash V$: Both are explicit stochastic objects, so we renormalise by defining a new object

$$\Psi \circleddash V \rightsquigarrow \Psi_{\circleddash V}$$

This can be defined via the limiting procedure

$$\left(\Psi_{\circleddash V}\right)_\delta := \Psi_\delta \circleddash V_\delta - 3C_\delta^{(2)}\, \mathrm{I}_\delta$$

$$\xrightarrow{\delta \to 0} \Psi_{\circleddash V} \in C_t C_x^{-\frac{1}{2}-},$$

where $C_\delta^{(2)}$ is a new diverging current, $C_\delta^{(2)} \sim \log\delta$.

(In a later lecture we will prove these stochastic steps; for this and the coming lecture, we take these as given facts and only really care about the spatial regularities.)

Let us define another stochastic object Y as

$$(\partial_t - \Delta) Y = V, \quad Y(t=0) = 0 \Rightarrow Y := \mathcal{L}^{-1}(V) \sim (-1-) + 2 = 1-.$$

$X \circleddash V$: Notice that X satisfies a linear equation with mild formulation (8)

$$X = S(t)X_0 - 3\int_0^t S(t-t')\left[(X+\Upsilon-\Psi)\circledcirc V\right](t')\,dt'.$$

<u>Idea</u>: The paraproduct $(X+\Upsilon-\Psi)\circledcirc V$ says the frequencies of V are much larger than those of $(X+\Upsilon-\Psi)$.

$\Rightarrow$ "Expect operators to mostly act on V."

i.e. expect

$$\underbrace{\int_0^t S(t-t')}\left[(X+\Upsilon-\Psi)\circledcirc V\right]dt' \underset{\text{like}}{\overset{\text{Behave}}{\approx}} (X+\Upsilon-\Psi)\circledcirc \underline{\underline{Y}}$$

So we write

$$\boxed{\begin{aligned} X &= -3(X+\Upsilon-\Psi)\circledcirc Y + \mathrm{com}_1(X,Y) \\ \mathrm{Com}_1(X,Y) &:= S(t)X_0 - 3\int_0^t S(t-t')\left[(X+\Upsilon-\Psi)\circledcirc V\right]dt' \\ &\quad + 3(X+\Upsilon-\Psi)\circledcirc Y \end{aligned}}$$

We expect the commutator $\underline{\mathrm{com}_1} \sim 1+$ (at least) since $Y \sim 1-$.

Now we write

$$X\circleddash V = \underbrace{-3\left[(X+\Upsilon-\Psi)\circledcirc \overset{1-}{Y}\right]\circleddash \overset{-1-}{V}}_{\text{Does not make sense!}} + \underbrace{\mathrm{com}_1(X,Y)\circleddash V}_{\substack{\text{Makes sense}\\ (1+2\varepsilon)+(-1-\varepsilon)=\varepsilon>0.}}$$

The high frequencies of Υ dominate in

$$(X+Y-\Psi)\circledless\Upsilon. \ (*)$$

Therefore, we expect

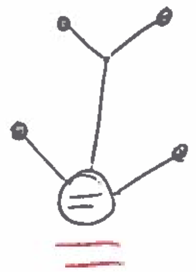
$$[(X+Y-\Psi)\circledless\Upsilon]\circleddash V \text{ "}\approx\text{" } (X+Y-\Psi)\circledless(\Upsilon\circleddash V).$$

$\left((*):\ \underset{n_1}{[f}\circledless\underset{n_2}{g]}\circleddash\underset{n_3}{h} \Rightarrow \left.\begin{matrix} n=n_1+n_2+n_3 \\ |n_1|<<|n_2| \\ |n_1+n_2|\sim|n_3| \end{matrix}\right\} \Rightarrow |n_2|\sim|n_3| \text{ so } [f\circledless g]\circleddash h \text{ "}=\text{" } f\circledless[g\circleddash h] \right)$

but

$$\underset{(1-)}{\Upsilon}\circleddash\underset{(-1-)}{V} \quad \overset{\text{Renormalise}}{\rightsquigarrow}$$

$(1-)+(-1-)<0.$



} New object

Define $[\circledless,\circleddash](f,g,h):=(f\circledless g)\circleddash h-f(g\circleddash h)$

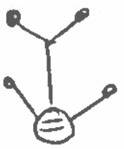
No $\circledless$ for convenience

and

$$\mathrm{Com}_2(X+Y):=[\circledless,\circleddash](-3(X+Y-\Psi),\Upsilon,V),$$

then we understand $X\circleddash V$ to be

$$\Big(X\circleddash V = -3[(X+Y-\Psi)\,\cdot\,]+\mathrm{Com}_2(X+Y)$$



$$+\mathrm{Com}_1(X,Y)\circleddash V. \Big)$$

? terms

Recall (pg 5) that we had two "ill-defined" terms (actually, just ambiguous based on regularity counting alone.) in the coefficients for Q. These were $\mathbf{I}\Upsilon$ and $\mathbf{I}(\Psi)^2$.

$\mathsf{I}\Psi$: The only bit to consider is the resonant product $\mathsf{I} \circleddash \Psi$.

$-\tfrac{1}{2}- \quad \tfrac{1}{2}- \Rightarrow \left(-\tfrac{1}{2}-\right)+\left(\tfrac{1}{2}-\right) = -\varepsilon < 0.$

Notice though that in terms of frequencies,

$$n = \underbrace{(n_1+n_2+n_3)}_{\Psi} + \underbrace{n_4}_{\mathsf{I}}$$

where $|n_1+n_2+n_3| \sim |n_4|$.

In defining Ψ, we essentially set $\left.\begin{matrix} n_1+n_2 \neq 0 \\ n_1+n_3 \neq 0 \\ n_2+n_3 \neq 0 \end{matrix}\right\}$.

For the resonant product, the only potential issue occurs if

$$n_3+n_4=0,$$

but this is forbidden when n is small by.

So no renormalisation necessary. We write

$$\mathsf{I}\Psi := \mathsf{I} \circledneq \Psi + \overset{\Psi}{\circleddash}\!{-}\mathsf{I}.$$

$\mathsf{I}(\Upsilon)^2$: Need to define $\mathsf{I} \circleddash (\Upsilon)^2$. Decompose into

$$\mathsf{I} \circleddash (\Upsilon)^2 = 2\,\mathsf{I} \circleddash [\Upsilon \circledless \Upsilon] + \mathsf{I} \circleddash \big[\underbrace{\Upsilon \circleddash \Upsilon}_{\left(\frac{1}{2}-\right)+\left(\frac{1}{2}-\right)=1-}\big]$$

$\left(-\tfrac{1}{2}-\right)+(1-) = \tfrac{1}{2}- > 0.$

$\Rightarrow$ Makes sense.

Define as

$$= 2\Upsilon \overset{\Upsilon}{\circleddash}\!{-}\mathsf{I} + 2[\circledless,\circleddash](\Upsilon,\Upsilon,\mathsf{I}) + \mathsf{I} \circleddash [\Upsilon \circleddash \Upsilon]$$

$$\Rightarrow \mathsf{I}(\Upsilon)^2 := \mathsf{I} \circledneq (\Upsilon)^2 + \mathsf{I} \circleddash [\Upsilon \circleddash \Upsilon] + 2\Upsilon \overset{\Upsilon}{\circleddash}\!{-}\mathsf{I} + 2[\circledless,\circleddash](\Upsilon,\Upsilon,\mathsf{I}).$$

We have arrived at the following system of equations: (11)

$$
(**)\quad
\begin{cases}
(\partial_t-\Delta)X = -3(X+Y-\Psi)\prec V \\[4pt]
(\partial_t-\Delta)Y = -(X+Y)^3 - 3Y\odot V + 3(\Psi\odot V) \\
\qquad + 9\left[(X+Y-\Psi)\,(\mathcal{Y}\odot V)\right] - 3\,\mathrm{com}_2(X+Y) \\
\qquad - 3\,\mathrm{com}_1(X,Y)\odot V + Q(X+Y) \\
\qquad -3(X+Y-\Psi)\succ V
\end{cases}
$$

Every term here now makes sense and we can now hope to complete a fixed point argument to obtain a local in-time solution.

Summary of Stochastic objects

We have 6 fundamental objects in (**).

Diagram	$\mathbf{1}$	V	Ψ	$(\Psi\odot\mathbf{1})$	$(\Psi\odot V)$	$(\mathcal{Y}\odot V)$
Regularity	$-\frac{1}{2}-\varepsilon$	$-1-\varepsilon$	$\frac{1}{2}-\varepsilon$	$-\varepsilon$	$-\frac{1}{2}-\varepsilon$	$-\varepsilon$

$$V_\delta = (\mathbf{1}_\delta)^2 - \underline{\underline{C_\delta^{(1)}}}\ , \qquad W_\delta = (\mathbf{1}_\delta)^3 - 3\underline{\underline{C_\delta^{(1)}}}\,\mathbf{1}_\delta$$

$$(\mathcal{Y}\odot V)_\delta = \mathcal{Y}_\delta\odot V_\delta - \underline{\underline{C_\delta^{(2)}}}\ , \qquad (\Psi\odot V) = \Psi_\delta\odot V_\delta - 3\,\underline{\underline{C_\delta^{(2)}}}\,\mathbf{1}_\delta$$

$$(\Psi\odot\mathbf{1}) = \Psi\odot\mathbf{1}$$

↳ Not renormalized.

Small remark (more on this in coming lectures)

Qn: In V_δ why is there a 3? Why is there only a 1 in V_δ?
Furthermore, why is there a Ψ_δ in V_δ? etc.....

Quick Answer: (Imprecise!) The Stochastic convolution Ψ is a mean zero Gaussian random variable with regularity $-\frac{1}{2}-$.

$$\Rightarrow \text{"} \Psi \sim \sum_{n\in\mathbb{Z}^3} \frac{g_n}{\langle n\rangle} e^{inx} \text{"} \text{ where } g_n \sim \mathcal{N}(0,1).$$

So

$$\Psi^3 = \sum_{n=n_1+n_2+n_3} \hat{\Psi}(n_1)\,\hat{\Psi}(n_2)\,\hat{\Psi}(n_3) \sim \sum_{n=n_1+n_2+n_3} \frac{g_{n_1} g_{n_2} g_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle}.$$

We have a resonance if $n_1+n_2=0$ or $n_1+n_3=0$ or $n_2+n_3=0$ in the above sum. So

$$\Psi^3 \text{"}\approx\text{"} \underbrace{\sum_{\substack{n=n_1+n_2+n_3\\ n_1+n_2\neq 0}} \frac{g_{n_1} g_{n_2} g_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle}}_{\text{Does not explode!}} + 3\sum_{\substack{n=n_1+n_2+n_3\\ n_1+n_2 = 0}} \frac{g_{n_1} g_{n_2} g_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle}$$

• 3 ways to make a resonance

$$= \sum_{\substack{n=n_1+n_2+n_3\\ n_1+n_2\neq 0}} \frac{g_{n_1} g_{n_2} g_{n_3}}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} + 3\underbrace{\Big(\sum_n \frac{|g_n|^2}{\langle n\rangle^2}\Big)}_{C_\delta^{(1)}}\underbrace{\Big(\sum_n \frac{g_n}{\langle n\rangle}\Big)}_{\approx \Psi}.$$

$$C_\delta^{(1)} \sim \mathbb{E}\Big[\sum_{|n|<1/\delta} \frac{|g_n|^2}{\langle n\rangle^2}\Big] \sim \sum_{|n|<1/\delta} \frac{1}{\langle n\rangle^2} \sim \frac{1}{\delta}.$$

So, informally, we can think of the renormalisation as the nonresonant contribution to Ψ^3. Another way to view this is:

n_1 n_2 n_3 e.g. $n_1+n_2=0$

$\Psi^3 =$  $+ \binom{3}{2}$ 

nonresonant

$\binom{3}{2}$ ways to connect two of three "Ψ".

Q$^{\underline{n}}$: How does the renormalised system (**) with mass term 'm' (appearing in (Q)) relate back to the original (Φ^4_3) equation? (13)

Suppose all terms in (**) have been mollified so we are interested in how (X_δ, Y_δ)-system relates to the u_δ-equation.

We will show that solving the (mollified) system (**) with mass $\underline{m}$ corresponds to a solution u_δ of the form

$$u_\delta = X_\delta + Y_\delta + \Psi_\delta - \Psi^{\vee}_\delta,$$

which satisfies

$$\begin{cases} \partial_t u_\delta = \Delta u_\delta - u_\delta^3 + m_\delta u_\delta + \xi_\delta, \\ u_\delta|_{t=0} = (u_0)_\delta \end{cases}$$

where

$$\boxed{m_\delta := m + 3C_\delta^{(1)} - 9C_\delta^{(2)}.} \quad (*_3)$$

i.e. the renormalised system (**) with given mass m amounts to solving (Φ^4_3) but with the <u>modified</u> mass $(*_3)$.

<u>Calculations</u>: For simplicity, we will drop all 'δ' subscripts indicating mollification in what follows.

Recall the following: $u = v + \Psi - \Psi^{\vee}$

$$v = X + Y$$

$$(\partial_t - \Delta)\Psi = \xi, \quad (\partial_t - \Delta)\Psi^{\vee} = \Psi^{3} = \Psi^3 - 3C_\delta^{(1)}\Psi$$

$$\dot{Q}(v) = -(v-\Psi)^3 + v^3 - 3(v-\Psi)^2\,\mathbf{i} + m u$$

(Compare ⊛ and the preceeding eq$^{=}$ on pg 5).

We begin:

$$(\partial_t-\Delta)u = (\partial_t-\Delta)\mathbf{i} - (\partial_t-\Delta)\Psi + (\partial_t-\Delta)X + (\partial_t-\Delta)Y$$

$$= \xi - \mathbf{i}^3 + 3C_\delta^{(1)}\mathbf{i} - 3(v-\Psi)\varolessthan v - v^3$$

$$-3(v-\Psi)\varogreaterthan v + 9\Big[(v-\Psi)$$  $$\Big] - 3\,\mathrm{com}_2(X+Y)$$

Expand out

$$-3\,\mathrm{com}_1(X,Y)\circeq v - 3Y\circeq v + 3$$  $$+ Q(v)$$

Expand out

$$= \xi - \mathbf{i}^3 - v^3 + (3C_\delta^{(1)} - 9C_\delta^{(2)})\mathbf{i} - 9C_\delta^{(2)}(v-\Psi) + Q(v)$$

$$-3(v-\Psi)\not\circeq v - 3(Y-\Psi)\circeq v$$

$$-3\Big[-3(v-\Psi)(Y\circeq v) + \mathrm{com}_1(X,Y)\circeq v + \underbrace{\mathrm{com}_2(X+Y)}_{\text{Use definition of } \mathrm{com}_2 \text{ (pg 9)}}\Big]$$

$$= \xi - \mathbf{i}^3 - v^3 + (3C_\delta^{(1)} - 9C_\delta^{(2)})\mathbf{i} - 9C_\delta^{(2)}(v-\Psi)$$

$$+ Q(v) - 3(v-\Psi)\not\circeq v - 3(Y-\Psi)\circeq v$$

$$-3\Big[\underbrace{-3\big[(v-\Psi)\varolessthan Y\big]\circeq v + \mathrm{com}_1(X,Y)\circeq v}_{= X\circeq v}\Big]$$

$$= \xi - \mathbf{i}^3 - v^3 + (3C_\delta^{(1)} - 9C_\delta^{(2)})\mathbf{i} - 3(v-\Psi)\underbrace{v}_{\text{Expand this}}$$

$$-9C_\delta^{(2)}(v-\Psi) + Q(v)$$

$$= \xi - \phi^3 - v^3 + \left(3C_\delta^{(1)} - 9C_\delta^{(2)}\right)(\underbrace{\phi + v - \Psi}_{=u})$$
$$- 3(v-\Psi)\phi^2 + Q(v)$$

Use $Q(v)$ (Top of pg 14)

$$= \xi + \left(3C_\delta^{(1)} - 9C_\delta^{(2)}\right) u - \phi^3 - 3(v-\Psi)\phi^2 \cancel{- v^3}$$
$$- (v-\Psi)^3 \cancel{+ v^3} - 3(v-\Psi)^2\phi + mu$$

$$= \xi + \left(m + 3C_\delta^{(1)} - 9C_\delta^{(2)}\right) u - [\underbrace{\phi^3 + 3(v-\Psi)\phi^2 + 3(v-\Psi)^2\phi + (v-\Psi)^3}_{=(v+\phi-\Psi)^3 = u^3}]$$

$$= \xi + \underbrace{\left(m + 3C_\delta^{(1)} - 9C_\delta^{(2)}\right)}_{=: m_\delta} u - u^3$$

Lecture 5, 14/03/18

In the previous lecture we reduced the Φ^4_3-model on $\mathbb{T}^3$,

$$\text{"}\partial_t u = \Delta u - u^3 + m u + \xi\text{"}, \qquad (\Phi^4_3)$$

to a system of renormalised equations for X and Y,

$$\begin{cases} (\partial_t - \Delta)\tilde{X} = F(\tilde{X}+\tilde{Y}) \\ (\partial_t - \Delta)\tilde{Y} = G(\tilde{X},\tilde{Y}), \end{cases} \qquad (*)$$

where

$$u = \tilde{X} + \tilde{Y} + \mathsf{1} - \curlyvee,$$

and

$$F(\tilde{X}+\tilde{Y}) = -3(\tilde{X}+\tilde{Y} - \curlyvee)\odot V,$$

$$G(\tilde{X},\tilde{Y}) = -(\tilde{X}+\tilde{Y})^3 - 3Y \circleddash V - 3(\tilde{X}+\tilde{Y} - \curlyvee)\odot V$$
$$+ p(X+Y) - 3\,\mathrm{com}(\tilde{X},\tilde{Y}),$$

$$p(\tilde{X}+\tilde{Y}) = \underbrace{a_0 + a_1 (\tilde{X}+\tilde{Y}) + a_2 (\tilde{X}+\tilde{Y})^2}_{\text{Stochastic terms } \sim -1/2-.}$$

$$\mathrm{com}(\tilde{X},\tilde{Y}) := \mathrm{com}_1(\tilde{X},\tilde{Y}) \circleddash V + \mathrm{com}_2(\tilde{X}+\tilde{Y}).$$

In this lecture, we provide a local well-posedness theory for the system

$$(**) \quad \begin{cases} (\partial_t - \Delta) X = F(X+Y) - cX & \to \text{Decay} \\ (\partial_t - \Delta) Y = G(X,Y) + cX, & \to \text{Growth.} \end{cases}$$

where $c > 0$ is a constant. Solutions to $(**)$ are equivalent to those of (Φ^4_3) as the <u>sum</u> $X+Y$ does <u>not</u> depend on c.

While this trick is not necessary for the local theory,

$$(\tilde{X}+\tilde{Y} = X+Y)$$

Mourrat-Weber crucially use it for their globalisation argument. The idea is that one can find a c sufficiently large such that solutions (X,Y) to (**) do not blow-up.

Duhamel for (**):

$$
\begin{cases}
X(t) = e^{t(\Delta-c)}X_0 + \int_0^t e^{(t-t')(\Delta-c)} F(X+Y)(t')\,dt' & (2\text{-}1)\\
Y(t) = e^{t\Delta}Y_0 + \int_0^t e^{(t-t')\Delta}\left[G(X,Y)(t') + cX(t')\right]dt' & (2\text{-}2)
\end{cases}
$$

(Note: The numbering scheme in this lecture follows that of the paper of Mourrat-Weber.)

We write $\quad B_p^s := B_{p,\infty}^s$.

We study (2-1)–(2-2) with data $(X_0,Y_0) \in B_\infty^{-3/5} \times B_\infty^{-3/5}$, although any $\# > -2/3$ also works.

Given $T \in (0,1]$, set

$$
\begin{aligned}
\mathbb{X}_T := &\left[C([0,T]; B_\infty^{-\frac{3}{5}}) \cap C((0,T]; B_\infty^{\frac{1}{2}+2\varepsilon}) \cap C^{1/8}((0,T]; L^\infty)\right]\\
&\times \left[C([0,T]; B_\infty^{-\frac{3}{5}}) \cap C((0,T]; B_\infty^{1+2\varepsilon}) \cap C^{1/8}((0,T]; L^\infty)\right].
\end{aligned}
$$

$\big(=(\text{Space for }X)\times(\text{Space for }Y)\big)$.

Notice the increase in regularity when $t \neq 0$.

with the norm

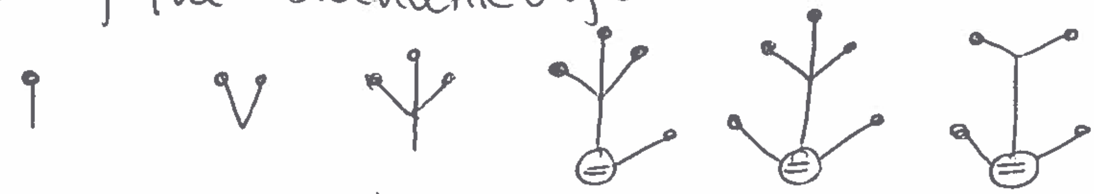
$$\|(X,Y)\|_{\mathfrak{X}_T} := \max\Big\{ \sup_{t\in[0,T]} \|X(t)\|_{B_\infty^{-3/5}},\ \sup_{t\in(0,T]} t^{3/5}\|X(t)\|_{B_\infty^{\frac12+2\varepsilon}},$$

$$\sup_{0<t_1<t_2\le T} \frac{t_1^{1/2}\|X(t_2)-X(t_1)\|_{L_x^\infty}}{|t_1-t_2|^{1/8}},\ \sup_{t\in[0,T]}\|Y(t)\|_{B_\infty^{-3/5}},$$

$$\sup_{t\in(0,T]} t^{\frac{17}{20}}\|Y(t)\|_{B_\infty^{1+2\varepsilon}},\ \sup_{0<t_1<t_2\le T} \frac{t_1^{1/2}\|Y(t_2)-Y(t_1)\|_{L_x^\infty}}{|t_1-t_2|^{1/8}} \Big\}.$$

Theorem 2-1: Let $\varepsilon>0$ sufficiently small, $K\ge 1$. Let τ denote one of the stochastic objects

τ



"Regularity" of τ $\quad \alpha_\tau$ $\quad -\frac12-\varepsilon \quad -1-\varepsilon \quad \frac12-\varepsilon \quad -\varepsilon \quad -\frac12-\varepsilon$

Assume $\tau\in C([0,1];B_\infty^{\alpha_\tau})$ and that

$$\sup_{0\le t\le 1}\|\tau(t)\|_{B_\infty^{\alpha_\tau}} \le K \quad \forall \tau.$$

Furthermore, assume that

$$\sup_{0\le t_1<t_2\le 1} \frac{\|\Psi(t_1)-\Psi(t_2)\|_{B_\infty^{1/4-\varepsilon}}}{|t_1-t_2|^{1/8}} \le K.$$

Then:

(i) For all $(X_0,Y_0)\in B_\infty^{-3/5}\times B_\infty^{-3/5}$, $\exists\, T^*\in(0,1]$ and $\exists!\,(X,Y)$ solutions to (2-1) and (2-2) on $(0,T^*]$

Blowup alternative:

Either $T^* = 1$ ($\Rightarrow (X, Y) \in \mathbb{X}_1$)

OR

$$\lim_{t \to (T^*)^-} \|(X(t), Y(t))\|_{B_\infty^{-3/5} \times B_\infty^{-3/5}} = \infty.$$

(ii) If $(X_0, Y_0) \in B_\infty^{1/2+2\varepsilon} \times B_\infty^{1+2\varepsilon}$, then the solution (X, Y) is continuous up to time 0 (in $B_\infty^{1/2+2\varepsilon} \times B_\infty^{1+2\varepsilon}$).

i.e. $(X, Y) \in [C([0,T]; B_\infty^{\frac{1}{2}+2\varepsilon}) \cap C^{1/8}([0,T]; L_x^\infty)]$
$\times [C([0,1]; B_\infty^{1+2\varepsilon}) \cap C^{1/8}([0,T]; L_x^\infty)]$

Rmk: We prove (i) but not (ii).

Rmk: The consistency of the exponents for the additional assumption $\Psi \in C_t^{1/8} B_\infty^{1/4-\varepsilon}$ can be argued by the fact that $\Psi \in C^0 B_\infty^{\frac{1}{2}-\varepsilon}$, so we are essentially exchanging $1/4$ spatial derivatives for $1/8$-temporal regularity where the exchange rate is determined by the parabolic scaling.

The proof of Thm 2.1 will be given in this lecture, assuming the following results hold. The proofs for these results are deferred to an upcoming lecture.

Proposition A.13: Let $\alpha, \beta \in \mathbb{R}$ and $p \in [1,\infty]$.

i) If $\alpha \geq \beta$, then there exists $C>0$ s.t. uniformly over $t>0$,
$$\|e^{t\Delta} f\|_{B_p^\alpha} \leq C t^{\underbrace{\frac{\beta-\alpha}{2}}_{\leq 0}} \|f\|_{B_p^\beta}$$

ii) If $0 \leq \beta - \alpha \leq 2$,
$$\|(1-e^{t\Delta}) f\|_{B_p^\alpha} \leq C t^{\underbrace{\frac{\beta-\alpha}{2}}_{\geq 0.}} \|f\|_{B_p^\beta}$$

Remark: We will use Prop^n A.13 with $e^{t(\Delta-c)}$ replacing $e^{t\Delta}$. The same estimates hold since $e^{t(\Delta-c)} f = e^{-ct}(e^{t\Delta} f)$.

Proposition 2.2 (1st commutator estimate)

Let $\varepsilon>0$, $\beta \in (4\varepsilon, 1+2\varepsilon]$, $p \in [1,\infty]$ and $T>0$. Then
$$\|\mathrm{Com}_1(X,Y)(t) - e^{t\Delta} X_0\|_{B_p^{1+2\varepsilon}} \lesssim K^2 + \int_0^t \frac{K}{(t-s)^{1+2\varepsilon-\beta/2}} \|(X(s),Y(s))\|_{B_p^\beta \times B_p^\beta}\, ds$$
$$+ \int_0^t \frac{K}{(t-s)^{1+2\varepsilon}} \|\delta_{s,t}(X+Y)\|_{L^p}\, ds$$
$$(\delta_{s,t} f := f(t) - f(s)).$$

So $\mathrm{com}_1(X,Y) - e^{t\Delta} X_0 \in B_p^{1+}$ as expected back in lecture 4 (pg. 8).

The term with $\delta_{s,t}$ necessitates a little bit of Hölder regularity.

Proposition A.9: Let $\alpha<1$, $\beta,\gamma \in \mathbb{R}$, $p, p_1, p_2, p_3 \in [1,\infty]$ s.t.
$$\beta+\gamma<0, \quad \alpha+\beta+\gamma>0, \quad \frac{1}{p} = \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}.$$

Then the mapping
$$[\circledless, \circleddash] : (f,g,h) \longmapsto (f \circledless g) \circleddash h - f(g \circleddash h),$$
extends to a continuous trilinear map
$$B_{p_1}^\alpha \times B_{p_2}^\beta \times B_{p_3}^\gamma \longmapsto B_p^{\alpha+\beta+\gamma}.$$

Remark: We apply Prop^{n} A.9. when estimating $\mathrm{Com}_2(X+Y)$. Our corresponding inputs (f,g,h) have regularities

$$\left(\tfrac{1}{2}-, 1-, -1-\right)$$

$$\Downarrow \text{Prop A.9}$$

$$\text{Commutator} \in B_\infty^{\frac{1}{2}-}.$$

The point of Prop^{n} A.9 is that the commutator takes into account the regularity of the "low component" f.

Recall that $f \circleddash g \sim \mathrm{Reg}(g) \Rightarrow$ Lost information about $\mathrm{Reg}(f)$.

Lemma 2.3: There exists a constant $C = C(c, K) > 0$, such that for all $M \geq \max(1, \|X_0\|_{B_\infty^{-3/5}}, \|Y_0\|_{B_\infty^{-3/5}})$,

$T \in (0,1]$, $(X,Y) \in \mathcal{X}_{T,M} =: B_M \subset \mathcal{X}_T$ (closed ball of radius M)

and $s \in (0,T]$, we have

$$\|F(X+Y)(s)\|_{B_\infty^{-1-\varepsilon}} \leq C M s^{-\frac{33}{100}} \tag{2-10}$$

$$\|G(X,Y)(s) + cX(s)\|_{B_\infty^{-\frac{1}{2}-\varepsilon}} \leq C M^3 s^{-\frac{99}{100}}.$$

For part (ii) of Th^{m} 2-1, we would use: If
$$M \geq \max(1, \|X_0\|_{B_\infty^{\frac{1}{2}+2\varepsilon}}, \|Y_0\|_{B_\infty^{1+2\varepsilon}}),$$
then (2-10) and (2-11) hold without the terms $s^{-33/100}$ & $s^{-99/100}$.

Proof of Th^{m} 2-1: Denote by $\Phi^X[X,Y]$ and $\Phi^Y[X,Y]$ the RHS of (2-1) and (2-2), respectively.

We establish that $(\Phi^X[X,Y], \Phi^Y[X,Y])$ is a contraction (for small enough $T>0$) on $\mathcal{X}_{T,M}$

Bounds for X: For $\beta \in \{-\frac{3}{5}, \frac{1}{2}+2\varepsilon\}$, we have by (2-1),

$$\|\Psi^X[X,Y](t)\|_{B^\beta_\infty} \underset{\substack{(\text{Prop}\\ A\text{-}13)}}{\leq} \|e^{t(\Delta-c)}X_0\|_{B^\beta_\infty} + \int_0^t \frac{1}{(t-s)^{\frac{\beta+1+\varepsilon}{2}}} \|F(X+Y)(s)\|_{B_\infty^{-1-\varepsilon}} ds$$

$$\underset{\substack{(\text{Prop } A\text{-}13)\\ + \\ (2\text{-}10)}}{\lesssim} t^{-\frac{1}{2}(\beta+\frac{3}{5})} \|X_0\|_{B_\infty^{-3/5}} + M \int_0^t \frac{1}{(t-s)^{\frac{\beta+1+\varepsilon}{2}}} \frac{1}{s^{\frac{33}{100}}} ds.$$

$$= t^{-\frac{1}{2}(\beta+3/5)} \|X_0\|_{B_\infty^{-3/5}} + M t^{\frac{17}{100}-\frac{\beta}{2}-\frac{\varepsilon}{2}} \cdot t^{\frac{\beta}{2}+\frac{\varepsilon}{2}-\frac{17}{100}} \times \int_0^t \frac{ds}{(t-s)^{\frac{\beta+1+\varepsilon}{2}} s^{\frac{33}{100}}}.$$

(Recall the Beta Function Fact from Lecture 2 (pg-3):

$$t^{\alpha_1} \int_0^t (t-s)^{\alpha_2} (s)^{\alpha_3} ds = B(\alpha_2+1, \alpha_3+1) < \infty \quad \text{(Best)}$$

when $\alpha_1+\alpha_2+\alpha_3 = -1$, $\alpha_2, \alpha_3 > -1$.)

Note: $\frac{\beta+1+\varepsilon}{2} \in \{\frac{1}{5}+, \frac{3}{4}+\}$, and $1 = \left(\frac{\beta+1+\varepsilon}{2}\right) + \frac{33}{100} + \left(\frac{17}{100} - \frac{\beta+\varepsilon}{2}\right).$

$$\underset{(B\text{-est})}{\lesssim} t^{-\frac{1}{2}(\beta+3/5)} \|X_0\|_{B_\infty^{-3/5}} + M t^{\frac{17}{100}-\frac{\beta}{2}-\frac{\varepsilon}{2}}.$$

When $\beta = -3/5$, $\|\Psi^X[X,Y](t)\|_{B_\infty^{-3/5}} \lesssim \|X_0\|_{B_\infty^{-3/5}} + M \underbrace{t^{\frac{47}{100}-\frac{\varepsilon}{2}}}_{\leq 1}$

$$\leq 2M.$$

When $\beta = \frac{1}{2}+2\varepsilon$, we multiply by $t^{3/5}$,

$$t^{3/5} \|\Psi^X[X,Y](t)\|_{B^{1/2+2\varepsilon}} \leq t^{\frac{1}{20}-\varepsilon} \|X_0\|_{B_\infty^{-3/5}} + t^{\overbrace{\frac{3}{5}-\frac{2}{25}-\frac{3}{2}\varepsilon}^{>0}} M$$

In summary, 8

$$
① \begin{cases} \|\Psi^X[X,Y]\|_{C_T B_\infty^{-3/5}} \lesssim \|X_0\|_{B_\infty^{-3/5}} + T^\theta \mathcal{M}, \quad \theta>0. \\ \sup_{t\in(0,T]} t^{3/5} \|\Psi^X[X,Y]\|_{B_\infty^{\frac{1}{2}+2\varepsilon}} \lesssim T^\theta \|X_0\|_{B^{-3/5}} + T^\theta \mathcal{M}. \end{cases}
$$

Hölder bound for X: We write $\Psi^X(t) := \Psi^X[X,Y](t)$. Have

$$
\begin{aligned}
\Psi^X(t) - \Psi^X(s) &= \left(e^{(t-s)(\Delta-c)} - \mathrm{Id}\right) e^{s(\Delta-c)} X_0 \\
\underline{t\geq s} \qquad &+ \left(e^{(t-s)(\Delta-c)} - \mathrm{Id}\right) \int_0^s e^{(s-r)(\Delta-c)} F(X+Y)(r)\,dr \\
&+ \int_s^t e^{(t-r)(\Delta-c)} F(X+Y)(r)\,dr.
\end{aligned}
$$

(Recall the embeddings $B^\varepsilon_{\infty,\infty} \subset B^0_{\infty,1} \subset L^\infty$
Geometric series $\qquad f = \sum_N P_N f$.)

$$
\begin{aligned}
\Rightarrow \left\| \left(e^{(t-s)(\Delta-c)} - \mathrm{Id}\right) e^{s(\Delta-c)} X_0 \right\|_{L^\infty_x} &\lesssim \| \text{———} \|_{B^\varepsilon_{\infty,\infty}} \\
&\underset{\text{(Prop A.13 (ii))}}{\lesssim} (t-s)^{1/8} \|e^{s(\Delta-c)} X_0\|_{B_\infty^{1/4+\varepsilon}} \\
&\underset{\text{(Prop A.13 (i))}}{\lesssim} (t-s)^{1/8} s^{-\frac{17}{40}-\frac{\varepsilon}{2}} \|X_0\|_{B_\infty^{-3/5}}
\end{aligned}
$$

By similar arguments,

$$
\begin{aligned}
\|\Psi^X(t) - \Psi^X(s)\|_{L^\infty_x} &\lesssim (t-s)^{1/8} s^{-\frac{17}{40}-\varepsilon} \|X_0\|_{B_\infty^{-3/5}} \\
&+ (t-s)^{1/8} \int_0^s \frac{1}{(s-r)^{\frac{1}{2}(\frac{1}{4}+1+2\varepsilon)}} \|F(X+Y)(r)\|_{B_\infty^{-1-\varepsilon}}\,dr \\
&+ \int_s^t \frac{1}{(t-r)^{\frac{1}{2}(1+2\varepsilon)}} \|F(X+Y)(r)\|_{B_\infty^{-1-\varepsilon}}\,dr.
\end{aligned}
$$

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Consider the second term. By (2-10), we bound by

$$M(t-s)^{1/8}\int_0^s \frac{1}{(s-r)^{\frac{1}{2}(\frac{1}{4}+1+2\varepsilon)}}\frac{1}{r^{\frac{33}{100}}}\,dr$$

$$\lesssim M(t-s)^{1/8}\, s^{\frac{9}{200}-\varepsilon}\underbrace{\frac{1}{s^{\frac{9}{200}-\varepsilon}}\int_0^s \frac{1}{(s-r)^{\frac{1}{2}(\frac{1}{4}+1+2\varepsilon)}}\frac{1}{r^{\frac{33}{100}}}\,dr}_{\text{by (B-e1t) } \lesssim 1}$$

$$\lesssim M(t-s)^{1/8}\, t^{\frac{9}{200}-\varepsilon}$$

For the third term we cannot use the Beta function fact directly and must proceed differently. Using the inequality

$$r^{-\frac{33}{100}} \le (r-s)^{-\frac{75}{200}+\varepsilon}\, t^{\frac{9}{200}-\varepsilon}, \quad (s<r<t)$$

(a consequence of $(r-s)^{\theta}\le r^{\theta_1}t^{\theta_2}$, $\theta=\theta_1+\theta_2$) and (2-10) gives

$$\lesssim M t^{\frac{9}{200}-\varepsilon}\int_s^t \frac{1}{(t-r)^{\frac{1}{2}+\varepsilon}}\frac{1}{(r-s)^{\frac{75}{200}-\varepsilon}}\,dr$$

Put $\left(\tau=\frac{r-s}{t-s}\right)$

$$\lesssim M t^{\frac{9}{200}-\varepsilon}(t-s)^{1-\frac{1}{2}-\varepsilon-\frac{75}{200}+\varepsilon}\underbrace{\int_0^1 \frac{1}{(1-\tau)^{\frac{1}{2}+\varepsilon}}\frac{1}{\tau^{\frac{75}{200}-\varepsilon}}\,d\tau}_{\lesssim 1 \text{ as } \frac{1}{2}+\varepsilon+\frac{75}{200}-\varepsilon<1.}.$$

$$\lesssim M t^{\frac{9}{200}-\varepsilon}(t-s)^{1/8}$$

Putting these together gives

$$\frac{\|\Psi^X(t)-\Psi^X(s)\|_{L^\infty_x}\, s^{1/2}}{(t-s)^{1/8}} \lesssim s^{\frac{3}{40}-\varepsilon}\|X_0\|_{B^{-3/5}_\infty} + M t^{\frac{9}{200}-\varepsilon} s^{1/2},$$

$$② \Rightarrow \sup_{0\le s<t\le T}\left(s^{1/2}\frac{\|\Psi^X(t)-\Psi^X(s)\|_{L^\infty_x}}{|t-s|^{1/8}}\right)\lesssim s^{\theta}\|X_0\|_{B^{-3/5}_\infty}+s^{\theta}t^{\theta}M.$$

Bounds for Y: $\beta \in \{-\frac{3}{5}, 1+2\varepsilon\}$. (10)

$$\|\Phi^Y(t)\|_{B^\beta_\infty} \overset{\text{Prop}^n\text{A.13 (ii)}}{\lesssim} t^{-\frac{1}{2}(\beta+\frac{3}{5})}\|Y_0\|_{B^{-3/5}_\infty} + \int_0^t \frac{1}{(t-s)^{\frac{1}{2}(\beta+\frac{1}{2}+\varepsilon)}} \times \|G(X,Y)(s)+cX(s)\|_{B^{-\frac{1}{2}-2\varepsilon}_\infty}\,ds.$$

$$\overset{(2.11)}{\lesssim} t^{-\frac{1}{2}(\beta+\frac{3}{5})}\|Y_0\|_{B^{-3/5}_\infty} + M^3\int_0^t \frac{1}{(t-s)^{\frac{1}{2}(\beta+\frac{1}{2}+\varepsilon)}}\frac{1}{s^{\frac{99}{100}}}\,ds.$$

When $\beta=-3/5$,

$$\|\Phi^Y(t)\|_{B^{-3/5}_\infty} \lesssim \|Y_0\|_{B^{-3/5}_\infty} + M^3\int_0^t \frac{(t-s)^{\frac{1}{20}-\frac{\varepsilon}{2}}}{s^{99/100}}\,ds$$

$$\lesssim \|Y_0\|_{B^{-3/5}_\infty} + t^{\frac{1}{20}+\frac{1}{100}-\frac{\varepsilon}{2}}M^3.$$

Could have $\rightarrow$ just dropped the linear operator $e^{(t-s)\Delta}$.

When $\beta=1+2\varepsilon$,

$$\|\Phi^Y(t)\|_{B^{1+2\varepsilon}_\infty} \lesssim t^{-\frac{4}{5}-\varepsilon}\|Y_0\|_{B^{-3/5}_\infty} + M^3\int_0^t \frac{1}{(t-s)^{\frac{3}{4}+\varepsilon(\frac{3}{2})}}\frac{1}{s^{99/100}}\,ds$$

$$\lesssim t^{-4/5-\varepsilon}\|Y_0\|_{B^{-3/5}_\infty} + M^3 t^{-\frac{37}{50}-\frac{3}{2}\varepsilon}$$

$$\Rightarrow t^{\frac{17}{20}}\|\Phi^Y(t)\|_{B^{1+2\varepsilon}_\infty} \lesssim t^{\frac{1}{20}-\varepsilon}\|Y_0\|_{B^{-3/5}_\infty} + M^3 t^{\frac{11}{100}-\frac{3}{2}\varepsilon}.$$

Hence,

$$(3)\begin{cases} \sup_{t\in[0,T]}\|\Phi^Y(t)\|_{B^{-3/5}_\infty} \lesssim \|Y_0\|_{B^{-3/5}_\infty} + t^\theta M^3. \\ \sup_{t\in[0,T]}\left(\|\Phi^Y(t)\|_{B^{1+2\varepsilon}_\infty}\cdot t^{17/20}\right) \lesssim t^\theta\|Y_0\|_{B^{-3/5}_\infty} + t^\theta M^3. \end{cases}$$

Hölder bound for Y:

(11)

In a similar manner as for the Hölder bound for X we have

$$\|\Phi^Y(t)-\Phi^Y(s)\|_{L^\infty_x} \lesssim (t-s)^{1/8} s^{-\frac{17}{40}-\varepsilon} \|Y_0\|_{B_\infty^{-3/5}}$$

(4')

$$+ (t-s)^{1/8} M^3 s^{\frac{1}{100}-\frac{3}{8}-2\varepsilon}$$

$$+ \int_s^t \frac{1}{(t-r)^{\frac{1}{4}+2\varepsilon}} M^3 r^{-\frac{99}{100}} dr \quad , \quad t \geq s.$$

The first term is treated in an identical manner as the first term for $\Phi^X(t)-\Phi^X(s)$.

For the second & third terms we also proceed similarly:

2nd term: $L^\infty_x \to B_\infty^{1/4+\varepsilon} \to B_\infty^{-\frac{1}{2}-2\varepsilon}$

Picks out $(t-s)^{1/8}$ — Use Prop A.13 — Use (2-11)

$$\lesssim M^3 (t-s)^{1/8} \int_0^s \frac{1}{(s-r)^{\frac{3}{8}+\frac{3}{2}\varepsilon}} \frac{dr}{r^{\frac{99}{100}}}$$

Use Beta function fact.

(2nd term above)

3rd term: $L^\infty_x \to B_\infty^{\varepsilon} \xrightarrow{\text{(Prop. A.13)}} B_\infty^{-\frac{1}{2}-2\varepsilon} \to$ (2-11) +

$$r^{-\frac{99}{100}} \leq (r-s)^{-\frac{5}{8}+2\varepsilon} s^{\frac{1}{100}-\frac{3}{8}-2\varepsilon}$$

3rd term above becomes

$$\lesssim s^{\frac{1}{100}-\frac{3}{8}-2\varepsilon} \int_s^t \frac{1}{(t-r)^{2/8-2\varepsilon}} \frac{1}{(r-s)^{5/8-2\varepsilon}} dr \cdot M^3$$

$$\lesssim s^{\frac{1}{100}-\frac{3}{8}-2\varepsilon} (t-s)^{1/8} \underbrace{\frac{1}{(t-s)^{1/8}} \int_s^t \frac{1}{(t-r)^{2/8-2\varepsilon}} \frac{1}{(r-s)^{5/8-2\varepsilon}} dr}_{\text{Change of variables } \tau = \frac{r-s}{t-s}.} \cdot M^3$$

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$$\lesssim s^{\frac{1}{100}-\frac{3}{8}-2\varepsilon}(t-s)^{1/8}M^3.$$

Applying these in (4'), multiplying both sides by $s^{1/2}$ and dividing by $(t-s)^{1/8}$, implies

$$(4)\quad \sup_{0\le s<t\le T}\left(\frac{s^{1/2}\|\Phi^Y(t)-\Phi^Y(s)\|_{L^\infty}}{(t-s)^{1/8}}\right)\lesssim s^{\theta}\|Y_0\|_{B^{-3/5}_{\infty}}+s^{\theta}\tau^{\theta}M^3.$$

Combining (1), (2), (3) and (4) shows $(\Phi^X[X,Y],\Phi^Y[X,Y]):\mathcal{X}_{T,M}\to\mathcal{X}_{T,M}$

Difference estimates (for the contraction property) hold by similar arguments. $\boxtimes$.

Q^n: What is the relationship of this approach to Φ^4_3 (Th^m 2-1) to Paracontrolled distributions?

Paracontrolled distributions: Gubinelli-Imkeller-Perkowski '15 FMπ

There, we say v is paracontrolled by w if we can write

$$v=v_1\circledless w+(\text{Smoother term}).$$

This, roughly, says that the irregular behaviour of v (or high frequency behaviour) is governed by the explicit w. The smoother term will play almost no role since it will have "strong" decay of its high frequencies.

For Φ^4_3, we wrote $v=X+Y$ and v solved

$$(\partial_t-\Delta)v=\underbrace{\square\circledless V}_{(\partial_t-\Delta)X.}+\underbrace{(\partial_t-\Delta)Y}_{\text{Smoother}}$$

$$\Rightarrow(\partial_t-\Delta)(u-\Psi+\Psi^{\Psi})=(\partial_t-\Delta)v \text{ is paracontrolled by } V.$$

Lecture 6 21/03/18 ①

Proof of Lemma 2.3:

As $(X,Y) \in \mathbb{X}_{T,M}$,

$$\|X(s)\|_{B_\infty^{-3/5}} \le M, \quad \|X(s)\|_{B_\infty^{\frac{1}{2}+2\varepsilon}} \le M s^{-3/5}.$$

By interpolation

$$\Rightarrow \quad \|X(s)\|_{B_\infty^\gamma} \le M s^{-\frac{3}{5}\left(\frac{10\gamma+6}{11+20\varepsilon}\right)}, \quad \forall \gamma \in \left[-\tfrac{3}{5}, \tfrac{1}{2}+2\varepsilon\right]$$

$$\left(\begin{array}{l} \gamma = (1-\theta)(-3/5) + \theta(\tfrac{1}{2}+2\varepsilon) \\ \Rightarrow \theta = \dfrac{10\gamma+6}{11+20\varepsilon}. \end{array} \right.$$

In particular, with $\gamma=\varepsilon$,

$$\|X(s)\|_{L^\infty} \lesssim \|X(s)\|_{B^0_{\infty,1}} \lesssim \|X(s)\|_{B^\varepsilon_{\infty,\infty}}$$

$$\lesssim M s^{-\frac{3}{5}\left(\frac{10\varepsilon+6}{11+20\varepsilon}\right)}$$

$$= M s^{-\frac{18+30\varepsilon}{55+100\varepsilon}}$$

$$\lesssim M s^{-\frac{33}{100}} \qquad (\varepsilon \ll 1)$$

Similarly,

$$\|Y(s)\|_{B_\infty^{-3/5}} \le M, \quad \|Y(s)\|_{B_\infty^{1+2\varepsilon}} \le M s^{-\frac{17}{20}},$$

By (interpolation) $\Rightarrow$

$$\|Y(s)\|_{B_\infty^\gamma} \lesssim M s^{-\frac{17}{20}\tilde{\theta}}, \quad \gamma = (1-\tilde{\theta})(-\tfrac{3}{5}) + (1+2\varepsilon)\tilde{\theta}.$$

$$\Rightarrow \|Y(s)\|_{L^\infty} \lesssim M s^{-\frac{33}{100}} \;\&\; \|Y(s)\|_{B_\infty^{\frac{1}{2}+2\varepsilon}} \lesssim M s^{-3/5}.$$

Note: In L^∞ and $B_\infty^{\frac{1}{2}+2\varepsilon}$, X and Y have the same bounds. This is for convenience.

$$\Rightarrow \|F(X+Y)(s)\|_{B_\infty^{-1-\varepsilon}} \sim \|(X+Y-\Psi)\circledcirc V\|_{B_\infty^{-1-\varepsilon}}$$

$$\lesssim \|X+Y-\Psi\|_{L^\infty}\,\|V\|_{B_\infty^{-1-\varepsilon}}$$

Recall $\|f\circledcirc g\|_{B_p^\beta} \le \|f\|_{L^{p_1}}\|g\|_{B_{p_2}^\beta}$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$

$$\lesssim \left(Ms^{-\frac{33}{100}}+K\right)\cdot K$$

$\to$ Theorem 2-1 assumption.

$$\lesssim M\underbrace{K^2}_{C=C(c,K)}s^{-\frac{33}{100}}$$

$C=C(c,K)$ from statement of Lemma 2-3.

Now estimate $G(X+Y)+cX$ term.

- $\|cX(s)\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \lesssim \|X(s)\|_{L^\infty} \lesssim Ms^{-\frac{33}{100}} \lesssim M^3 s^{-\frac{99}{100}}$ $\quad (s<1)$.

<u>$G(X+Y)$:</u>

$$G(X+Y) = -(X+Y)^3 - 3\,\mathrm{com}(X,Y) - 3Y\circledcirc V - 3(X+Y-\Psi)\circledcirc V + P(X+Y)$$

$$P(X+Y) = \underbrace{a_0}+\underbrace{a_1}(X+Y)+\underbrace{a_2}(X+Y)^2$$

Stochastic terms $\sim -\frac{1}{2}-$.

$$\begin{cases} a_0 = b_0 + 3\Psi\circledcirc V - 9\,\Psi\,(\mathcal{I}(\curlyvee)\circledcirc\curlyvee) \\ a_1 = b_1 + 9\,(\mathcal{I}(\curlyvee)\circledcirc\curlyvee), \\ a_2 = b_2 \end{cases}$$

$$\begin{cases} b_0 = m(\phi-\Psi)+(\curlyvee)^3-3\phi(\curlyvee)^2 & \sim -\frac{1}{2}- \\ b_1 = m+6\Psi\phi-3(\curlyvee)^2 & \sim -\frac{1}{2}- \\ b_2 = -3\phi+3\curlyvee & \sim -\frac{1}{2}- \end{cases}$$

(See lecture 4).

We use the triangle inequality and bound each of the terms of G separately.

$\underline{P(X+Y)}$

- $$\|a_2(X+Y)^2_{(t)}\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \lesssim \|a_2\|_{B_\infty^{-\frac{1}{2}-\varepsilon}} \|(X+Y)^2\|_{B_\infty^{\frac{1}{2}+2\varepsilon}}$$
$-\frac{1}{2}-\varepsilon$
$$\lesssim \|a_2\|_{B_\infty^{-\frac{1}{2}-\varepsilon}} \|X+Y\|_{B_\infty^{\frac{1}{2}+2\varepsilon}} \|X+Y\|_{L_X^\infty}$$
$$\lesssim KM^2 s^{-\frac{93}{100}}.$$

- $$\|a_1(X+Y)+a_0\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \leq \|a_1\|_{B_\infty^{-\frac{1}{2}-\varepsilon}} \|X+Y\|_{B_\infty^{\frac{1}{2}+2\varepsilon}} + \|a_0\|_{B_\infty^{-\frac{1}{2}-\varepsilon}}$$
$$\lesssim C(K)Ms^{-3/5}.$$

- $$\|(\underbrace{X+Y-\Psi}_{\beta})\odot \underbrace{V}_{\alpha}\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \lesssim \|X+Y-\Psi\|_{B_\infty^{\frac{1}{2}-\varepsilon}} \|V\|_{B_\infty^{-1-\varepsilon}}$$
$\alpha+\beta$
$$\lesssim C(K)Ms^{-3/5}.$$

- $$\|(X+Y)^3\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \lesssim \|(X+Y)^3\|_{L^\infty} \lesssim \left(Ms^{-\frac{33}{100}}\right)^3 = M^3 s^{-\frac{99}{100}}.$$

- $$\|Y\odot V\|_{B_\infty^{-\frac{1}{2}-2\varepsilon}} \leq \|Y\odot V\|_{B_\infty^{\varepsilon}}$$
$$\lesssim \|Y\|_{B_\infty^{1+2\varepsilon}} \|V\|_{B_\infty^{-1-\varepsilon}}$$
$$\lesssim KMs^{-17/20}$$
$$\lesssim KMs^{-\frac{99}{100}}. \qquad (s<1)$$

It remains to estimate

$$\mathrm{com}(X,Y) := \mathrm{com}_1(X,Y) \circleddash V + \mathrm{com}_2(X+Y).$$

(4)

$$\mathrm{com}_2(X+Y) = [\circledcirc, \circleddash]\big(\underbrace{-3(X+Y-Y)}_{\substack{\frac{1}{2}-\varepsilon \\ \alpha}}, \underbrace{Y}_{\substack{1-\varepsilon \\ \beta}}, \underbrace{V}_{\substack{-1-\varepsilon \\ \gamma}}\big)$$

Recall Prop^n A.9 from lecture 5 $\Leftarrow$ $\alpha+\beta+\gamma = \frac{1}{2}-3\varepsilon > 0$, $\beta+\gamma = -2\varepsilon < 0$.

$\Downarrow$

Suffices to apply Prop^n A.9 with $\alpha = 3\varepsilon$.

$$\|\mathrm{com}_2(X+Y)(s)\|_{L^\infty_x} \le C(K)\big(1 + \|X+Y\|_{B_\infty^{\boxed{3\varepsilon}}}\big)$$

$$\lesssim C(K) M s^{-3/5}$$

$$\|\mathrm{com}_1(X,Y)\circleddash V(s)\|_{L^\infty} \lesssim \underbrace{\|\mathrm{com}_1(X,Y)\|_{B_\infty^{1+2\varepsilon}}}_{\substack{\text{Use Prop}^n\ 2.2 \\ \text{with } \beta = \frac{1}{2}+2\varepsilon.}} \underbrace{\|V\|_{B_\infty^{-1-\varepsilon}}}_{\le K}$$

$$\lesssim \big(\|e^{s\Delta}X_0\|_{B_\infty^{1+2\varepsilon}} + \|\mathrm{com}_1(X,Y)(s) - e^{s\Delta}X_0\|_{B_\infty^{1+2\varepsilon}}\big) K$$

$$\lesssim K\|e^{s\Delta}X_0\|_{B_\infty^{1+2\varepsilon}} + K^3 + K^2 M\int_0^s \frac{1}{(s-r)^{\frac{3}{4}+\varepsilon}} \frac{1}{r^{3/5}}\,dr$$

$$+ K^2 M\int_0^s \frac{1}{(s-r)^{1+2\varepsilon}} \underbrace{(r-s)^{1/8}}\, r^{-1/2}\,dr$$

$\|\delta_{s,t}(X+Y)\|_{L^\infty} \le r^{-1/2}(r-s)^{1/8}$. Hölder bound.

$\left(\sup \dfrac{s^{1/2}\|\delta_{st} f\|_{L^\infty}}{|s-t|^{1/8}} \le M\right.$

$1+2\varepsilon \mapsto -3/5.$

$$\lesssim KMs^{-4/5-\varepsilon} + K^2 M s^{-\frac{7}{20}-\varepsilon} \underbrace{s^{\frac{7}{20}+\varepsilon}\int_0^s \frac{1}{(s-r)^{3/4+\varepsilon}} \frac{1}{r^{3/5}}\,dr}$$

$$+ K^2 M s^{-\frac{3}{8}-2\varepsilon} \underbrace{s^{\frac{3}{8}+2\varepsilon}\int_0^s \frac{1}{(s-r)^{\frac{7}{8}+2\varepsilon}}\frac{1}{r^{1/2}}\,dr}$$ (5)

For this and first integral term use the Beta function estimate (e.g. Page 7 of Lecture 5)

$\lesssim 1$.

$$\lesssim KMs^{-\frac{4}{5}-\varepsilon} + K^3 + K^2Ms^{-\frac{7}{20}-\varepsilon} + K^2Ms^{-\frac{3}{8}-2\varepsilon}.$$

$$\leq C(c,K)M^3 s^{-99/100}$$ ☒

We now prove Proposition 2-2. We make use of Proposition A-9, for which we give only a heuristic argument of its validity, deferring the rigorous proof to that given in Mourrat-Weber.

Prop A-9: $\alpha<1$, $\beta,\gamma\in\mathbb{R}$, $p, p_1, p_2, p_3 \in [1,\infty]$ s.t.

$$\beta+\gamma<0, \quad \alpha+\beta+\gamma>0, \quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$$

Then

$$\|[\circledcirc,\circleddash](f,g,h)\|_{B_p^{\alpha+\beta+\gamma}} \lesssim \|f\|_{B^\alpha_{p_1}}\|g\|_{B^\beta_{p_2}}\|h\|_{B^\gamma_{p_3}}. \quad (*)$$

$$\|(f\circledcirc g)\circleddash h - f(g\circleddash h)\|_{B_p^{\alpha+\beta+\gamma}}$$

For 'proving' (*), we look at estimating instead

$$\|[\circledcirc,\circleddash]((\nabla)^{-\alpha}f,(\nabla)^{-\beta}g,(\nabla)^{-\gamma}h)\|_{B_p^{\alpha+\beta+\gamma}}.$$

The condition $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$ comes about because of products in L^p spaces (Hölder's inequality).

Recalling definitions of $\textcircled{<}$ and $\textcircled{=}$ from Lecture 1, (6)
we have

$$\mathcal{F}\{(f \mathbin{\textcircled{<}} g)\mathbin{\textcircled{=}} h - f(g \mathbin{\textcircled{=}} h)\}(n)$$

$$= \sum_{|j-k|\le 2}\ \sum_{n=n_1+n_2+n_3} \frac{\langle n\rangle^{\alpha+\beta+\gamma}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta}\langle n_3\rangle^{\gamma}}\varphi_k(n_3)\Bigg[\Bigg(\sum_{\substack{l\le q-2\\ q\ge 0}}\varphi_l(n_1)\varphi_q(n_2)\Bigg)\times\varphi_j(n_1+n_2) - \varphi_j(n_2)\Bigg]$$

$$\frac{\langle n\rangle^{\alpha+\beta+\gamma}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta}\langle n_3\rangle^{\gamma}} =: m(\bar{n}).$$

$$\times \hat{f}(n_1)\hat{g}(n_2)\hat{h}(n_3).$$

$$= \sum_{n=n_1+n_2+n_3}\Bigg(\underbrace{\sum_{|j-k|\ge 2} m(\bar{n})\varphi_k(n_3)\Bigg[\Bigg(\sum_{\substack{l\le q-2\\ q\ge 0}}\varphi_l(n_1)\varphi_q(n_2)\Bigg)\varphi_j(n_1+n_2) - \varphi_j(n_2)\Bigg]}\Bigg)$$
$$\times \hat{f}(n_1)\hat{g}(n_2)\hat{h}(n_3).$$

Roughly speaking, we want to be able to bound this multiplier.
Write it as (forgetting the sum over j,k),

$$m(\bar{n})\varphi_k(n_3)\Bigg[\sum_{\substack{l\le q-2\\ q\ge 0}}\varphi_l(n_1)\varphi_q(n_2)\Bigg]\Big[\varphi_j(n_1+n_2) - \varphi_j(n_2)\Big]$$

$$+ m(\bar{n})\varphi_k(n_3)\Bigg[\sum_{\substack{l\le q-2\\ q\ge 0}}\varphi_l(n_1)\varphi_q(n_2) - 1\Bigg]\varphi_j(n_2).$$

$$= (I) + (II).$$

$\underline{\mathrm{I}}$: Considering the supports of the $\varphi_{(\cdot)}(\cdot)$'s, we have in this case, $|n_1| << |n_2|$, $|n_2| \sim |n_3|$. (7)

By the Mean Value Th$^{\underline{m}}$;

$$|\varphi_j(n_1+n_2) - \varphi_j(n_2)| = \left|\varphi\left(\frac{n_1+n_2}{2^j}\right) - \varphi\left(\frac{n_2}{2^j}\right)\right|$$
$$\leq |\varphi'(n^*)| \frac{|n_1|}{2^j}$$
$$\lesssim \frac{|n_1|}{|n_2|} \quad (|n_2| \sim 2^j).$$

$$\Rightarrow |(\mathrm{I})| \lesssim \frac{|m(\bar{n})||n_1|}{|n_2|} \; [\, \underbrace{|n_1| << |n_2| \sim |n_3|}_{\Rightarrow\ |n| \lesssim \max(|n_2|, |n_3|).} \,]$$

$$\approx \frac{\langle n\rangle^{\alpha+\beta+\gamma} \langle n_1\rangle}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta+1}\langle n_3\rangle^{\gamma}}$$

$$\boxed{\alpha+\beta+\gamma \geq 0} \quad \lesssim \frac{\langle n_2\rangle^{\alpha+\beta+\gamma}\langle n_1\rangle^{1-\alpha}}{\langle n_2\rangle^{\beta+\gamma+1}} \lesssim \left(\frac{\langle n_1\rangle}{\langle n_2\rangle}\right)^{1-\alpha} \lesssim 1 \quad \boxed{\alpha < 1} \quad |n_1| << |n_2|.$$

$\underline{\mathrm{II}}$: We write (II) as $\left(\sum_{l,q} \varphi_l \varphi_q = 1\right)$.

$$(\mathrm{II}) = m(\bar{n})\varphi_k(n_3)\varphi_j(n_2)\left(\sum_{l,q} \varphi_l(n_1)\varphi_q(n_2)(1 - [\,l \leq q-2\,])\right)$$

$$\sim m(\bar{n})\varphi_k(n_3)\varphi_j(n_2) \underbrace{\sum_{|l| \gtrsim |q|} \varphi_l(n_1)\varphi_q(n_2)}_{\text{encodes } |n_1| \gtrsim |n_2|}.$$

$\Rightarrow$ For (II), we have $\underline{|n_1| \gtrsim |n_2| \sim |n_3|} \Rightarrow |n| \lesssim |n_1|$.

So $|(\mathrm{II})| \lesssim \dfrac{\langle n\rangle^{\alpha+\beta+\gamma}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta+\gamma}} \underset{\boxed{\alpha+\beta+\gamma>0}}{\lesssim} \dfrac{\langle n_1\rangle^{\alpha+\beta+\gamma}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\beta+\gamma}}$ (8)

$$\underset{\boxed{\substack{\beta+\gamma<0 \\ \Rightarrow -(\beta+\gamma)>0}}}{\lesssim} \frac{\langle n_2\rangle^{-(\beta+\gamma)}}{\langle n_1\rangle^{-(\beta+\gamma)}}$$

$$\sim \left(\frac{\langle n_2\rangle}{\langle n_1\rangle}\right)^{-(\beta+\gamma)} \underset{|n_2|\lesssim|n_1|}{\lesssim} 1$$

Proof of Proposition 2.2:

Define the commutator $[e^{t\Delta}, \circledcirc]$ by

$$[e^{t\Delta}, \circledcirc] : (f,g) \longmapsto e^{t\Delta}(f \circledcirc g) - f \circledcirc (e^{t\Delta} g).$$

Recall (Lecture 4, pg. 8),

$$\mathrm{com}_1(X,Y)(t) - e^{t\Delta}X_0 = -3\int_0^t S(t-t')\left[(X+Y-\Psi)\circledcirc V\right]dt' + 3(X+Y-\Psi)\circledcirc Y.$$

$$= -3\int_0^t [e^{(t-t')\Delta}, \circledcirc](X+Y-\Psi, V)(t')\,dt'$$

$$-3\int_0^t (X+Y-\Psi)\circledcirc(e^{(t-t')\Delta}V)\,dt'$$

$$+3(X+Y-\Psi)(t)\circledcirc Y(t)$$

$$= 3\int_0^t [\delta_{t't}(X+Y-\Psi)]\circledcirc[e^{(t-t')\Delta}V(t')]\,dt'$$

$$- 3\int_0^t [e^{(t-t')\Delta}, \varolessthan]\,(X+Y-\Psi, V)(t')\,dt'$$
$$=: (I) + (II).$$

The idea here is to create two commutator terms, the first being $[e^{(t-t')\Delta}, \varolessthan]$ and the second a "commutator with $\int_0^t$" i.e.

$$(X+Y-\Psi)(t) \varolessthan \underbrace{Y(t)}_{=\int_0^t e^{(t-t')\Delta} V(t')dt'} - \boxed{\int_0^t}(X+Y-\Psi)(t') \varolessthan [e^{(t-t')\Delta} V(t')]\,dt'$$

"Bring onto this term"

The first commutation puts $e^{(t-t')\Delta}$ onto V, in preparation of

II) We use

Proposition A-16: $\alpha<1$, $\beta\in\mathbb{R}$, $\gamma\geq\alpha+\beta$, $p, p_1, p_2\in[1,\infty]$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

Then
$$\|[e^{t\Delta}, \varolessthan](f,g)\|_{B^\gamma_{p,\infty}} \leq C t^{\underbrace{\frac{\alpha+\beta-\gamma}{2}}_{\leq 0}} \|f\|_{B^\alpha_{p_1,\infty}} \|g\|_{B^\beta_{p_2,\infty}}$$

$$\Rightarrow \left\|\int_0^t [e^{(t-t')\Delta}, \varolessthan](\Psi(t'), V(t'))\,dt'\right\|_{B_p^{1+2\varepsilon}}$$

Use Prop. A.16 with: $\gamma=1+2\varepsilon$, $\alpha=\frac{1}{2}-\varepsilon$, $\beta=-1-\varepsilon$

$$\lesssim \int_0^t \|[e^{(t-t')\Delta}, \varolessthan](\Psi(t'), V(t'))\|_{B_p^{1+2\varepsilon}}\,dt'$$

$$\lesssim K^2 \int_0^t \frac{1}{(t-t')^{\frac{3}{4}+2\varepsilon}}\,dt' \lesssim K^2$$

(10)

$$\left\| \int_0^t \left[e^{(t-t')\Delta}, \circledcirc \right] \left((X+Y)(t'), V(t') \right) dt' \right\|_{B_p^{1+2\varepsilon}}$$

Use Prop^n A-16 with: $\begin{pmatrix} \gamma = 1+2\varepsilon \\ \alpha = \beta \\ \beta = -1-\varepsilon \end{pmatrix}$.

$$\lesssim \int_0^t \frac{K}{(t-t')^{\frac{2+3\varepsilon-\beta}{2}}} \| (X+Y)(t') \|_{B_p^\beta} dt'.$$

I/ $$\left\| \int_0^t [\delta_{tt'}(X+Y-\Psi)] \circledcirc \left(e^{(t-t')\Delta} V(t') \right) dt' \right\|_{B_p^{1+2\varepsilon}}.$$

Prop^n A.7 f⊚g bounding
$$\lesssim \int_0^t \| \delta_{tt'}(X+Y-\Psi)(t') \|_{L^p} \| e^{(t-t')\Delta} V(t') \|_{B_\infty^{1+2\varepsilon}} dt'$$

Go to $-1-\varepsilon$ any : Heat estimate

$$\lesssim \int_0^t \frac{K}{(t-t')^{1+\frac{3}{2}\varepsilon}} \| \delta_{t't}(X+Y-\Psi)(t') \|_{L^p} dt'$$

$$\lesssim \int_0^t \frac{K}{(t-t')^{1+3/2\varepsilon}} \left(\| \delta_{t't}(X+Y)(t') \|_{L^p} + \underbrace{\| \delta_{t't} \Psi \|_{L^p}} \right) dt'.$$

$\frac{1}{(t-t')^{1+}}$ not integrable alone over $(0,t)$.
$\Rightarrow$ Need to lose a power of t on Ψ i.e. need Hölder regularity assumption

Theorem 2-1 assumption:
$$\| \delta_{t't} \Psi \|_{B_\infty^{1/4-\varepsilon}} \leq K |t-t'|^{1/8}.$$
$\Leftarrow$

$\left(\|\cdot\|_{L^p} \lesssim \|\cdot\|_{L^\infty} \lesssim \|\cdot\|_{B_\infty^\varepsilon} \lesssim \|\cdot\|_{B_\infty^{1/4-\varepsilon}} \right)$.

$$\lesssim \int_0^t \frac{K}{(t-t')^{1+\frac{3}{2}\varepsilon}} \| \delta_{t't}(X+Y)(t') \|_{L^p} dt' + K^2.$$

$\boxtimes$

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We complete the local well-posedness argument by proving the remaining commutator estimate.

Prop$^{\underline{n}}$ A.16: Let $\alpha<1$, $\beta\in\mathbb{R}$, $\gamma\geq\alpha+\beta$, $p,p_1,p_2\in[1,\infty]$ s.t.
$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}.$$
For every $t\geq 0$, define
$$[e^{t\Delta},\varolessthan]:(f,g)\longmapsto e^{t\Delta}(f\varolessthan g)-f\varolessthan(e^{t\Delta}g).$$
Then, there exists $C<\infty$ s.t. uniformly over $t>0$,
$$(*)\quad \|[e^{t\Delta},\varolessthan](f,g)\|_{B^{\gamma}_{p,\infty}}\leq C\,t^{\frac{\alpha+\beta-\gamma}{2}}\|f\|_{B^{\alpha}_{p_1,\infty}}\|g\|_{B^{\beta}_{p_2,\infty}}$$

Proof: We will show $(*)$ with $\|\nabla f\|_{B^{\alpha-1}_{p_1\infty}}$, which suffices since
$$\|\nabla f\|_{B^{\alpha-1}_{p_1,\infty}}\leq\|\langle\nabla\rangle f\|_{B^{\alpha-1}_{p_1,\infty}}=\|f\|_{B^{\alpha}_{p_1,\infty}}.$$
Write
$$[e^{t\Delta},\varolessthan](f,g)=\sum_{k=0}^{\infty}h_k,$$
where
$$h_k:=e^{t\Delta}(\underbrace{S_{k-1}f\cdot P_kg}_{\varolessthan})-\underbrace{S_{k-1}f\cdot P_k(e^{t\Delta}g)}_{\varolessthan},$$
$$S_k=\sum_{j<k}P_j,\quad \left(\begin{array}{l}P_j:\\ S_k:\end{array}\right.$$  $)$.

Let $\mathcal{A}=$ annulus $B_{10/3}\setminus B_{1/2}=\{|\xi|\sim 1\}$. Then
$$\operatorname{supp}\hat{h}_k\subset c\,2^k\mathcal{A}.$$

We show

$$\left\| 2^{k\gamma} \|h_k\|_{L^p_x} \right\|_{\ell^\infty_k} \lesssim t^{\frac{\alpha+\beta-\gamma}{2}} \|\nabla f\|_{B^{\alpha-1}_{p_1}} \|g\|_{B^{\beta}_{p_2}}.$$

Let $\phi \in C_c^\infty$ with $\operatorname{supp}\phi \subset \tilde{A}$, so that $\phi \equiv 1$ on A (annulus).

Set

$$G_{k,t} = \mathcal{F}^{-1}\left\{ \phi\left(\tfrac{\cdot}{2^k}\right) e^{-t|\cdot|^2} \right\}.$$

Thus if $\operatorname{supp}\hat{h} \subset 2^k A$, then

$$e^{t\Delta} h = G_{k,t} * h = \mathcal{F}^{-1}\left\{ \phi\left(\tfrac{\xi}{2^k}\right) e^{-t|\xi|^2} \hat{h}(\xi) \right\}$$

$$\Rightarrow h_k(x) = G_{k,t} * (S_{k-1}f \cdot P_k g) - S_{k-1}f \, (G_{k,t} * P_k g)$$

$$= -\int G_{k,t}(y) (P_k g)(x-y) \underbrace{\left[(S_{k-1}f)(x) - (S_{k-1}f)(x-y) \right]}_{\text{Mean value Thm: } = -\int_0^1 \nabla S_{k-1}f(x-sy)\cdot y \, ds} dy$$

$$= \int_0^1 \int P_k g(x-y) \, \tilde{G}_{k,t}(y) \cdot \nabla S_{k-1} f(x-sy) \, dy \, ds$$

$$= \int_0^1 h_{k,s}(x) \, ds,$$

where we have defined $\tilde{G}_{k,t}(y) = y \, G_{k,t}(y)$ and

$$h_{k,s}(x) = \int P_k g(x-y) \, \tilde{G}_{k,t}(y) \cdot \nabla S_{k-1} f(x-sy) \, dy.$$

We prove

$$(**) \quad \|h_{k,s}\|_{L^p} \lesssim \|\tilde{G}_{k,t}\|_{L^1} \|\nabla S_{k-1} f\|_{L^{p_1}} \|P_k g\|_{L^{p_2}}.$$

uniformly in $s \in [0,1]$.

(Minkowski's inequality would then imply $\|h_k\|_{L^p} = \|\int_0^1 h_{k,s} ds\|_{L^p} \le \int_0^1 \|h_{k,s}\|_{L^p} ds$).

Thinking of $|\tilde{G}_{k,t}|$ as $|\tilde{G}_{k,t}|^{\frac{1}{p}+\frac{1}{p'}}$, Hölder's inequality implies

$$|h_{k,s}(x)| \lesssim \|\tilde{G}_{k,t}\|_{L^1}^{1-1/p} \Big(\int |\tilde{G}_{k,t}(y)| |P_k g(x-y)|^p \times |\nabla S_{k-1} f(x-sy)|^p dy \Big)^{1/p}.$$

Taking the p^{th}-power of both sides and integrating in x implies

$$\|h_{k,s}\|_{L^p_x}^p \le \|\tilde{G}_{k,t}\|_{L^1}^{p-1} \iint |\tilde{G}_{k,t}(y)| |P_k g(x-y)|^p |\nabla S_{k-1} f(x-sy)|^p dy\, dx$$

$$\lesssim \|\tilde{G}_{k,t}\|_{L^1}^{p-1} \int |\tilde{G}_{k,t}(y)| \int |P_k g(x-y)|^p |\nabla S_{k-1} f(x-sy)|^p dx\, dy$$

$$\lesssim \|\tilde{G}_{k,t}\|_{L^1}^{p-1} \|\tilde{G}_{k,t}\|_{L^1} \|P_k g\|_{L^{p_2}}^p \|\nabla S_{k-1} f\|_{L^{p_1}}^p,$$

which is (**).

It remains to study $\|\tilde{G}_{k,t}(y)\|_{L^1_y}$.

It suffices to study $y_1 G_{k,t}(y)$. $\quad (y_1, \dots, y_d) = y$.

In the Fourier side,

$$\xi \longmapsto \partial_{\xi_1} \Big(\phi\big(\tfrac{\xi}{2^k}\big) e^{-t|\xi|^2} \Big)$$

$$= \frac{1}{2^k} \Big\{ (\partial_{\xi_1} \phi)\big(\tfrac{\xi}{2^k}\big) - 2\xi_1 t \phi\big(\tfrac{\xi}{2^k}\big) \Big\} e^{-t|\xi|^2}.$$

Recall Lemma' (p.g. 4) of Lecture 2; which for us says

Lemma': For every $\tilde{\phi} \in C^\infty$ with support in A, we have

$$\boxed{\left\| \mathcal{F}^{-1}\left\{ \tilde{\phi}\left(\frac{\cdot}{2^k}\right) e^{-t|\cdot|^2} \right\} \right\|_{L^1_x} \lesssim_{\tilde{\phi}} e^{-ct2^{2k}}.}$$

As supp $\partial_{\xi_1}\phi \subset A$, we apply Lemma' and get

$$\|\widehat{G}_{k,t}\|_{L^1} \lesssim 2^{-k}(1+ \underbrace{t2^{2k}}_{\substack{\text{Lemma}' \Rightarrow t2^{k} \text{ as } \xi_1 \sim 2^k \\ = 2^{-k}(t2^{2k})}})e^{-ct2^{2k}}$$

Using $\|P_k g\|_{L^{p_2}} \lesssim 2^{-k\beta}\|g\|_{B^{\beta}_{p_2}}$, and using that $\alpha<1$, implies

$$\begin{aligned}
\|\nabla S_{k-1} f\|_{L^{p_1}} &\leq \sum_{j\leq k-1} \|P_j(\nabla f)\|_{L^{p_1}} \\
&= \sum_{j\leq k-1} 2^{j(1-\alpha)} 2^{j(\alpha-1)} \|P_j(\nabla f)\|_{L^{p_1}} \\
&\leq \Big(\sum_{j\leq k-1} 2^{j(1-\alpha)}\Big) \sup_j 2^{j(\alpha-1)} \|P_j(\nabla f)\|_{L^{p_1}} \\
&\lesssim 2^{k(1-\alpha)} \|\nabla f\|_{B^{\alpha-1}_{p_1}},
\end{aligned}$$

we have

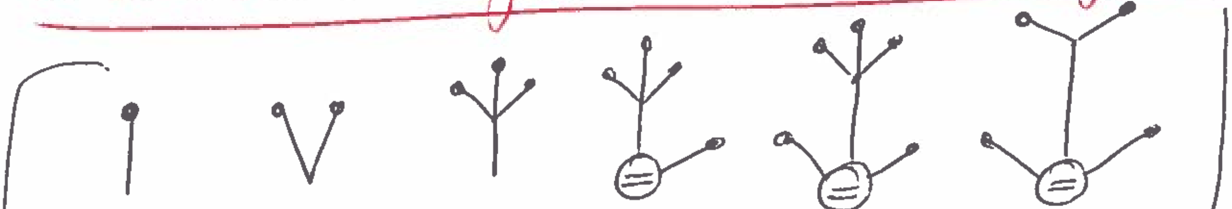
$$\begin{aligned}
2^{k\gamma}\|h_{k,s}\|_{L^p} &\lesssim 2^{k(\gamma-\alpha-\beta)}(1+(t2^{2k}))e^{-ct2^{2k}} \|\nabla f\|_{B^{\alpha-1}_{p_1}} \|g\|_{B^{\beta}_{p_2}} \\
&\lesssim t^{\frac{\alpha+\beta-\gamma}{2}} \left[(t2^{2k})^{\frac{\gamma-\alpha-\beta}{2}} (1+t2^{2k}) e^{-ct2^{2k}} \right] \|\nabla f\|_{B^{\alpha-1}_{p_1}} \|g\|_{B^{\beta}_{p_2}},
\end{aligned}$$

Noticing that since $\gamma \geq \alpha+\beta$,

$$x^{\frac{\gamma-\alpha-\beta}{2}}(1+x)e^{-cx} \lesssim 1 \quad \text{for } x \geq 0,$$

completes the proof. $\square$.

Construction of the Stochastic Objects



$-\frac{1}{2}-$ $\quad -1- \quad$ $\frac{1}{2}- \quad$ $0- \quad$ $-\frac{1}{2}- \quad$ $0-$.

We will show how these diagrams are constructed and prove their stated regularities.

The discussion will be based on

Mourrat-Weber-Xu, "Construction of Φ^4_3 diagrams for pedestrians"

Unpublished lecture notes: arXiv 1610.08897v2 or Weber's website.

Proposition 3.6: Let $\tau : \mathbb{R}_t \to \mathcal{S}'_x(\mathbb{T}^d)$ „$C^\infty_x(\mathbb{T}^d)$" be a random process in

$$\mathcal{H}_{\leq K} = \bigoplus_{j=0}^{K} \mathcal{H}_j \to \text{Homogeneous Wiener Chaoses of degree } j.$$

Assume that τ is stationary in space, i.e. for every $x \in \mathbb{T}^d$ the processes $(\tau(t,\cdot))_{t\in\mathbb{R}}$ and $(\tau(t,x+\cdot))_{t\in\mathbb{R}}$ have the same law.

① Suppose that for some $t \in \mathbb{R}$, $\alpha \in \mathbb{R}$, that for every ⑥
$n \in \mathbb{Z}^d$, we have

$$\mathbb{E}\left[|\hat{\tau}(t,n)|^2\right] \lesssim \langle n \rangle^{-d-2\alpha}. \qquad \text{(3.13)}$$

(Spacial Fourier coefficient frequency n.)

Then we have $\tau(t) \in C_x^\beta(\mathbb{T}^d)$, $\beta < \alpha$, with

$$\mathbb{E}\left[\|\tau(t)\|^p_{C_x^\beta}\right] < \infty. \qquad \text{(3.14)} \quad (1 \le p < \infty).$$

② In addition to ①, if we also have

$$\mathbb{E}\left[|\hat{\tau}(t,n) - \hat{\tau}(s,n)|^2\right] \lesssim \langle n \rangle^{-d-2\alpha+2\lambda} |t-s|^\lambda \qquad \text{(3.15)}$$

for some $\lambda \in (0,1)$ and uniformly for $0 < |t-s| < 1$, $n \in \mathbb{Z}^d$, then

$$\sup_{0<|t-s|<1} \frac{\mathbb{E}\left[\|\tau(t) - \tau(s)\|^p_{C_x^\beta}\right]}{|t-s|^{\lambda p/2}} < \infty, \qquad \text{(3.16)}$$

for every $\beta < \alpha - \lambda$. In particular, $\tau \in C(\mathbb{R}; C^\beta(\mathbb{T}^d))$.

The point of this lemma is that for τ one of the processes on the previous page, we can find its regularity by just verifying (3-13).
That is, for fixed $t \in \mathbb{R}$, we need only compute moments of the Fourier transform $\hat{\tau}(t,n)$ and show it decays

sufficiently.

Contrast this to the method we used in Lecture 3 (following the approach of GKO) to compute regularities of Wick powers of the stochastic convolution. There we computed the moments of the norms "by hand" on the physical side.

Rmk: ⚠: In (3.15), the coefficient of λ (i.e. 2) depends on the scaling of the underlying equation and hence on the equation you're working with (because the stochastic objects τ you create do).

Proof of Prop^{n} 3.6:

Claim: For every $s,t\in\mathbb{R}$ and $n,n'\in\mathbb{Z}^d$,

$$\mathbb{E}[\hat{\tau}(s,n)\,\hat{\tau}(t,n')] = 0,$$

unless $n+n'=0$.

Pf of claim: We have

$$\mathbb{E}[\hat{\tau}(s,n)\,\hat{\tau}(t,n')] = \iint_{\mathbb{T}^d\times\mathbb{T}^d} \mathbb{E}[\tau(s,x)\tau(t,y)]\, e^{-2\pi i(n\cdot x + n'\cdot y)}\,dx\,dy$$

$$= \iint_{\mathbb{T}^d\times\mathbb{T}^d} \mathbb{E}[\tau(s,x)\tau(t,y)]\, e^{-2\pi i(n+n')\cdot x} \times e^{-2\pi i n'\cdot(y-x)}\,dx\,dy.$$

By the stationarity assumption, is just a function of the difference $(y-x)$, that is

$$\mathbb{E}[\tau(s,x)\tau(t,y)] = F_{s,t}(y-x),$$

⑧

for some function $F_{s,t}$.

$$\Rightarrow \mathbb{E}[\hat{\tau}(s,n)\hat{\tau}(t,n')] = \int_{\mathbb{T}^d} e^{-2\pi i(n+n')\cdot x} \int_{\mathbb{T}^d} F_{s,t}(y-x) e^{-2\pi i n'\cdot(y-x)} \, dy \, dx$$

$$= \hat{F}_{s,t}(n') \int_{\mathbb{T}^d} e^{-2\pi i(n+n')\cdot x} \, dx$$

$$= 0$$

unless $n+n'=0$.

We will only prove ② as the proof of ① is similar.

Let $\tau_{s,t} := \tau(t) - \tau(s)$ and define

$$P_K \tau_{s,t}(x) = \sum_n \varphi_K(n) \hat{\tau}_{s,t}(n) e^{2\pi i n\cdot x}$$

where $\varphi_K(n) = \varphi\left(\frac{n}{2^K}\right)$ is smooth and even.

As τ is real-valued, which implies

$$\hat{\tau}(t,-n) = \overline{\hat{\tau}(t,n)},$$

φ is even and using the claim, we have

$$\mathbb{E}[|P_K \tau_{s,t}(x)|^2] = \sum_{n,n'\in\mathbb{Z}^d} \varphi_K(n)\varphi_K(n') \mathbb{E}[\hat{\tau}_{s,t}(n)\hat{\tau}_{s,t}(n')] \times e^{2\pi i(n+n')\cdot x}$$

$$= \sum_{n\in\mathbb{Z}^d} (\varphi_K(n))^2 \mathbb{E}[|\hat{\tau}_{s,t}(n)|^2].$$

by (3.15),

$$\lesssim \sum_{n\in\mathbb{Z}^d} \underbrace{(\varphi_k(n))^2}_{\substack{\text{supported}\\ \text{on } |n|\sim 2^k}} \langle n\rangle^{-d-2\alpha+2\lambda} |t-s|^\lambda$$

$$\lesssim 2^{k(-2\alpha+2\lambda)}\, 2^{-dk}\Big(\sum_{|n|\sim 2^k} 1\Big)\, |t-s|^\lambda$$

$$\lesssim 2^{k(-2\alpha+2\lambda)} |t-s|^\lambda.$$

By the Wiener Chaos estimate (corollary on pg. 7 of lecture 2)

$$\mathbb{E}[\|P_k \tau_{s,t}\|^p_{L^p_x}] \le \sup_{x\in\mathbb{T}^d} \mathbb{E}[|P_k \tau_{s,t}(x)|^p]$$

$$\lesssim \sup_{x\in\mathbb{T}^d} \left(\mathbb{E}[|P_k\tau_{s,t}(x)|^2]\right)^{p/2}$$

$$\lesssim |t-s|^{\frac{\lambda p}{2}}\, 2^{-k(\alpha-\lambda)p}.$$

We now use the following proposition.

Proposition 2.6: Let $\beta<\alpha-d/p$. There exists $C<\infty$ s.t. for every random distribution f on $\mathbb{T}^d$, we have

$$\mathbb{E}[\|f\|^p_{C^\beta}] \le C \sup_{k\ge 0} 2^{\alpha k p}\, \mathbb{E}[\|P_k f\|^p_{L^p}].$$

Therefore

$$\mathbb{E}[\|\tau_{s,t}\|^p_{C^\beta}] \underset{\substack{\downarrow\\ \gamma=\alpha-\lambda}}{\overset{\text{(Prop}^n\ 2.6)}{\lesssim}} \sup_{k\ge 0} 2^{(\alpha-\lambda)kp}\, \mathbb{E}[\|P_k\tau_{s,t}\|^p_{L^p}] \text{ , provided } \beta<\alpha-\lambda-\frac{d}{p}$$

$$\lesssim \sup_k 2^{kp(\alpha-\lambda-(\alpha-\lambda))}|t-s|^{\frac{\lambda p}{2}}$$

$$\sim |t-s|^{\frac{\lambda p}{2}}.$$

Taking p large, we get (3-16) for $\beta<\alpha-\lambda$. $\square$

Lecture 8, 4/4/18 (1)

We begin by proving Proposition 2-6 which was used last lecture in the proof of proposition 3-6.

Prop^{n} 2-6: Let $\beta < \alpha - d/p$. Then

$$\mathbb{E}[\|f\|^p_{C^\beta}] \leq C \sup_{k\geq 0} 2^{\alpha k p} \mathbb{E}[\|P_k f\|^p_{L^p}].$$

Proof: $\|f\|^p_{C^\beta} = \sup_{k\geq 0} 2^{\beta k p} \|P_k f\|^p_{L^\infty_x}$

$C^\beta = B^\beta_{\infty,\infty}$

Bernstein/Sobolev $\alpha-\beta \geq d/p$.

$$\lesssim \sup_{k\geq 0} 2^{(\beta p+d)k} \|P_k f\|^p_{L^p}$$

$$= C \sup_{k\geq 0} 2^{\underbrace{(\beta p+d-\alpha p)k}_{\leq 0}} 2^{\alpha k p} \|P_k f\|^p_{L^p}.$$

If $\alpha-\beta = \frac{d}{p}$, then taking expectations would give only

$$\mathbb{E}[\|f\|^p_{C^\beta}] \leq C \,\mathbb{E}[\sup_{k\geq 0} 2^{\alpha k p} \|P_k f\|^p_{L^p}].$$

We cannot switch the $\mathbb{E}$ with the $\sup_{k\geq 0}$!

If $\alpha-\beta > d/p$, then we can make use of the dyadics by replacing the $\sup_{k\geq 0}$ with ℓ^1_k, i.e.

$$\mathbb{E}[\|f\|^p_{C^\beta}] \leq C \sum_{k\geq 0} 2^{k\underbrace{(\beta p-\alpha p+d)}_{<0}} 2^{\alpha k p} \mathbb{E}[\|P_k f\|^p_{L^p}]$$

$$\leq \tilde{C} \sup_{k\geq 0} 2^{\alpha k p} \mathbb{E}(\|P_k f\|^p_{L^p})$$

$\square$

Construction of the Stochastic objects cont.

Note: In order to remain consistent with the presentation in Mourrat-Weber-Xu, we work with the rescaled Brownian motion: $\{\beta(\cdot,n)\}_{n\in\mathbb{Z}^3}$ complex-valued Brownian motions, independent but conditioned so that

$\beta(t,n) = \overline{\beta(t,-n)}$ and with variance (co-)

$$\mathbb{E}[\beta(t,n)\beta(t,n')] = \begin{cases} |t| & , \text{if } n+n'=0 \\ 0 & , \text{otherwise} \end{cases}$$

Previously, had a factor of 2.

$\tau = \mathbf{!}$: $\quad (\partial_t - \Delta + 1)\mathbf{!} = \xi$ (Had $(\partial_t-\Delta)\mathbf{!}=\xi$ '+1' for convenience)

$$\Rightarrow \widehat{\mathbf{!}}(t,n) = \int_{-\infty}^{t} S_{t-t'}(n)\, d\beta(t',n)$$

where

- $$S_t(n) = \begin{cases} e^{-t\langle n\rangle^2}, & t\geq 0 \\ 0, & t<0. \end{cases}$$ → Extend backwards by zero.

- Starting at $t=-\infty$, also for convenience. If we show $\int_{-\infty}^{t}\cdot \in C^\beta$ for any t (and some β), then clearly $\int_0^t = \int_{-\infty}^{t} - \int_{-\infty}^{0} \in C^\beta$ as well.

Graphical version:

$\widehat{\Psi}(t,n)$ =

(t,n).

● (t,n) : Represents the time t and frequency n we are evaluating at (Base node)

○ : An instance of the white noise $d\beta(t',n)$, where t' is a dummy variable.

↑ : Represents $\int_{-\infty}^{t} S_{t-t'}(n)$; the "Duhamel part"

This graphical notation at a fixed time and frequency will be useful for computations involving more complicated stochastic objects.

→ We want to check condition ① of Prop^n 3-6.

To this end, we study $\mathbb{E}[\widehat{\Psi}(t,n)\underbrace{\widehat{\Psi}(t',-n)}_{\overline{\widehat{\Psi}(t,n)}}]$, $t,t'\in\mathbb{R}$.

By properties of Wiener integrals,

$$\mathbb{E}[\widehat{\Psi}(t,n)\widehat{\Psi}(t',-n)] = \int_{\mathbb{R}} S_{t-u}(n)S_{t'-u}(n)\,du$$

$$= e^{-(t+t')\langle n\rangle^2}\int_{(-\infty,t]\cap(-\infty,t']} e^{2u\langle n\rangle^2}\,du.$$

$$\left(\underline{\text{If } t\geq t'}\right) \quad = e^{-(t+t')\langle n\rangle^2}\int_{-\infty}^{t'} e^{2u\langle n\rangle^2}\,du$$

$$= \frac{e^{-(t-t')\langle n\rangle^2}}{2\langle n\rangle^2}.$$

So in general,

$$\mathbb{E}[\hat{\Psi}(t,n)\hat{\Psi}(t',-n)] = \frac{e^{-|t-t'|\langle n\rangle^2}}{2\langle n\rangle^2} \leq \langle n\rangle^{-3+1}$$

$d=3$ $\quad -2\alpha=1$.

$$\Rightarrow \Psi(t) \in C_x^{-\frac{1}{2}-}.$$

We could also compute the temporal regularity but we have seen this many times before (e.g. lecture 2 where we use the mean value theorem).

We also have (using the exponential decay)

$$\mathbb{E}[\hat{\Psi}(t,n)\overline{\hat{\Psi}(t',n)}] \lesssim \frac{1}{\langle n\rangle^2}\left(\frac{1}{|t-t'|\langle n\rangle^2}\right)^{\gamma} \qquad (4.23)$$

for all $\gamma \geq 0$ and $t \neq t'$.

$\tau = V$: Recall $V_N = (\Psi_N)^2 - C_N$ → Frequency cutoff.

$$\widehat{\Psi_N^2}(t,n) = \sum_{\substack{n=n_1+n_2 \\ |n_j|\leq N}} \hat{\Psi}(t,n_1)\hat{\Psi}(t,n_2)$$

$$= \sum_{\substack{n=n_1+n_2 \\ |n_j| \le N}}' \underbrace{\left(\int_{-\infty}^{t} S_{t-u_1}(n_1)\, d\beta(u_1,n_1)\right)}_{=:X_1} \underbrace{\left(\int_{-\infty}^{t} S_{t-u_2}(n_2)\, d\beta(u_2,n_2)\right)}_{=:X_2.}.$$

Define $F = X_1 X_2$. Then by Itô's lemma,

$$dF = X_1\, dX_2 + X_2\, dX_1 + \underbrace{dX_1\, dX_2}_{\text{symmetry, } \frac{2}{2}=1.}$$

$$\left(\text{"} dX_j(t) = S_{t-u_j}(n_j)\, d\beta(u_j,n_j). \text{"} \right)$$

$$= 2X_2\, dX_1 + dX_1\, dX_2 \quad \text{(by symmetry).}$$

Integrating and using

$$d\beta(n_1)\, d\beta(n_2) = [n_1+n_2=0]\, du,$$

gives $(F(0)=0)$

$$F(t) = \left(\int_{-\infty}^{t} S_{t-u_1}(n_1)\, d\beta(u_1,n_1)\right)\left(\int_{-\infty}^{t} S_{t-u_2}(n_2)\, d\beta(u_2,n_2)\right).$$

$$= 2\int_{-\infty}^{t} \left[\int_{-\infty}^{u_1} S_{t-u_1}(n_1) S_{t-u_2}(n_2)\, d\beta(u_2,n_2)\right] d\beta(u_1,n_1)$$

$$+ \mathbb{1}_{\{n_1=-n_2\}} \int_{-\infty}^{t} S_{t-u}(n_1) S_{t-u}(n_2)\, du.$$

where the iterated integral above is to be understood as as an <u>iterated Wiener-Itô integral</u> (e.g. see Kuo, "Introduction to Stochastic Integration", Chapter 9).

Recalling that we extended $S_t(\cdot)$ by zero when $t<0$, we have

$$\widehat{\rho}^{\,2}_N(t,n) = \sum_{\substack{n=n_1+n_2\\ |n_j|\le N}} \left\{ \int_{\mathbb{R}^2} S_{t-u_1}(n_1) S_{t-u_2}(n_2)\, d\beta(u_2,n_2)\, d\beta(u_1,n_1) \right.$$

$$(*) \qquad \left. + \; \mathbb{1}_{\{n_1+n_2=0\}} \int_{\mathbb{R}} |S_{t-u}(n_1)|^2\, du \right\}.$$

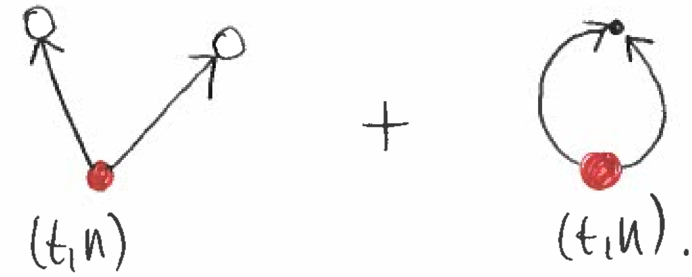
Notice that the second term is

$$\sum_{|n_1|\le N} \int_{-\infty}^{t} |S_{t-u}(n_1)|^2 du = \sum_{|n_1|\le N} \frac{1}{2\langle n_1\rangle^2} \sim N$$

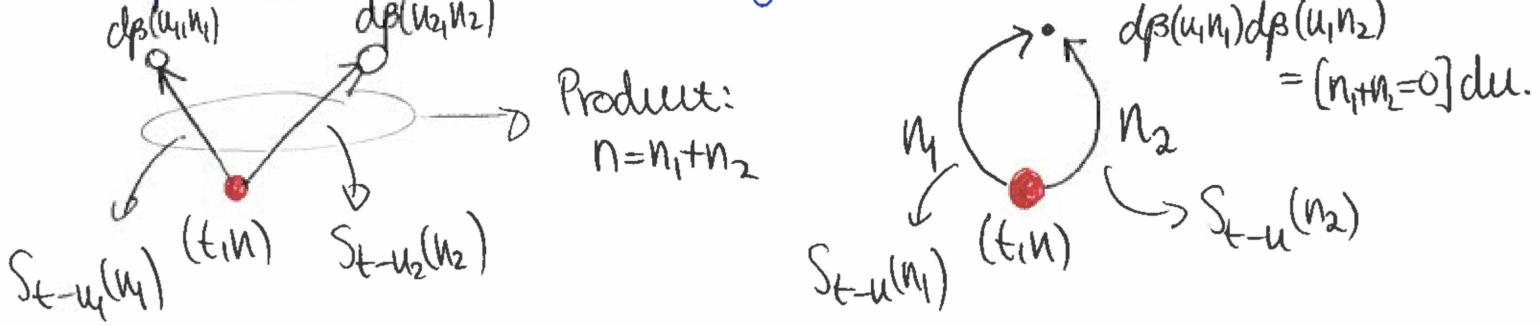
$$\text{i.e.} \Rightarrow \underline{C_N := \mathbb{E}[\rho_N^2(t)]} \sim N$$

In the graphical notation $(*)$ can be represented as:

$$\widehat{\rho}^{\,2}_N(t,n) =$$



• : Represents the "contracting" process $n_1+n_2=0$.



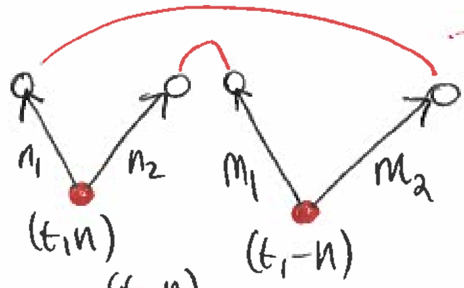
So

$$\widehat{V}(t,n) := \sum_{n=n_1+n_2} 2 \int_{-\infty}^{t} \left[\int_{-\infty}^{u_1} S_{t-u_1}(n_1) S_{t-u_2}(n_2)\, d\beta(u_2,n_2) \right] d\beta(u_1,n_1).$$

Dropped restriction.

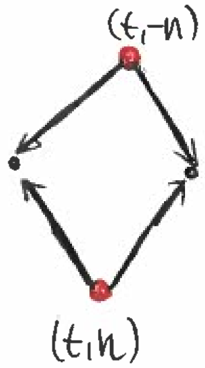
and

$$\mathbb{E}\left[|\widehat{V}(t,n)|^2 \right] = \mathbb{E}\left[V(t,n)\, V(t,-n) \right]$$

$$= \mathbb{E}\Big[$$ 

$$\Big].$$

Pairings of frequencies.

$$= 2 \times$$ 

2! ways to pair i.e. # of ways to perform a contraction

e.g. $\begin{pmatrix} n_1+m_1=0 \\ n_2+m_2=0 \end{pmatrix}, \begin{pmatrix} n_1+m_2=0 \\ n_2+m_1=0 \end{pmatrix}$.

$$= 2 \sum_{n=n_1+n_2} \int_{\mathbb{R}^2} |S_{t-u_1}(n_1)|^2 |S_{t-u_2}(n_2)|^2 \, du_1\, du_2$$

$$= 2 \sum_{n=n_1+n_2} \frac{1}{2\langle n_1\rangle^2} \frac{1}{2\langle n_2\rangle^2}$$

$$\lesssim \frac{1}{\langle n\rangle} = \langle n\rangle^{-3-2(-1)} \Rightarrow V(t) \in C_x^{-1-}.$$

Lemma 4.1

Lemma 4-1: Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy (8)

$$\alpha + \beta > d \quad \text{and} \quad \alpha, \beta < d.$$

Then

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}^d \\ n = n_1 + n_2}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \frac{1}{\langle n \rangle^{\alpha+\beta-d}}.$$

Lemma 4.2: As in Lemma 4-1 but instead suppose only $\alpha + \beta > d$.

Then

(Resonant case)
$$\sum_{\substack{n = n_1 + n_2 \\ |n_1| \sim |n_2|}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \frac{1}{\langle n \rangle^{\alpha+\beta-d}}.$$

$\tau = \Psi$: Recall $\Psi_N = (\partial_t - \Delta + 1)^{-1}\left((\Psi_N)^3 - 3C_N \Psi_N \right).$

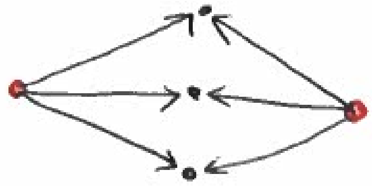
As in the previous case, we have

$$\widehat{\Psi_N^3}(t,n) = \sum_{\substack{n = n_1 + n_2 + n_3 \\ |n_j| \leq N}} 6 \int_{-\infty}^{t} \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} S_{t-u_1}(n_1) S_{t-u_2}(n_2) S_{t-u_3}(n_3) \times d\beta(u_3, n_3)\, d\beta(u_2, n_2)\, d\beta(u_1, n_1)$$

$$+ \; 3 C_N \widehat{\Psi_N}(t,n)$$

In this case, we use Itô-lemma with $F = X_1 X_2 X_3$

$$\Rightarrow dF = X_1 X_2 dX_3 + X_1 X_3 dX_2 + X_2 X_3 dX_1 \rightsquigarrow \text{"}6 X_3 dX_2 dX_1\text{"}$$
$$+ X_3 dX_1 dX_2 + X_1 dX_2 dX_3 + X_2 dX_1 dX_3 \rightsquigarrow \text{"}3 X_3 dX_1 dX_2\text{"}$$

$$\Rightarrow \mathbb{E}\left[\left|\widehat{\vee}(t,n)\right|^2\right] = \overset{\overset{6}{\shortparallel}}{3!} \times (t,n)$$  $(t,-n)$

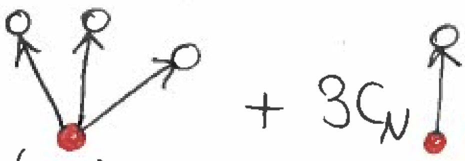
$$= 6 \underbrace{\sum_{n=n_1+n_2+n_3} \prod_{j=1}^{3}}_{} \frac{1}{2\langle n_j\rangle^2}$$

Two sums, apply Lemma 4.1 iteratively

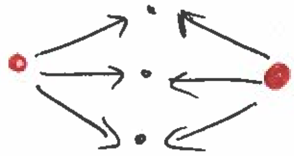
$$\lesssim 6 \sum_{n_1} \frac{1}{\langle n_1\rangle^2 \langle n_1 - n\rangle^1} \longrightarrow 1+2=3!$$

$\Downarrow$

Diverges logarithmically!

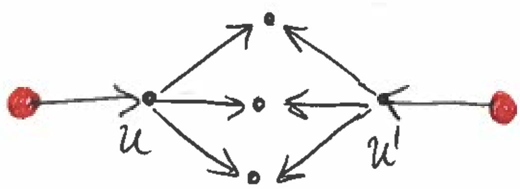
$$\left(\widehat{I_N^3}(t,n) = \right.$$  $$\left. + \; 3C_N \right).$$

Consider, for $t \neq t'$,

$$\mathbb{E}\left[\widehat{\vee}(t,n)\,\widehat{\vee}(t',-n)\right] = 6 \times (t,n)$$  $(t',-n)$

$$\underset{\text{by (4.23)}}{\lesssim} \frac{1}{|t-t'|^\gamma} \sum_{n=n_1+n_2+n_3} \frac{1}{\langle n_1\rangle^{2+2\gamma}} \frac{1}{\langle n_2\rangle^2 \langle n_3\rangle^2}.$$

$$\underset{\text{Lemma 4.1}}{\lesssim} \frac{1}{|t-t'|^\gamma} \frac{1}{\langle n\rangle^{2\gamma}}.$$

$$\Rightarrow \mathbb{E}\left[\left|\widehat{\Upsilon}(t,n)\right|^2\right] = 6 \times (t,n)$$  $(t,-n)$

$$\int_0^t S_{t-u}(n)\,\widehat{\vee}(u,n)\,du.$$

$$\lesssim \int_{\mathbb{R}^2} S_{t-u}(x)\, S_{t-u'}(x) \frac{1}{|u-u'|^{\gamma}} \frac{1}{\langle x\rangle^{2\gamma}}\, du\, du'$$

$$\lesssim \frac{1}{\langle x\rangle^{2\gamma}} \underbrace{\|S_{t-\cdot}(x)\|_{L^2}}_{\sim \left(\frac{1}{\langle x\rangle^{2}}\right)^{1/2} \sim \frac{1}{\langle x\rangle}} \left\| S_{t-\cdot} * \frac{1}{|\cdot|^{\gamma}} \right\|_{L^2_u}. \to \text{1-dim integral}$$

$$\lesssim \frac{1}{\langle x\rangle^{1+2\gamma}} \left\| S_{t-\cdot} * \frac{1}{|\cdot|^{\gamma}} \right\|_{L^2_u}$$

Hardy-Littlewood-Sobolev inequality (in $d=1$)(HLS)

$$1+\frac{1}{2} = \frac{1}{1/\gamma} + \frac{1}{q}$$

$$\Rightarrow \frac{1}{q} = \frac{3}{2} - \gamma.$$

Chose $\gamma<1$ for (HLS).

$$\lesssim \frac{1}{\langle x\rangle^{1+2\gamma}} \|S_{t-u}\|_{L^q_u}$$

$$\lesssim \frac{1}{\langle x\rangle^{1+2\gamma}} \frac{1}{\langle x\rangle^{3-2\gamma}} = \langle x\rangle^{-3-2(\frac{1}{2})}.$$

$$\Rightarrow \Psi(t) \in C_x^{\frac{1}{2}-}.$$

Lecture 9, 10/4/18

Back to iterated Wiener-Ito integrals and Wiener chaoses

Space-time white noise ξ: $\xi(\varphi) := \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}} \hat{\varphi}(t,n)\, d\beta(t,n)$,

where $\varphi \in L^2(\mathbb{R}\times\mathbb{T}^d)$.

Informally, we write

$$\xi(\varphi) = \int_{\mathbb{R}\times\mathbb{T}^d} \varphi(z)\,\xi(dz), \quad z = (t,x) \in \mathbb{R}\times\mathbb{T}^d.$$

For each $k \geq 1$, and $\varphi \in L^2((\mathbb{R}\times\mathbb{T}^d)^k)$, we denote the Iterated Wiener-Ito integral by:

$$(*) \quad \ldots \quad \xi^{\otimes k}(\varphi) = \int_{(\mathbb{R}\times\mathbb{T}^d)^k} \varphi(z_1,\ldots,z_k)\,\xi(dz_1)\cdots\xi(dz_k).$$

Let $\tilde{\varphi}$ denote the symmetrization of φ, i.e.

$$\tilde{\varphi}(z_1,\ldots,z_k) = \frac{1}{k!}\sum_{\sigma\in S_k} \varphi(z_{\sigma(1)},\ldots,z_{\sigma(k)}),$$

where S_k is the permutation group on $\{1,\ldots,k\}$.
Then $(*)$ can be realised as an "iterated" integral since

$$\xi^{\otimes k}(\varphi) = \xi^{\otimes k}(\tilde{\varphi}) = \frac{1}{k!}\int_{(\mathbb{R}\times\mathbb{T}^d)^k} \tilde{\varphi}(z_1,\ldots,z_k)\,\mathbb{1}_{\{t_1<\cdots<t_k\}}\,\xi(dz_1)\cdots\xi(dz_k).$$

and we have the isometry property ②

$$\mathbb{E}\left[|\xi^{\otimes k}(\varphi)|^2\right] = \mathbb{E}\left[|\xi^{\otimes k}(\tilde{\varphi})|^2\right]$$

isometry property. $\rightarrow = k! \int_{(\mathbb{R}\times\mathbb{T}^d)^k} |\tilde{\varphi}(z_1,\dots,z_k)|^2 dz_1\cdots dz_k.$

Now by Jensen's inequality,

$$|\tilde{\varphi}|^2 = \left(\frac{1}{k!}\sum_{\sigma\in S_k}\varphi(\vec{z}_{(\sigma)})\right)^2$$

$\left(\frac{1}{k!}\sum_{S_k}$ is a normalized counting measure$\right)$ $\rightarrow$ $\leq \frac{1}{k!}\sum_{\sigma\in S_k}|\varphi(\vec{z}_{(\sigma)})|^2$

$$\Rightarrow \mathbb{E}\left[|\xi^{\otimes k}(\varphi)|^2\right] \leq k! \int_{(\mathbb{R}\times\mathbb{T}^d)^k} |\varphi(z_1,\dots,z_k)|^2 dz_1\cdots dz_k.$$

Recall the Wiener chaos of order k:

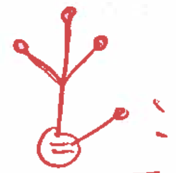
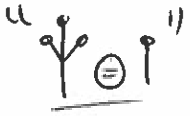
$$\begin{cases} \mathcal{H}_k := \left\{\xi^{\otimes k}(\varphi) : \varphi\in L^2((\mathbb{R}\times\mathbb{T}^d)^k)\right\}. \\ \mathcal{H}_0 = \mathbb{R} \end{cases}$$

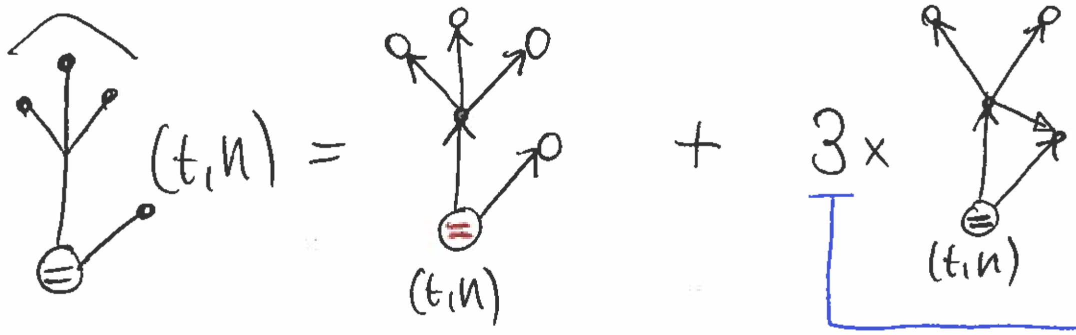
Then (Lemma 3.2 in Mourrat-Weber-Xu)

$$\mathcal{H}_{\leq n} := \bigoplus_{k=0}^{\infty}\mathcal{H}_k$$

$$= \overline{\operatorname{span}\left\{\xi(\varphi_1)\cdots\xi(\varphi_k) : \varphi_1,\dots,\varphi_k\in L^2(\mathbb{R}\times\mathbb{T}^d)\right\}}^{L^2(\Omega)}$$

The point to be made here is that for appropriate φ, we can decompose the Fourier transform of each of our stochastic objects τ in terms of Wiener chaoses. Then by orthogonality of $\mathcal{H}_n$ in L^2, computing $\mathbb{E}[\,|\hat{\tau}(t,n)|^2]$ is equivalent to computing the variance of each projection of $\hat{\tau}(t,n)$ onto $\mathcal{H}_k$ and summing these.

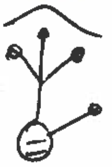
$\tau =$  : "  "



3 ways to perform a contraction with V & /.

$$=: I(t,n) + II(t,n)$$
$$\quad\;\; \cap \qquad\quad\; \cap$$
$$\quad\;\; \mathcal{H}_4 \qquad\;\; \mathcal{H}_2.$$

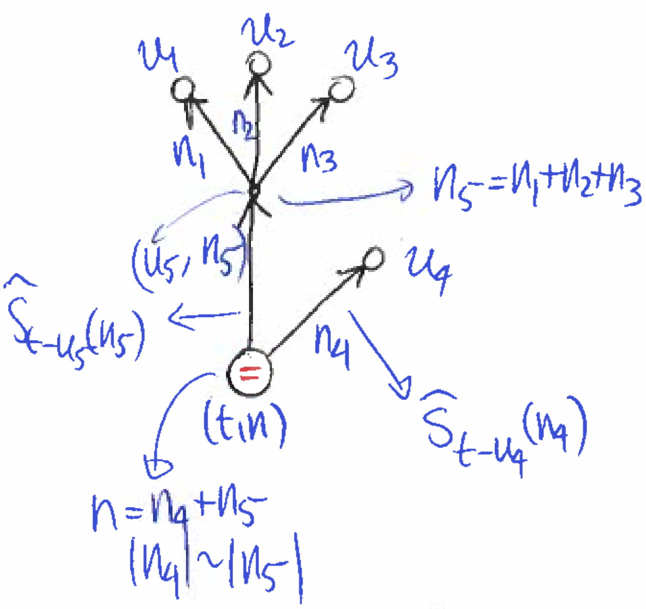
By orthogonality of $\mathcal{H}_k$'s in L^2,

$\mathbb{E}\Big[\,\Big|$  $(t,n)\Big|^2\Big] = \mathbb{E}\big[|I(t,n)|^2\big] + \mathbb{E}\big[|II(t,n)|^2\big].$

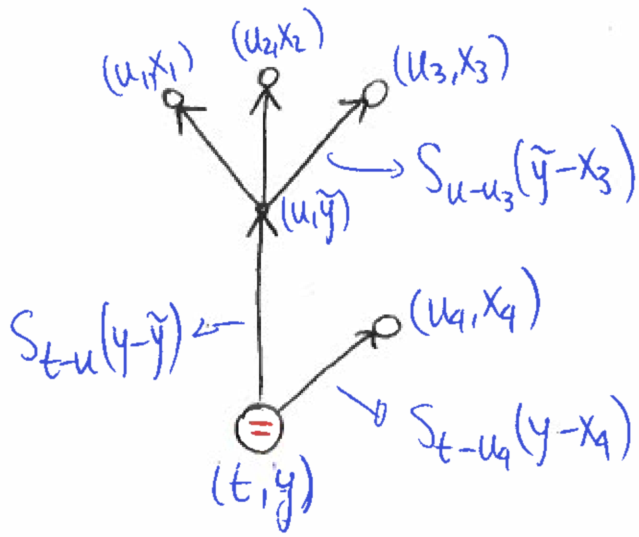
Rmk: Notice the following diagrams do not appear

These diverging diagrams are forbidden because we are working with  .

$I(t,n)$: We can write $\mathbb{E}[|I(t,n)|^2] = \mathbb{E}[|\xi^{\otimes 4}(\varphi)|^2]$ for some appropriate $\varphi \in L^2((\mathbb{R}\times\mathbb{T}^3)^4)$.



Frequency side



Physical side

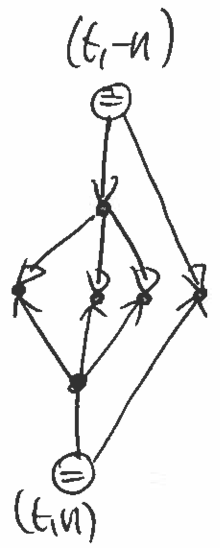
From these diagrams, we can write

$$
\begin{aligned}
I(t,n) = \sum_{\substack{n=n_4+n_5\\ |n_4|\sim|n_5|}} \Bigg[\int_0^t \hat{S}_{t-u_5}(n_5) \sum_{n_5=n_1+n_2+n_3} 6 \int_{-\infty}^{u_5} \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} \hat{S}_{u_5-u_1}(n_1) \\
\times \hat{S}_{u_5-u_2}(n_2)\, \hat{S}_{u_5-u_3}(n_3)\, d\beta(u_3,n_3)\, d\beta(u_2,n_2)\, d\beta(u_1,n_1) \\
\times du_5 \Bigg] \\
\times \int_0^t \hat{S}_{t-u_4}(n_4)\, d\beta(u_4,n_4).
\end{aligned}
$$

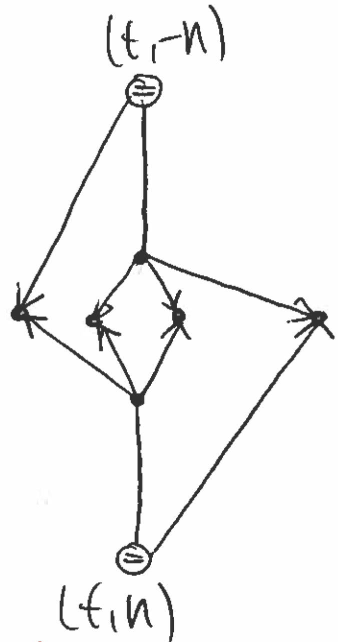
$$= \int_{(\mathbb{R}\times\mathbb{T}^3)^4} \Big[\int_{y\in\mathbb{T}^3} \int_{\mathbb{R}\times\mathbb{T}^3} S_{t-u}(y-\tilde{y})\, S_{u-u_1}(\tilde{y}-x_1)\, S_{u-u_2}(\tilde{y}-x_2) \times S_{u-u_3}(\tilde{y}-x_3)\, du\, d\tilde{y} \ominus S_{t-u_4}(y-x_4)\, e^{-2\pi i n\cdot y} \times dy \Big]\, \xi(du_1,dx_1)\,\xi(du_2,dx_2)\,\xi(du_3,dx_3)\,\xi(du_4,dx_4).$$

Duhamel integral operator

$=: \varphi$

$\mathbb{E}\big[|I(t,n)|^2\big] = 6 \times$ 

(Tree 1)

$+ \; 3\cdot 3! \times$ 

(Tree 2)

Claim: (Tree 2) $\lesssim$ (Tree 1)

To see this, note that Tree 1 is of the form

$$\int_{(\mathbb{R}\times\mathbb{T}^3)^4} \varphi(x_1,x_2,x_3,x_4)\,\overline{\varphi(x_1,x_2,x_3,x_4)}$$

$$= \sum_{n=n_1+n_2+n_3+n_4} \hat{\varphi}(n_1,n_2,n_3,n_4)\,\overline{\hat{\varphi}(n_1,n_2,n_3,n_4)}.$$

$$= \sum_{n=n_1+n_2+n_3+n_4} |\hat{\varphi}(n_1,n_2,n_3,n_4)|^2.$$

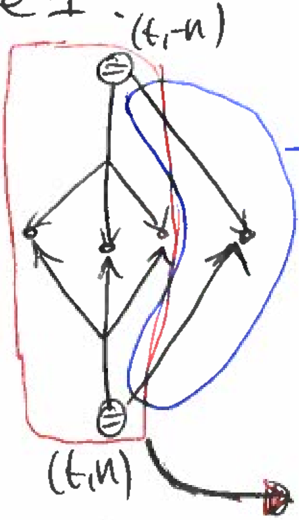
whilst (a version of) tree 2 is of the form

$$\sum_{n=n_1+n_2+n_3+n_4} \overline{\widehat{\varphi}(n_1,n_2,n_3,n_4)\,\widehat{\varphi}(n_1,\underline{n_4},\underline{n_3},\underline{n_2})}$$

$$\overset{C\text{-}S}{\le} \sum_{n=n_1+n_2+n_3+n_4} |\widehat{\varphi}(n_1,n_2,n_3,n_4)|^2$$

$\sim$ Tree 1

$$\Rightarrow \mathbb{E}\left[|I(t,n)|^2\right] \lesssim$$



$$\to \int_{-\infty}^{t} |S_{t-u_2}(n_2)|^2 du_2 \sim \frac{1}{\langle n_2\rangle^2}.$$

$\to$ Estimated last lecture (pp. 9–10).

$$\lesssim \frac{1}{\langle n_1\rangle^4}.$$

$$\lesssim \sum_{\substack{n=n_1+n_2\\ |n_1|\sim|n_2|}} \frac{1}{\langle n_1\rangle^4\langle n_2\rangle^2}$$

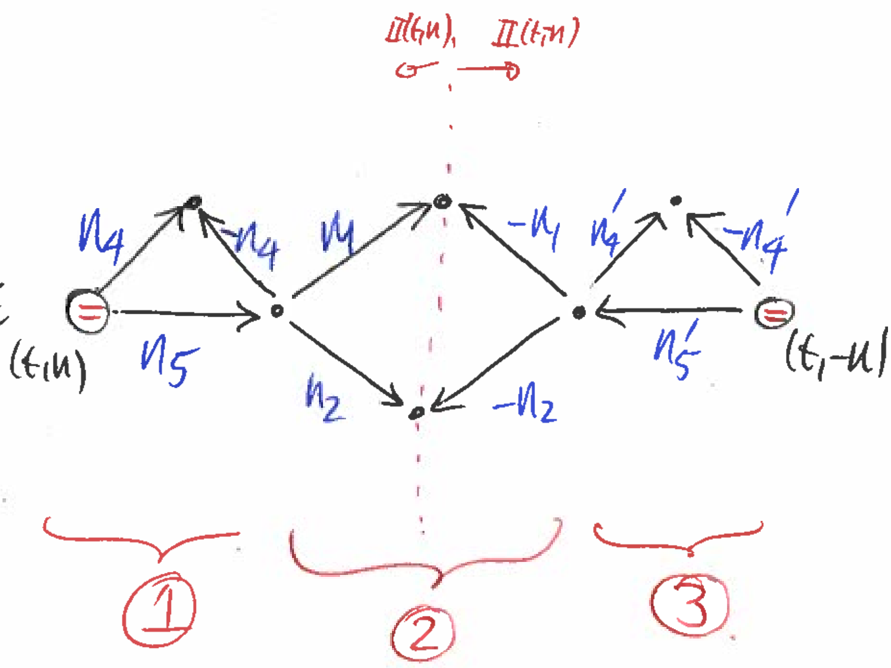
$$\overset{\text{(Lemma 4-2)}}{\lesssim} \frac{1}{\langle n\rangle^{6-3}} = \frac{1}{\langle n\rangle^3} = \langle n\rangle^{-3-2\cdot 0}.$$

$$\overset{\text{(Prop}^{\text{n}}\text{ 3-6)}}{\Rightarrow} \check{I}(t) \in C_x^{-\varepsilon} \quad a\text{-}s.$$

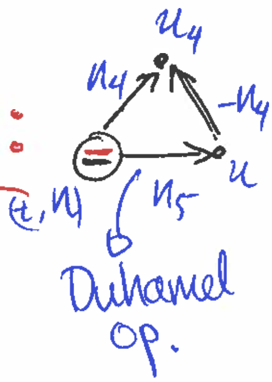
$\mathbb{II}(t,n)$

We have

$$\mathbb{E}\left[|\mathbb{II}(t,n)|^2\right] \lesssim$$



① :

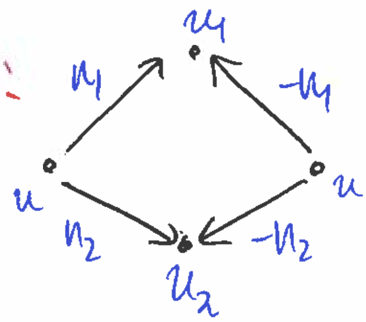


$$\to \sum_{n=n_4+n_5} \int_{\mathbb{R}} \hat{S}_{t-u}(n_5)\left(\int_{\mathbb{R}} |\hat{S}_{t-u_4}(n_4)|^2 du_4\right) du$$

$$\lesssim \sum_{n=n_4+n_5} \frac{1}{\langle n_5\rangle^2}\frac{1}{\langle n_4\rangle^2} \underset{\text{(Lemma 4.1)}}{\lesssim} \frac{1}{\langle n\rangle}$$

③ : Same as ①; gives another $\frac{1}{\langle n\rangle}$ factor.

② :



$$\to \sum_{n=n_1+n_2} \left(\int_{\mathbb{R}} |\hat{S}_{u-u_1}(n_1)|^2 du_1\right)\left(\int_{\mathbb{R}} |\hat{S}_{u-u_2}(n_2)|^2 du_2\right)$$

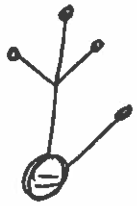
$$\lesssim \sum_{n=n_1+n_2} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2} \lesssim \frac{1}{\langle n\rangle}$$

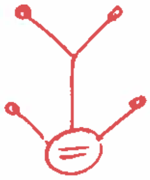
In total,

$$\mathbb{E}\left[|\mathbb{II}(t,n)|^2\right] \lesssim \frac{1}{\langle n\rangle^3} = \langle n\rangle^{-3-2\cdot 0}$$

$$\Rightarrow \check{\mathbb{II}}(t) \in C_x^{-\varepsilon} \text{ a-s.}$$

Putting together I and II implies

 $(t) \in C_x^{-\varepsilon}$, a.s.

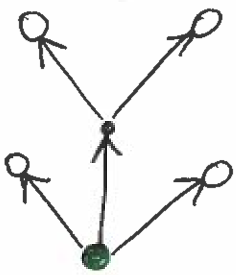
$\tau =$  : "Yov."

Forgetting about the resonant product $\circleddash$, let us consider

$$I(\varphi_N^2)\,\varphi_N^2 \in \mathcal{H}_{\leq 4}.$$

By the Wiener Chaos decomposition it has components in $\mathcal{H}_4$, $\mathcal{H}_2$ and $\mathcal{H}_0$.

$\mathcal{H}_4$ component:



(t,n)

Representations:

Real space:

$$\int_{(\mathbb{R}\times\mathbb{T}^3)^4} \Big[\int_{y\in\mathbb{T}^3} e^{-2\pi i n y} \int_{\mathbb{R}\times\mathbb{T}^3} S_{t-u}(y-\tilde{y})\, S_{u-u_1}(\tilde{y}-x_1)\, S_{u-u_2}(\tilde{y}-x_2)$$

$$\times\, S_{t-u_3}(y-x_3)\, S_{t-u_4}(y-x_4)\, du\, d\tilde{y}\, dy\Big]$$

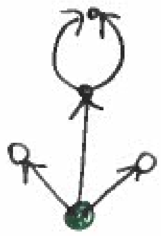
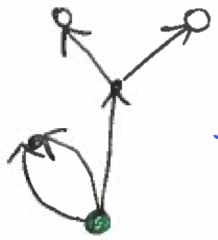
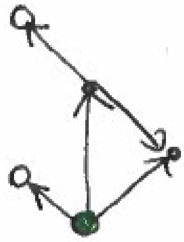
$$\times\, \xi(du_1,dx_1)\,\xi(du_2,dx_2)\,\xi(du_3,dx_3)\,\xi(du_4,dx_4)$$

Fourier space:

$$\sum_{\substack{n_1,\dots,n_5\in\mathbb{Z}^3\\ n_1+n_2=n_5\\ n=n_4+n_3+n_5}}' \int_{\mathbb{R}^4} \Big(\int_{\mathbb{R}} \hat{S}_{u-u_1}(n_1)\,\hat{S}_{u-u_2}(n_2)\,\hat{S}_{t-u_3}(n_3)\,\hat{S}_{t-u_4}(n_4)$$

$$\times\, \hat{S}_{t-u}(n_5)\, du\Big)\, d\beta(u_1,n_1)\, d\beta(u_2,n_2)\, d\beta(u_3,n_3)\, d\beta(u_4,n_4)$$

$\mathcal{H}_2$ components:

1x  + 1x  + 4x 

The first of these is

$$\sum_{\substack{n_1,\dots,n_5\\ n_5=n_1+n_2\\ n=n_3+n_4+n_5\\ n_1+n_2=0\\ n_3+n_4\neq 0}} \int_{\mathbb{R}^2} \hat S_{t-u_3}(n_3)\hat S_{t-u_4}(n_4)\left(\int_{\mathbb{R}^2}\hat S_{t-u_5}(n_5)\hat S_{u_5-u_4}(n_1)\bar{\hat S}_{u_5-u_4}(n_2)\,du_4\,du_5\right)\times d\beta(u_3,n_3)\,d\beta(u_4,n_4).$$

$\Rightarrow n_5=0$

$\Rightarrow n=n_3+n_4$

$$=\left(\sum_{n_1}\int_{\mathbb{R}^2}\hat S_{u_5-u_4}(n_1)\hat S_{u_5-u_4}(-n_1)\hat S_{t-u_5}(0)\,du_5\,du_4\right)$$

$$\times \sum_{\substack{n_3,n_4\\ n=n_3+n_4\\ n_3+n_4\neq 0}}\int_{\mathbb{R}^2}\underbrace{\hat S_{t-u_3}(n_3)\hat S_{t-u_4}(n_4)\,d\beta(u_3,n_3)\,d\beta(u_4,n_4)}_{\text{Convolution}}$$

Convolution $\Rightarrow$ Product on physical side but without the resonance $n_3+n_4=0$

$\Rightarrow$ Is $\hat V(t,n)$.

$$=\left(\sum_{n_1} e^{-t}\int_{-\infty}^{t} e^{u_5}\int_{-\infty}^{u_5} e^{-2u_5\langle n_1\rangle^2}e^{2u_4\langle n_1\rangle^2}\,du_4\,du_5\right)\times\hat V(t,n)$$

$$=\left(\sum_{n_1}\frac{1}{2\langle n_1\rangle^2}\,e^{-t}\int_{-\infty}^{t}e^{u_5}\,du_5\right)\hat V(t,n)$$

$$= \left(\sum_{n_1} \frac{1}{2\langle n_1\rangle^2}\right)\cdot \hat{V}(t,n)$$

$$= C_N\, \hat{V}(t,n).$$

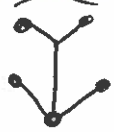
$\hookrightarrow \sim N$

- <u>Note:</u>
  - C_N is independent of t.
  - $C_N = \mathbb{E}\left[(\psi_N(t))^2\right] = \sum_{|m|<N} \int_{-\infty}^{t} e^{-2(t-u)\langle n\rangle^2}\,du = \underbrace{\sum_{|n|<N} \frac{1}{2\langle n\rangle^2}}_{\text{indep of } t}$.

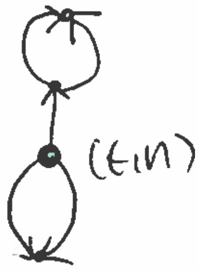
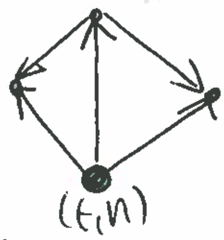
The 2^{nd} term is:

$$C_N\, \widehat{I(V)}(t,n)$$

The third term belongs to the Wiener chaos expansion of

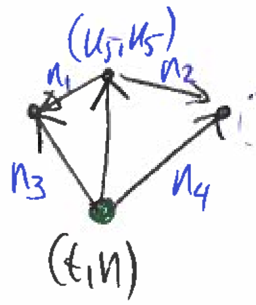
 (t,n).

$\mathcal{H}_0$ components:

$1\times$  $+\ 2\times$ 

The first term is clearly $I(C_N)C_N = C_N^2$, since $I(1)=1$, so diverges like $\sim N^2$.

The second term is more subtle; it diverges only logarithmically!



$$2^{nd}\text{ term} = \sum_{\substack{n_1,\dots,n_5 \\ n=n_3+n_4+n_5}} \int_{\mathbb{R}} \hat{S}_{t-u_5}(n_5)\left(\int_{\mathbb{R}} \hat{S}_{u_5-u_1}(n_1)\hat{S}_{t-u_1}(-n_1)\,du_1\right) \times \left(\int_{\mathbb{R}} \hat{S}_{u_5-u_2}(n_2)\hat{S}_{t-u_2}(-n_2)\,du_2\right)du_5$$

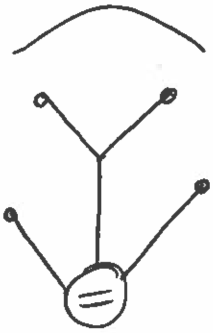
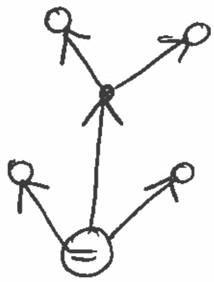
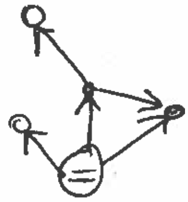
$$\underline{\underline{n=0}} \Leftarrow \begin{cases} n_1+n_3=0 \\ n_2+n_4=0 \\ n_5=n_1+n_2 \end{cases}$$

$$= \mathbb{1}_{\{n=0\}} \sum_{\substack{n_1,n_2,n_5 \\ n_5=n_1+n_2}} \int_{\mathbb{R}} \hat{S}_{t-u_5}(n_5)\, \frac{e^{-|t-u_5|\langle n_1\rangle^2}}{2\langle n_1\rangle^2}\, \frac{e^{-|t-u_5|\langle n_2\rangle^2}}{2\langle n_2\rangle^2}$$

$$= \mathbb{1}_{\{n=0\}} \frac{1}{4} \sum_{\substack{n_1,n_2,n_5 \\ n_5=n_1+n_2}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2} \int_{-\infty}^{t} e^{-(t-u_5)(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_5\rangle^2)}\,du_5$$

$$= \mathbb{1}_{\{n=0\}} \frac{1}{4} \underbrace{\sum_{\substack{n_1,n_2,n_5 \\ n_5=n_1+n_2}}}_{\text{Two full sums; "total power just like 6"}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_5\rangle^2)}$$

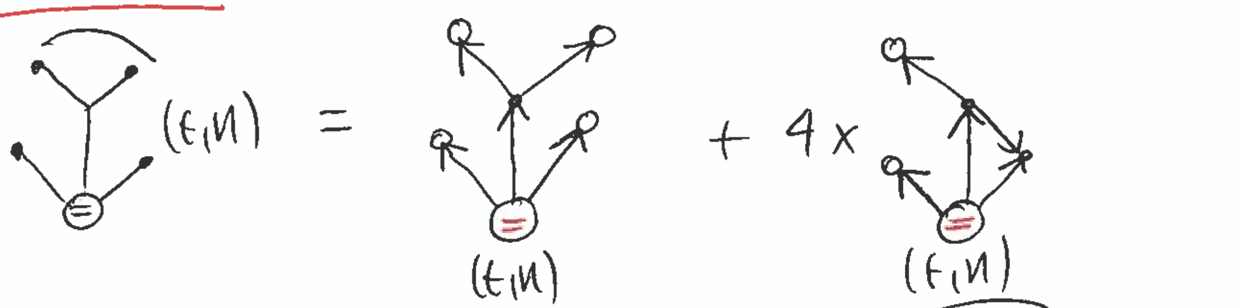
$\Rightarrow$ log divergence.

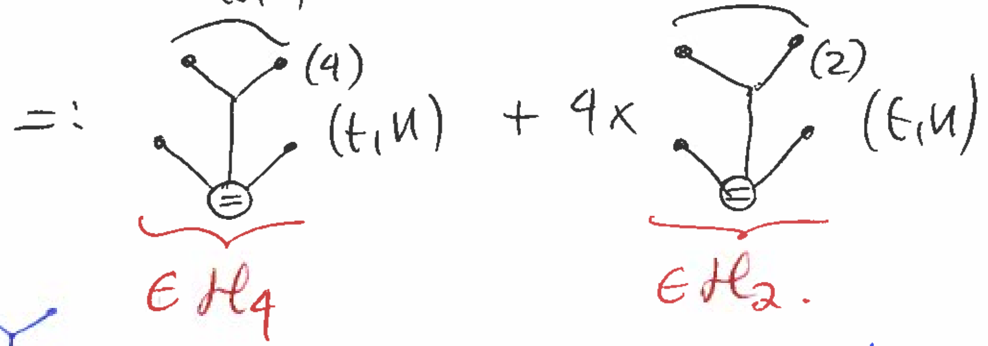
$\Rightarrow$  $(t,n) =$  $+\ 4$ 

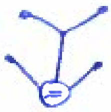
$\Uparrow$ $\mathcal{H}_4$ $\qquad\qquad$ $\Uparrow$ $\mathcal{H}_2$

Lecture 10 17/4/18 (and 20/4/18)

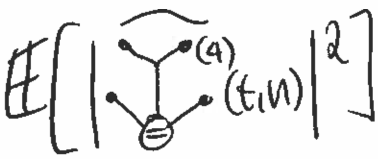
$\tau =$  : In the previous lecture, we saw that

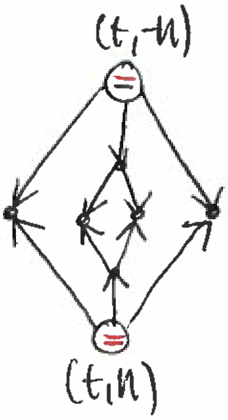




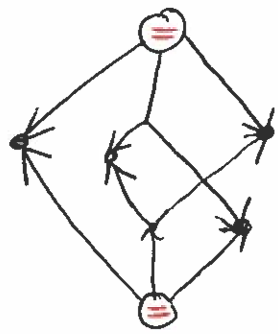
Notice that  is not just a projection onto the Wiener Chaos of highest degree.

This was the case for V as $V_N = \Phi_N^2 - C_N \in \mathcal{H}_2$.



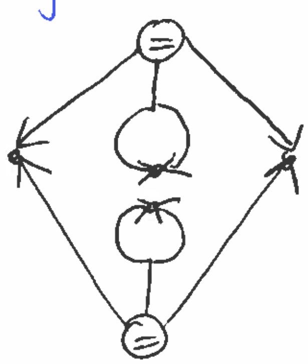


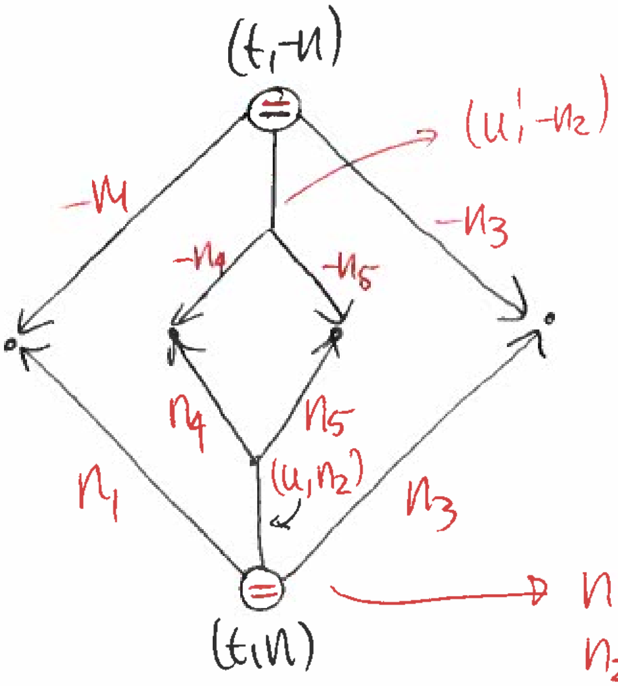
+ terms like



By Jensen's inequality as in the previous lecture these contributions can be bounded by the "diagonal" case.

• Also notice there are no "self interactions" because of the renormalization; e.g. no diagrams like

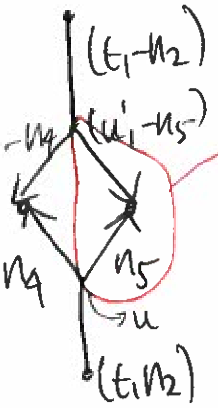




$n = n_1 + n_2 + n_3$

$n_2 = n_4 + n_5$

Resonance condition: $|n_1+n_3| \sim |n_2|$.



$$\mathbb{E}\left[\hat{\varphi}(u,n_4)\,\hat{\varphi}(u',-n_4)\right] = \int \tilde{S}_{u-u_4}(n_4)\,\tilde{S}_{u'-u_4}(-n_4)\,du_4 = \frac{e^{-|u-u'|\langle n_4\rangle^2}}{2\langle n_4\rangle^2}.$$

$\Rightarrow$ This diagram $\sim \displaystyle\int_{\mathbb{R}^2} \hat{S}_{t-u}(n_2)\,\hat{S}_{t-u'}(-n_2) \sum_{n_2=n_4+n_5} \frac{1}{|u-u'|^{\gamma}} \frac{1}{\langle n_4\rangle^{2+2\gamma}} \times \frac{1}{\langle n_5\rangle^2}\, du\, du'.$

Sum:

$0<\gamma<1/2$:

$$\sum_{n_2=n_4+n_5} \frac{1}{\langle n_4\rangle^{2+2\gamma}} \frac{1}{\langle n_5\rangle^2} \lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}}.$$

$$\Rightarrow \lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}} \int_{\mathbb{R}^2} \frac{\hat{S}_{t-u}(n_2)\,\hat{S}_{t-u'}(-n_2)}{|u-u'|^{\gamma}}\, du\, du'.$$

$$= \frac{1}{\langle n_2\rangle^{1+2\gamma}} \int_{\mathbb{R}} \hat{S}_{t-u}(n_2)\Big(\hat{S}_{t-\cdot}(-n_2) * \frac{1}{|\cdot|^{\gamma}}\Big)(u)\, du$$

$$\leq \frac{1}{\langle n_2\rangle^{1+2\gamma}} \big\|\hat{S}_{t-u}(n_2)\big\|_{L^2_u} \Big\|\frac{1}{|\cdot|^{\gamma}} * \hat{S}_{t-\cdot}(n_2)\Big\|_{L^2_u}$$

Hardy-Littlewood-Sobolev ineq

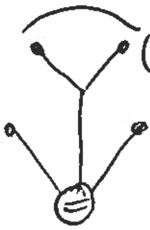
$$\frac{1}{2}+1 = \frac{1}{(1/\gamma)} + \frac{1}{q} \quad (0<\gamma<1$$

$$\Rightarrow \frac{2}{q} = 3-2\gamma$$

$$\lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}} \frac{1}{\langle n_2\rangle} \big\|\hat{S}_{t-u}(n_2)\big\|_{L^q_u}$$

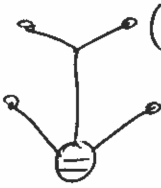
$$\lesssim \frac{1}{\langle n_2\rangle^{1+2\gamma}} \frac{1}{\langle n_2\rangle} \frac{1}{\langle n_2\rangle^{2/q}}$$

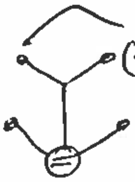
$$\lesssim \frac{1}{\langle n_2\rangle^{5}}.$$

$\Rightarrow \mathbb{E}\Big[\Big|$  $^{(4)}(t,n)\Big|^2\Big] \lesssim \displaystyle\sum_{\substack{n=n_1+n_2+n_3 \\ |n_1+n_3|\sim|n_2|}} \frac{1}{\langle n_2\rangle^5}\frac{1}{\langle n_1\rangle^2\langle n_3\rangle^2}$

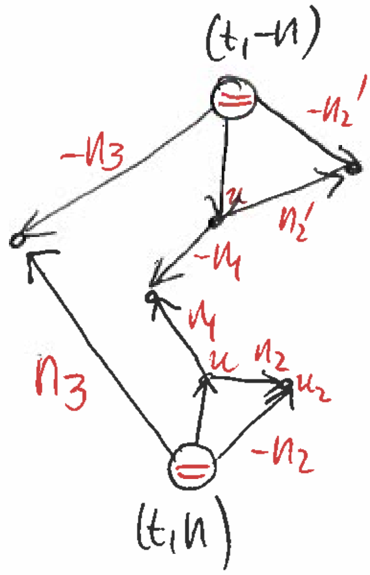
(Sum in n_1 with Lemma 4-1.
$$\lesssim \sum_{|n-n_2|\sim|n_2|} \frac{1}{\langle n_2\rangle^5}\frac{1}{\langle n-n_2\rangle}$$

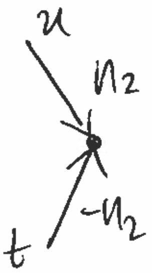
Lemma 4-2. $\lesssim \langle n\rangle^{-3} = \langle n\rangle^{-3-2\cdot(0)}.$

$\Rightarrow$  $^{(4)}(t) \in C_x^{0-}$ a.s.

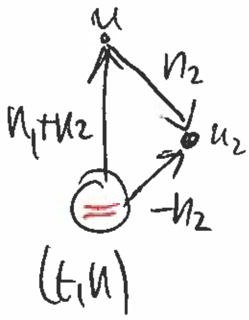
$\mathbb{E}\Big[\Big|$ 

$(t,n)\Big|^2\Big] \underset{\text{Jensen's ineq}}{\lesssim}$





$\longrightarrow \lesssim \dfrac{e^{-|u-t|\langle n_2\rangle^2}}{2\langle n_2\rangle^2}$

$\Rightarrow$ 

$$\to \int_{\mathbb{R}} \hat{S}_{t-u}(n_1+n_2)\frac{e^{-|t-u|\langle n_2\rangle^2}}{2\langle n_2\rangle^2}\,du$$

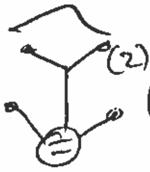
$$= \int_{-\infty}^{t} \frac{e^{-(t-u)(\langle n_2\rangle^2+\langle n_1+n_2\rangle^2)}}{2\langle n_2\rangle^2}\,du$$

$$\lesssim \frac{1}{\langle n_2\rangle^2}\,\frac{1}{\langle n_2\rangle^2+\langle n_1+n_2\rangle^2}$$

Summation Conditions:

- $-n = (-n_3)+(-n_2')+(-n_1+n_2') = -(n_1+n_3)$
- $n = n_3 + (-n_2) + (n_1+n_2) = n_1+n_3$
- $|n_3-n_2| \sim |n_1+n_2|$
- $|n_3+n_2'| \sim |n_2'-n_1|$.

5

$$\mathbb{E}\Big[\Big|$$  $$^{(2)}(t,n)\Big|^2\Big] \lesssim \sum_{\substack{n_1,n_2,n_2',n_3 \\ n=n_1+n_3 \\ |n_3-n_2|\sim|n_1+n_2| \\ |n_3+n_2'|\sim|n_2'-n_1|}} \frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_2\rangle^2}\frac{1}{\langle n_2'\rangle^2}\frac{1}{\langle n_3\rangle^2}\frac{1}{\langle n_2'\rangle^2+\langle n_2'-n_1\rangle^2} \times \frac{1}{\langle n_2\rangle^2+\langle n_1+n_2\rangle^2}$$

$$= \sum_{\substack{n_1,n_3 \\ n=n_1+n_3}} \frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_3\rangle^2} \underbrace{\sum_{\substack{n_2 \\ |n_2+n_1|\sim|n_3-n_2|}} \frac{1}{\langle n_2\rangle^2}\frac{1}{\langle n_2\rangle^2+\langle n_1+n_2\rangle^2}}_{4\text{-}1\ \sim\ \frac{1}{\langle n_1\rangle}} \underbrace{\sum_{\substack{n_2' \\ |n_3+n_2'|\sim|n_2'-n_1|}} \frac{1}{\langle n_2'\rangle^2}\frac{1}{\langle n_2'\rangle^2+\langle n_2'-n_1\rangle^2}}_{4\text{-}1\ \sim\ \frac{1}{\langle n_1\rangle}}$$

$$=: \underline{\mathrm{SUM}} \qquad \to \qquad \sum_{\substack{n_1,n_3 \\ n=n_1+n_3}} \frac{1}{\langle n_1\rangle^4}\frac{1}{\langle n_3\rangle^2}$$

$\Rightarrow$ The sum is finite (by power counting) but directly applying Lemma 4-1 we lose one power of $\langle n\rangle$ and would only get $\lesssim \langle n\rangle^{-2}$ not $\lesssim \langle n\rangle^{-3}$.

We proceed with more care.

From the resonance conditions we have

- $|n_1+n_2| = |n+n_2-n_3| \geq |n| - |n_2-n_3| \geq |n| - c|n_1+n_2|$

$$\Rightarrow |n_1+n_2| \gtrsim |n| \quad \ldots (1)$$

Similarly,

$$|n_2'+n_3| \gtrsim |n| \quad \ldots (2).$$

$$|n_2-n_3| \gtrsim |n| \quad \ldots (3)$$

$$|n_1-n_2'| \gtrsim |n| \quad \ldots (4)$$

Case 1: $\langle n\rangle \gg \min(\langle n_1\rangle, \langle n_3\rangle)$.

Case 1.1: $\min(\langle n_1\rangle, \langle n_3\rangle) = \langle n_3\rangle$.

Then by the resonance conditions, and (1) we get

$$|n_2| \geq |n_2 - n_3| - |n_3|$$
$$\geq C_1 |n_1 + n_2| - C_0 |n|$$
$$\geq \tilde{C}_1 |n| - C_0 |n| \qquad (\tilde{C}_1 \gg C_0.)$$
$$\Rightarrow |n_2| \gtrsim |n|.$$

Likewise, using (2), we get $|n_2'| \gtrsim |n|$.

$$\Rightarrow \text{SUM} \lesssim \sum_{n=n_1+n_3} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_3\rangle^2} \sum_{|n| \lesssim |n_2|} \frac{1}{\langle n_2\rangle^4} \sum_{|n| \lesssim |n_2'|} \frac{1}{\langle n_2'\rangle^4}$$

$$\lesssim \frac{1}{\langle n\rangle^2} \sum_{n=n_1+n_3} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_3\rangle^2}$$

(Lemma 4.1) $\lesssim \dfrac{1}{\langle n\rangle^3}$.

Case 1.2: $\min(\langle n_1\rangle, \langle n_3\rangle) = \langle n_1\rangle$.

Using (2) and (3) give: $|n_2| \gtrsim |n|$, $|n_2'| \gtrsim |n|$ and we proceed as in Case 1.1.

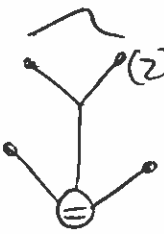
Case 2 : $\langle n \rangle \lesssim \min(\langle n_1 \rangle, \langle n_3 \rangle)$

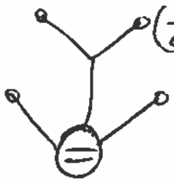
$$\sum_{n_2} \frac{1}{\langle n_2 \rangle^2} \frac{1}{\langle n_2 \rangle^2 + \langle n_1 + n_2 \rangle^2} = \sum_{|n_2| \ll |n_1|} \cdot \; + \sum_{|n_2| \gtrsim |n_1|}.$$

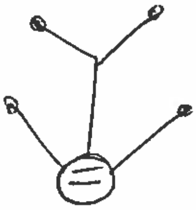
$$\lesssim \sum_{|n_2| \ll |n_1|} \frac{1}{\langle n_2 \rangle^2} \frac{1}{\langle n_1 \rangle^2} + \sum_{|n_2| \gtrsim |n_1|} \frac{1}{\langle n_2 \rangle^4}$$

$$\lesssim \frac{1}{\langle n_1 \rangle} + \frac{1}{\langle n_1 \rangle} \approx \frac{1}{\langle n_1 \rangle}.$$

$$\Rightarrow \text{SUM} \lesssim \sum_{\substack{n = n_1 + n_3 \\ \langle n \rangle \lesssim \langle n_1 \rangle}} \frac{1}{\langle n_1 \rangle^4} \frac{1}{\langle n_3 \rangle^2} \lesssim \frac{1}{\langle n \rangle^2} \sum_{n = n_1 + n_3} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_3 \rangle^2} \lesssim \frac{1}{\langle n \rangle^3}.$$

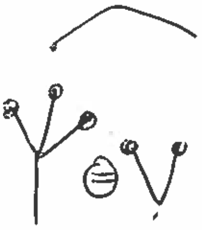
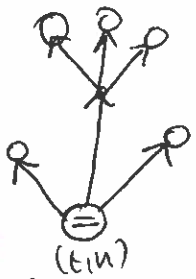
$\Rightarrow \mathbb{E}\Big[\Big|$  $^{(2)}(t,n)\Big|^2\Big] \lesssim \langle n \rangle^{-3} = \langle n \rangle^{-3-2(0)}$

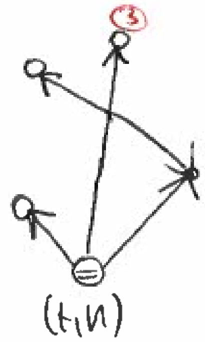
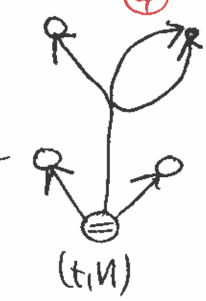
$\Rightarrow$  $^{(2)}(t) \in C_x^{0-}$ a.s.

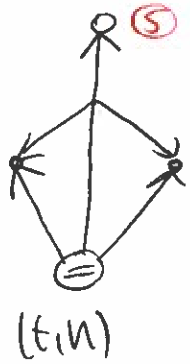
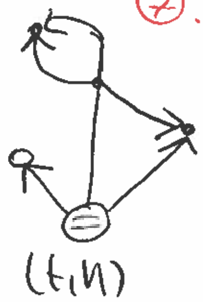
$\Rightarrow$  $(t) \in C_x^{0-}$ a-s

$\tau =$  ; $\rightsquigarrow \mathcal{Y}_N \ominus V_N - 6C_N' \Psi_N$

 $\sim \log N$

 $(t,N) =$ (1)  $\in \mathcal{H}_5$.

$+$ (2)  $+ 6$ (3)  $+ 2$ (4)  $\in \mathcal{H}_3$

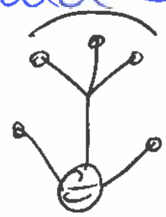
$+ 6$ (5)  $+ 2$ (6)  $+ 3$ (7)  $\in \mathcal{H}_1$.

Alternatively, forgetting about the '$\ominus$', we can expand $I(\Psi_N^3)\Psi_N^2$

$$I(\Psi_N^3)\Psi_N^2 = \underbrace{I(\Psi_N^3 - 3C_N\Psi_N)(\Psi_N^2 - C_N)}_{①+③+⑤} + \underbrace{3I(C_N\Psi_N)(\Psi_N^2 - C_N)}_{④}$$

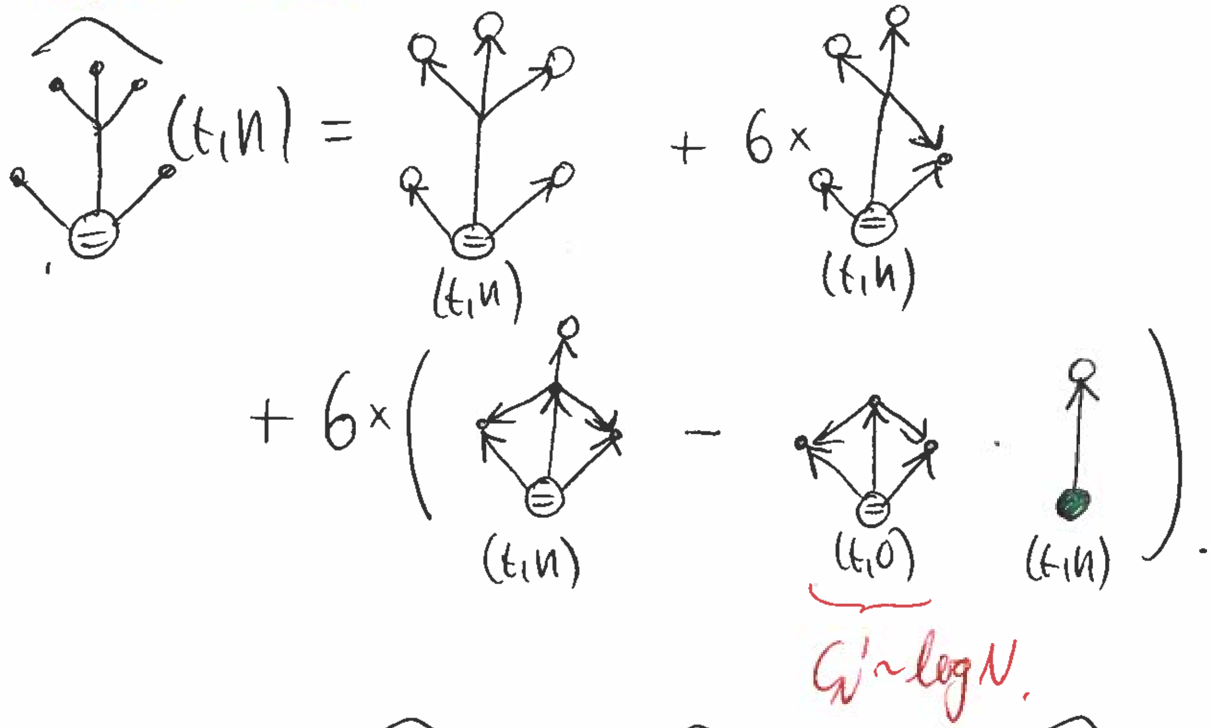
$$+ \underbrace{I(\Psi_N^3 - 3C_N\Psi_N)C_N}_{②} + \underbrace{I(3C_N\Psi_N)C_N}_{⑥+⑦}.$$

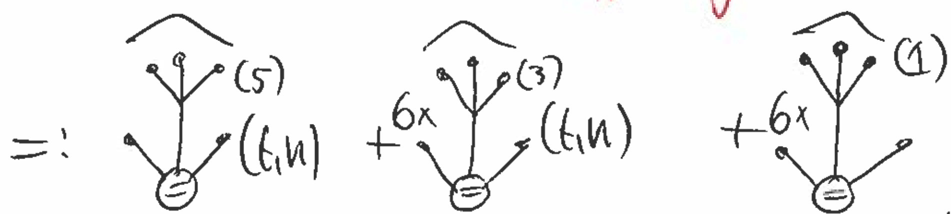
Thus, we would expect

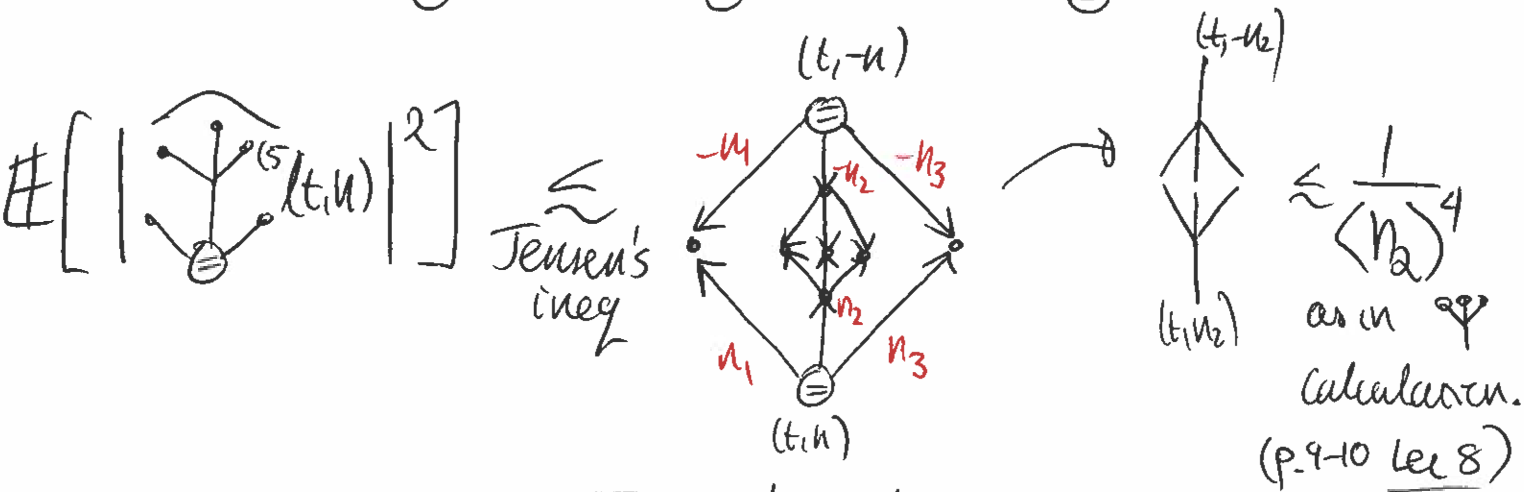
 (t,N) " $:=$ " $\underset{\mathcal{H}_5}{\underset{\Cap}{①}} + \underset{\mathcal{H}_3}{\underset{\Cap}{③}} + \underset{\mathcal{H}_1}{\underset{\Cap}{⑤}}$.

However, as we will see later, (5) diverges logarithmically due to the contribution  .

One is then lead to define

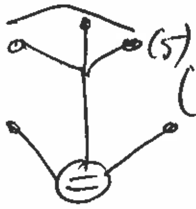


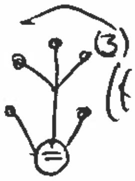


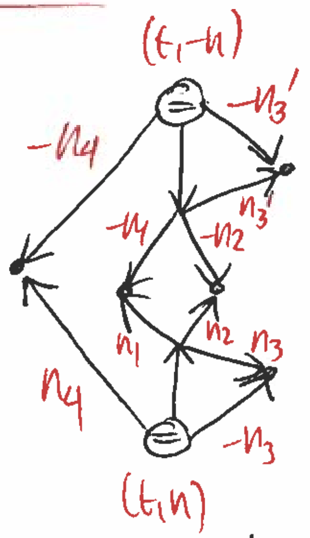


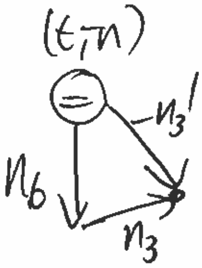
$$\lesssim \sum_{\substack{n=n_1+n_2+n_3\\ |n_1+n_3|\sim|n_2|}} \frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_2\rangle^4}\frac{1}{\langle n_3\rangle^2}$$

(Summing Lemma 4-1) $\displaystyle\lesssim \sum_{\substack{n_2\\ |n-n_2|\sim|n_2|}} \frac{1}{\langle n_2\rangle^4}\frac{1}{\langle n-n_2\rangle} \lesssim \langle n\rangle^{-2} = \langle n\rangle^{-3-2(-1/2)}.$

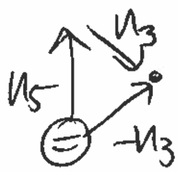
$\Rightarrow$  $(t,n) \in C_x^{-\frac{1}{2}-}$ a-s.

$\mathbb{E}\Big[\Big|$  $(t,n)\Big|^2\Big] \underset{\text{Jensen's ineq}}{\lesssim}$

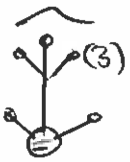




$$\lesssim \frac{1}{\langle n_3'\rangle^2}\frac{1}{\langle n_3'\rangle^2+\langle n_6\rangle^2} = \frac{1}{\langle n_3'\rangle^2}\frac{1}{\langle n_3'\rangle^2+\langle n_1+n_2-n_3'\rangle^2}$$



$$\lesssim \frac{1}{\langle n_3\rangle^2}\frac{1}{\langle n_3\rangle^2+\langle n_5\rangle^2} = \frac{1}{\langle n_3\rangle^2}\frac{1}{\langle n_3\rangle^2+\langle n_1+n_2+n_3\rangle^2}.$$

$\Rightarrow \mathbb{E}\Big[\Big|$  $(t,n)\Big|^2\Big]$

$$\lesssim \underbrace{\sum_{\substack{n=n_1+n_2+n_4\\ |n_1+n_2+n_3|\sim|n_4-n_3|\\ |n_1+n_2-n_3'|\sim|n_3'+n_4|}} \frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_2\rangle^2}\frac{1}{\langle n_4\rangle^2}\frac{1}{\langle n_3\rangle^2}\frac{1}{\langle n_3'\rangle^2}\frac{1}{\langle n_3\rangle^2+\langle n_1+n_2+n_3\rangle^2} \times \frac{1}{\langle n_3'\rangle^2+\langle n_1+n_2-n_3'\rangle^2}}$$

$=:$ SUM2, We will show

$$\text{SUM2} \lesssim \langle n\rangle^{-2}$$

Notation: $n_{jk} := n_j + n_k$, $n_{j-k} := n_j - n_k$

$n_{jk'} := n_j + n_k'$.

By symmetry, we may assume $|n_1| \geq |n_2|$.
From the resonance conditions, we have

$$|n_{123}| \geq |n| - |n_{3-4}| \geq |n| - c|n_{123}| \Rightarrow |n_{123}| \gtrsim |n|. \quad \text{...(1)}$$

$$|n_{12-3'}| = |n - n_{4-3'}| \geq |n| - |n_{43'}| \gtrsim |n| - c|n_{12-3'}| \Rightarrow |n_{12-3'}| \gtrsim |n| \quad \text{...(2)}$$

Case 1: $\langle n \rangle \gg \min(\langle n_{12} \rangle, \langle n_4 \rangle)$

Subcase 1-1: $\min(\langle n_{12} \rangle, \langle n_4 \rangle) = \langle n_{12} \rangle$

Using (1) and (2), we have

$$|n_3| \geq |n_{123}| - |n_{12}| \geq c_1|n| - \overset{\text{small}}{c_0}|n| \Rightarrow |n_3| \gtrsim |n| \quad \text{...(3)}$$

$$|n_3'| \geq |n_{12-3'}| - |n_{12}| \geq c_1|n| - \overset{\text{small}}{c_0}|n| \Rightarrow |n_3'| \gtrsim |n|. \quad \text{...(4)}$$

Subsubcase 1-1-2: $\langle n \rangle \gtrsim \langle n_4 \rangle$

$$\text{SUM2} \lesssim \sum_{n_1,n_2} \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \frac{1}{\langle n_4 \rangle^2} \sum_{|n_3|,|n_3'| \gtrsim |n| \gtrsim \langle n_4 \rangle} \frac{1}{\langle n_3 \rangle^4} \frac{1}{\langle n_3' \rangle^4}$$

$$\lesssim \sum_{|n_3|,|n_3'| \gtrsim |n|} \frac{1}{\langle n_3 \rangle^4} \frac{1}{\langle n_3' \rangle^{4-\varepsilon}} \sum_{n_1,n_2} \frac{1}{\langle n_1 \rangle^2} \frac{1}{\langle n_2 \rangle^2} \frac{1}{\langle n - n_1 - n_2 \rangle^{2+\varepsilon}}$$

$$\lesssim \frac{1}{\langle n\rangle}\frac{1}{\langle n\rangle^{1-\varepsilon}}\sum_{n_1}\frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n-n_1\rangle^{1+\varepsilon}}$$

$$\lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}}\frac{1}{\langle n\rangle^{\varepsilon}} \sim \frac{1}{\langle n\rangle^2}.$$

Subsubcase 1-1-2) $\langle n\rangle \ll \langle n_4\rangle$

$\Rightarrow \langle n_{12}\rangle \ll \langle n\rangle \ll \langle n_4\rangle$

$$\Rightarrow \text{SUM2} \lesssim \sum_{n_1,n_2}\frac{1}{\langle n_1\rangle^2}\frac{1}{\langle n_2\rangle^2}\frac{1}{\langle \underbrace{n_1+n_2}_{=n-n_4}\rangle^{\varepsilon}}\frac{1}{\langle n_4\rangle^2}\sum_{\substack{|n_3|,|n_3'|\gtrsim|n| \\ \gg\langle n_{12}\rangle}}\frac{1}{\langle n_3\rangle^4}\frac{1}{\langle n_3'\rangle^{4-\varepsilon}}$$

$$\lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}}\sum_{n_1}\frac{1}{\langle n_1\rangle^2}\sum_{n_2}\frac{1}{\langle n_2\rangle^2}\underbrace{\frac{1}{\langle n-n_4\rangle^{\varepsilon}}\frac{1}{\langle n_4\rangle^2}}_{\frac{1}{\langle n_4\rangle^{\varepsilon}}2}$$

$$\lesssim \frac{1}{\langle n\rangle^2}$$

Subcase 1-2) $\min(\langle n_{12}\rangle, \langle n_4\rangle) = \langle n_4\rangle$

Using the resonant conditions,

$$|n_3| \geq |n_{3-4}| - |n_4| \geq c_1|n_{123}| - |n_4|$$
$$\geq c_2|n| - c_3|n| \gtrsim |n|.$$

Similarly, $|n_3'| \gtrsim |n|$.

Proceed as in Subsubcase 1-1-2.

Case 2: $\langle n\rangle \lesssim \min(\langle n_{12}\rangle, \langle n_4\rangle)$

Subcase 2-1: $\langle n\rangle \ll \langle n_{12}\rangle$

$$\sum_{n_3} \frac{1}{\langle n_3\rangle^2} \frac{1}{\langle n_3\rangle^2 + \langle n_{12}+n_3\rangle^2} \lesssim \sum_{n_3} \frac{1}{\langle n_3\rangle^2} \frac{1}{\langle n_3+n_{12}\rangle^2} \lesssim \frac{1}{\langle n_{12}\rangle}.$$

$$\Rightarrow \text{SUM2} \lesssim \sum_{n_1 n_2} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_2\rangle^2} \frac{1}{\langle n_4\rangle^2} \frac{1}{\langle n_{12}\rangle^2} \quad \cdots (5)$$

$$\lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}} \sum_{n_1 n_2} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_2\rangle^2} \frac{1}{\langle n_4\rangle^2} \frac{1}{\langle n_{12}\rangle^{\varepsilon}}$$

$$\lesssim \frac{1}{\langle n\rangle^{2-\varepsilon}} \sum_{n_1 n_2} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_2\rangle^2} \frac{1}{\langle n_{12}\rangle^{2+\varepsilon}} \qquad \text{since } \langle n\rangle \ll \langle n_{12}\rangle.$$

$$\lesssim \frac{1}{\langle n\rangle^2}$$

Subcase 2-2: $\langle n\rangle \sim \langle n_{12}\rangle$

$$\text{SUM2} \lesssim (5) = \sum_{n_1 n_2} \frac{1}{\langle n_1\rangle\langle n_2\rangle} \frac{1}{\langle n_{12}\rangle^{\frac{3}{2}-\frac{\varepsilon}{2}}} \frac{1}{\langle n_{12}\rangle^{\frac{1}{2}+\frac{\varepsilon}{2}}} \quad \swarrow \langle n\rangle\sim\langle n_{12}\rangle$$

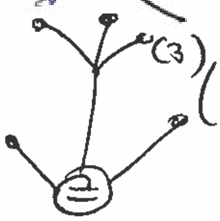
$$\times \frac{1}{\langle n_1\rangle\langle n_2\rangle} \frac{1}{\langle n_4\rangle^{\frac{3}{2}-\frac{\varepsilon}{2}}} \frac{1}{\langle n_4\rangle^{\frac{1}{2}+\frac{\varepsilon}{2}}}, \quad \swarrow \frac{1}{\langle n_4\rangle} \lesssim \frac{1}{\langle n\rangle} \text{ case 2}$$

$$\underset{\text{C-S}}{\lesssim} \frac{1}{\langle n\rangle^{1+\varepsilon}} \left(\sum_{n_1 n_2} \frac{1}{\langle n_1\rangle^2 \langle n_2\rangle^2} \frac{1}{\langle n_{12}\rangle^{3-\varepsilon}} \right)^{1/2}$$

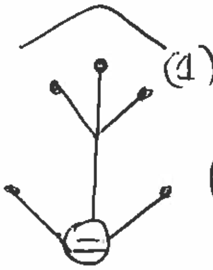
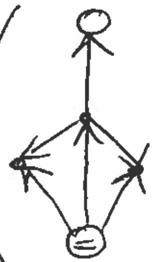
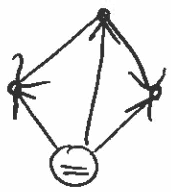
$$\times \left(\sum_{n_1 n_2} \frac{1}{\langle n_1\rangle^2} \frac{1}{\langle n_2\rangle^2} \frac{1}{\langle n_4\rangle^{3-\varepsilon}} \right)^{1/2}.$$

$$\lesssim \frac{1}{\langle n\rangle^{1+\varepsilon}} \frac{1}{\langle n\rangle^{\frac{1}{2}-\frac{\varepsilon}{2}}} \frac{1}{\langle n\rangle^{\frac{1}{2}-\frac{\varepsilon}{2}}} \lesssim \frac{1}{\langle n\rangle^2}.$$

This verifies $SUM2 \lesssim \langle n\rangle^{-2}$;

$\Rightarrow$  $^{(3)}(t) \in C_x^{-\frac{1}{2}-}$ a.s.

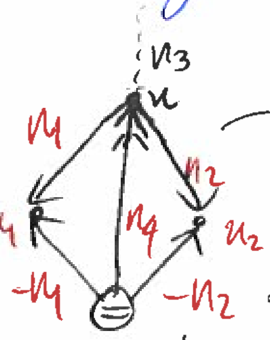
(Lecture 10, part 2, 2014/1/8)

 $^{(1)}(t,n) = 6\times\Big($  $-$  $\cdot$  $\Big)$

$(t,n) \qquad (t,0) \qquad (t,n)$

Remark: The presentation for this contribution in M-W-X is incorrect. The resonance condition stated there is incorrect and they miss a nontrivial term (see

Set

$\hat{K}_{t-u}(n) =$ 

$\rightarrow n_4 = n_1+n_2+n_3$

$\rightarrow n = (-n_1)+(-n_2)+(n_1+n_2+n_3) = n_3$

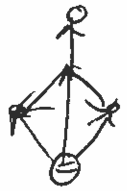
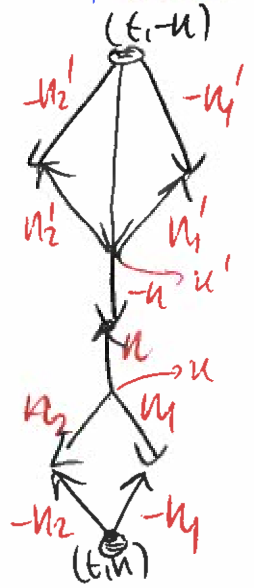
$\rightarrow |n_1+n_2| \sim |n_1+n_2+n|.$

$$= \sum_{\substack{n_1,n_2\\ |n_1+n_2|\sim|n_1+n_2+n|}} \hat{S}_{t-u}(n_1+n_2+n) \int_{-\infty}^{u} \hat{S}_{t-u_1}(-n_1)\hat{S}_{u-u_1}(n_1)\,du_1 \times \int_{-\infty}^{u} \hat{S}_{t-u_2}(-n_2)\hat{S}_{u-u_2}(n_2)\,du_2$$

$\Rightarrow u < t$

$$= \sum_{\substack{n_1 n_2 \\ |n_1+n_2| \sim |n+n_{12}|}} \frac{e^{-(t-u)(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n+n_{12}\rangle^2)}}{4\langle n_1\rangle^2\langle n_2\rangle^2}$$

Suppose we did not know that this term diverges logarithmically. Then, we would compute:

$\mathbb{E}\Big[\Big|$  $\Big|^2\Big] =$ 

$\diamond$ piece

$\diamond$ piece

$$\sim \int_{\mathbb{R}^2} \sum_{\substack{n_1 n_2 \\ |n_{12}| \sim |n+n_{12}|}} \frac{e^{-(t-u)(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n+n_{12}\rangle^2)}}{4\langle n_1\rangle^2\langle n_2\rangle^2} \cdot \sum_{\substack{n_1', n_2' \\ |n_{12}'| \sim |-n+n_{12}'|}} \frac{e^{-(t-u)(\langle n_1'\rangle^2+\langle n_2'\rangle^2+\langle n_{12}'-n\rangle^2)}}{4\langle n_1'\rangle^2\langle n_2'\rangle^2}$$

$$\times \frac{e^{-|u-u'|\langle n\rangle^2}}{2\langle n\rangle^2}\, du\, du'.$$

$\rightarrow$ ✗ piece.

Approach 1: $e^{-|u-u'|\langle n\rangle^2} \lesssim 1.$

Integrating over u, u', we have to sum essentially

$$\frac{1}{\langle n\rangle^2} \cdot \sum_{\substack{n_1 n_2 \\ |n_{12}| \sim |n+n_{12}|}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n+n_{12}\rangle^2)}$$

First of all, notice that the $\langle n\rangle^{-2}$ factor is useless here in trying to sum.
The resonance condition then gives

$$\underbrace{\sum_{n_1,n_2} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}\,\frac{1}{\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_{12}\rangle^2}}_{\text{power counting }=6\ \Rightarrow\ \text{Log-divergence!}}$$

Approach 2: Try to make use of the exponential $(\times 2)$, i.e. use

$$e^{-|u-u'|\langle n\rangle^2} \lesssim \frac{1}{|u-u'|^\gamma}\frac{1}{\langle n\rangle^{2\gamma}}, \quad \gamma>0.$$

Then, using the Hardy-Littlewood-Sobolev idea, we would have:

$$\sum_{\substack{n_1,n_2\\ n_1',n_2'}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2\langle n\rangle^{2+2\gamma}\langle n_1'\rangle^2\langle n_2'\rangle^2}\underbrace{\iint \frac{e^{-(t-u)(\cdots)}\,e^{-(t-u')(\cdots)}}{|u-u'|^\gamma}\,du\,du'}$$

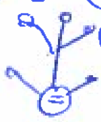
$$\lesssim \left\|e^{-(t-u)(\cdots)}\right\|_{L^2_u}\cdot\underbrace{\left\|\frac{1}{|\cdot|^\gamma}*e^{-(t-\cdot)(\cdots)}\right\|_{L^2_u}}_{\text{H-L-S},\ \frac{1}{q}=\frac{3}{2}-\gamma,\ 0<\gamma<1}$$

$$\lesssim \sum_{\substack{n_1,n_2\\ n_1',n_2'}} \frac{1}{\langle n\rangle^{2+2\gamma}\langle n_1\rangle^2\langle n_2\rangle^2\langle n_1'\rangle^2\langle n_2'\rangle^2}\,\frac{1}{\langle n_1\rangle+\langle n_2\rangle+\langle n+n_2\rangle}$$
$$\times\frac{1}{\left(\langle n_1'\rangle+\langle n_2'\rangle+\langle n+n_2'\rangle\right)^{3-2\gamma}}$$

Power counting: 4 sums $\rightsquigarrow$ need $4\times 3=12+$

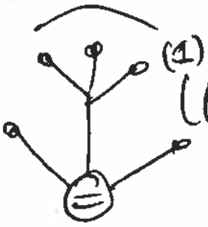
Have: $3-2\gamma+1+2+2+2+2 = 12-2\gamma < 12$!

We are actually worse off than if we had of just bounded by 1 (as in approach 1)!

This motivates subtracting off the log-divergent term in the definition of  $^{(1)}(t,n)$.

Let us see how this helps.

We write



$$^{(1)}(t,n) = \int_{-\infty}^{t} \widehat{K}_{t-u}(n)\big(\hat{\varphi}(u,n) - \hat{\varphi}(t,n)\big)\,du$$

add & subtracted

(M-W-X miss this term)

$$+ \int_{-\infty}^{t} \underbrace{\big(\widehat{K}_{t-u}(n) - \widehat{K}_{t-u}(0)\big)\,du}_{C_N' = \int_{-\infty}^{t} K_{t-u}(0)\,du \,\sim\, \log N} \cdot \hat{\varphi}(t,n)$$

$$=: I(t,n) + II(t,n).$$

$$\Rightarrow \mathbb{E}\left[|I(t,n)|^2\right] = \iint \widehat{K}_{t-u}(n)\,\widehat{K}_{t-u'}(-n) \times \underbrace{\mathbb{E}\left[\big(\hat{\varphi}(u,n) - \hat{\varphi}(t,n)\big)\big(\hat{\varphi}(u',-n) - \hat{\varphi}(t,-n)\big)\right]}_{\text{Use C-S and bound by } \|\cdot\|_{L^2(\Omega)}}\,du\,du'$$

$\underline{t > u}$

$$\mathbb{E}\left[|\hat{\varphi}(u,n) - \hat{\varphi}(t,n)|^2\right] = \mathbb{E}\left[\left|\int_u^t e^{-(t-\tau)\langle n\rangle^2}\,d\beta_n(\tau) + \int_{-\infty}^{u} \big(e^{-t\langle n\rangle^2} - e^{-u\langle n\rangle^2}\big) \times e^{\tau\langle n\rangle^2}\,d\beta_n(\tau)\right|^2\right]$$

(By independence)

$$= \mathbb{E}\left[|\cdot|^2\right] + \mathbb{E}\left[|\cdot|^2\right]$$

$$= \frac{1 - e^{-2(t-u)\langle n\rangle^2}}{2\langle n\rangle^2} + \frac{\big(e^{-t\langle n\rangle^2} - e^{-u\langle n\rangle^2}\big)^2 e^{2u\langle n\rangle^2}}{2\langle n\rangle^2}$$

$$\lesssim \frac{1 - e^{-2(t-u)\langle n\rangle^2}}{2\langle n\rangle^2}$$

By the Mean Value theorem,

$$\lesssim \frac{\min(1,\ |t-u|\langle n\rangle^2)}{\langle n\rangle^2}$$

$$\sim \min(\langle n\rangle^{-2}, |t-u|)$$

Hence, by interpolation,

$$\mathbb{E}[|\hat{\rho}(t,n)-\hat{\rho}(u,n)|^2] \lesssim (t-u)^{2\lambda}\langle n\rangle^{-2+4\lambda}. \quad , \lambda\in(0,1)$$

$$\Rightarrow \mathbb{E}[|I(t,n)|^2] \lesssim \langle n\rangle^{4\lambda-2}\Big(\int_{\mathbb{R}} \widehat{K}_{t-u}(n)(t-u)^{\lambda}du\Big)^2.$$

$$\lesssim \langle n\rangle^{4\lambda-2}\Big(\sum_{\substack{n_1,n_2\\ |n_{12}|\sim|n+n_{12}|}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}\frac{1}{(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n+n_{12}\rangle^2)^{\lambda+1}}\Big)^2$$

$$\lesssim \langle n\rangle^{4\lambda-2}\langle n\rangle^{-4\lambda} \sim \langle n\rangle^{-2} = \langle n\rangle^{-3-2(-1/2)}.$$

$$\mathbb{E}[|II(t,n)|^2] = \frac{1}{2\langle n\rangle^2}\Big(\underbrace{\int_{-\infty}^{t}\widehat{K}_{t-u}(n)-\widehat{K}_{t-u}(0)du}_{=:\,III(t,n)}\Big)^2.$$

It suffices to prove $|III(t,n)| \lesssim 1$ indep of n (and t).
Using the definition of $\widehat{K}_{t-\cdot}(n)$, we have

$$|III(t,n)| = \sum_{\substack{n_1,n_2\\ |n_{12}|\sim|n+n_{12}|}} \frac{1}{4\langle n_1\rangle^2\langle n_2\rangle^2}\left(\frac{1}{\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n+n_{12}\rangle^2} - \frac{1}{\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_{12}\rangle^2}\right).$$

Note that the resonance condition implies $|n|\lesssim|n_{12}|$
(as if $|n|>>|n_{12}|$, we have a contradiction).

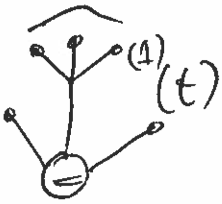
So $\quad |n| \lesssim |n_{12}| \lesssim \max(|n_1|,|n_2|) = |n_1|$

$\Downarrow$ By symmetry, can assume $|n_1| \geq |n_2|$.

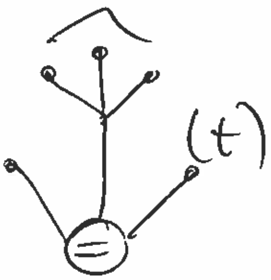
$$|III(t,n)| \lesssim \sum_{\substack{n_1,n_2\\ |n|\lesssim|n_1|}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}\frac{1}{(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_{12}\rangle^2)^2}\underbrace{\left|\langle n_{12}\rangle^2-\langle n+n_{12}\rangle^2\right|}_{\substack{\leq |n|^2+2|n||n_{12}|\\ \lesssim \langle n_{12}\rangle\langle n\rangle.}}.$$

$$\lesssim \sum_{\substack{n_1,n_2\\ |n|\lesssim|n_1|\\ |n_2|\leq|n_1|}} \frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}\frac{\langle n_{12}\rangle\langle n\rangle}{(\langle n_1\rangle^2+\langle n_2\rangle^2+\langle n_{12}\rangle^2)^2}$$

$$\lesssim \langle n\rangle \sum_{\substack{n_1,n_2\\ |n|\lesssim|n_1|\\ |n_2|\lesssim|n_1|}} \frac{1}{\langle n_1\rangle^5}\frac{1}{\langle n_2\rangle^2} \lesssim \langle n\rangle \sum_{|n|\lesssim|n_1|}\frac{1}{\langle n_1\rangle^4} \lesssim 1.$$

$\Rightarrow$  $\in C_x^{-\frac{1}{2}-}$ a-s.

$\Downarrow$

 $\in C_x^{-\frac{1}{2}-}$ a-s.

Time differences

Now we want to verify the temporal regularity of our Stochastic objects. We seek to apply part 2 of Proposition 3.6:

If $$\mathbb{E}\left[\,|\hat{\tau}(t_1,n)-\hat{\tau}(t_2,n)|^2\right] \lesssim |t_1-t_2|^{\lambda}\langle n\rangle^{-d-2\alpha+2\lambda} \quad \cdots \; (*)$$

for some $\lambda\in(0,1)$, uniformly in t_1,t_2, $n\in\mathbb{Z}^d$, $0<|t_1-t_2|<$

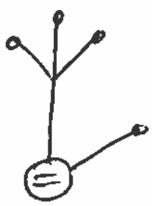
Then for all $\beta<\alpha-\lambda$, $\tau\in C(\mathbb{R}_+;C^{\beta}(\mathbb{T}^d))$ and

$$\sup_{0<|t_1-t_2|<1}\mathbb{E}\left[\frac{\|\tau(t_1)-\tau(t_2)\|^{p}_{C^{\beta}_x}}{|t_1-t_2|^{\lambda p/2}}\right]<\infty,\quad \forall p<\infty.$$

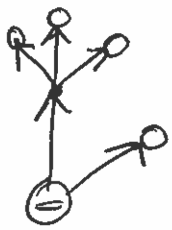
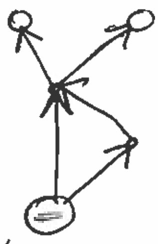
We need to obtain estimates of the form of $(*)$ for each of our Stochastic objects.

The idea is to exploit their multilinear structure and to use the Mean Value theorem.

As an example which illustrates essentially all the key methods, we consider

$\tau =$ 

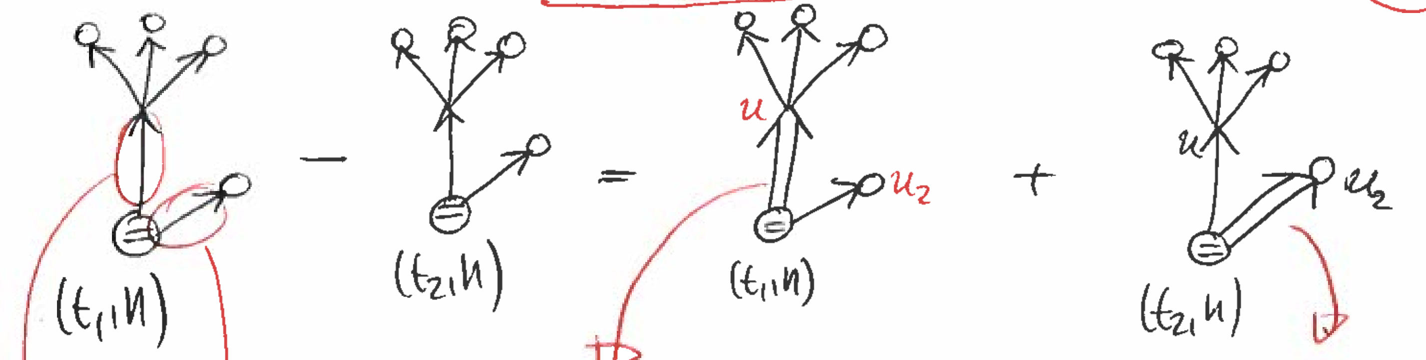
Recall,

$\hat{\tau}(t,n) =$  $+ \; 3\times$ 

(t,n) $\qquad$ (t,n)

Only study this first term in what follows.

Assume $t_1 \geq t_2$.



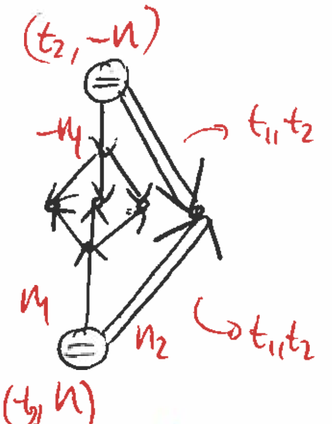
2 locations where t_1 appears.

$\Uparrow$: Denotes a time difference (in this case, $S_{t_1-u} - S_{t_2-u}$)

$S_{t_1-u_2} - S_{t_2-u_2}$.

$$=: I(t_1,t_2,n) + II(t_1,t_2,n).$$

$\mathbb{E}\left[|II(t_1,t_2,n)|^2\right] \underset{\text{Jensen's Ineq}}{\lesssim}$



As we have seen multiple times, the inner pattern  is estimated by $\frac{1}{\langle n_1\rangle^4}$.

Write differences as:
$$\int_{-\infty}^{t_1} \hat{S}_{t_1-u_2}(u_2)\, d\beta_{n_2}(u_2) - \int_{-\infty}^{t_2} \hat{S}_{t_2-u_2}(u_2)\, d\beta_{n_2}(u_2)$$
$$= \int_{t_2}^{t_1} \hat{S}_{t_1-u_2}(u_2)\, d\beta_{n_2}(u_2)$$
$$+ \int_{-\infty}^{t_2} \hat{S}_{t_1-u_2}(u_2) - \hat{S}_{t_2-u_2}(u_2)\, d\beta_{n_2}(u_2)$$

By independence

$$\mathbb{E}\Big[\Big|\ \cdot\ \Big|^2\Big] = \underbrace{\int_{t_2}^{t_1} |\hat{S}_{t_1-u_2}(n_2)|^2 du_2}_{1} + \underbrace{\int_{-\infty}^{t_2} |\hat{S}_{t_1-u_2}(n_2) - \hat{S}_{t_2-u_2}(n_2)|^2 du_2}_{2}$$

$$1 = \frac{1-e^{-2(t_1-t_2)\langle n_2\rangle^2}}{2\langle n_2\rangle^2} \lesssim \frac{\min(1, |t_1-t_2|^{\theta}\langle n_2\rangle^{2\theta})}{2\langle n_2\rangle^2}$$

2 : Write the difference as (using FTC):

$$\hat{S}_{t_1-u_2}(n_2) - \hat{S}_{t_2-u_2}(n_2) = -\langle n_2\rangle^2 \int_{t_2}^{t_1} e^{-(\tau-u_2)\langle n_2\rangle^2} d\tau. \qquad t_2 \le \tau \le t_1$$

By Minkowski's integral inequality

$$2 \le \langle n_2\rangle^4 \left(\int_{t_1}^{t_2} \left(\int_{-\infty}^{t_2} e^{-2(\tau-u_2)\langle n_2\rangle^2} du_2 \right)^{1/2} d\tau \right)^2$$

$$\lesssim \langle n_2\rangle^4 \left(\int_{t_1}^{t_2} \left(\int_{-\infty}^{\tau} e^{-2(\tau-u_2)\langle n_2\rangle^2} du_2 \right)^{1/2} d\tau \right)^2$$

$$\lesssim \frac{\min(\langle n_2\rangle^4 |t_1-t_2|^2, 1)}{\langle n_2\rangle^2} \lesssim \frac{|t_1-t_2|^{\lambda}\langle n_2\rangle^{2\lambda}}{\langle n_2\rangle^2}, \ \lambda\in[0,1]$$

$$\Rightarrow \mathbb{E}[|III(t_1,t_2,n)|^2] \lesssim \sum_{\substack{n_1,n_2\\ n=n_1+n_2}} \frac{1}{\langle n_1\rangle^4} \frac{1}{\langle n_2\rangle^{2-2\varepsilon}} |t_1-t_2|^{\varepsilon}$$

$$\text{(Lemma 4.2)} \lesssim |t_1-t_2|^{\varepsilon} \langle n\rangle^{-3+2\varepsilon\,(=-3-2(0)+2\varepsilon)} \qquad (\lambda=\theta=\varepsilon)$$

Applying Proposition 3-6,

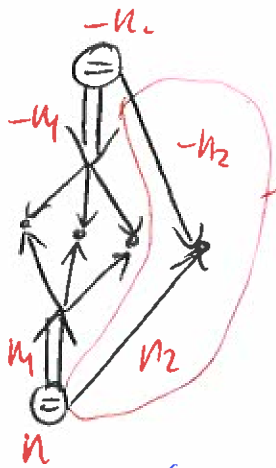
$$\Rightarrow \check{\mathbb{I}}(t) \in C(\mathbb{R}_+ ; C^{\beta}(\mathbb{T}^3)) \text{ for all } \beta < 0 - \varepsilon.$$

Taking ε arbitrarily small $\Rightarrow \beta < 0$.

By slightly modifying the argument in estimating piece ②, we can show:

Lemma: $$\| \hat{S}_{t_1-u}(n) - \hat{S}_{t_2-u}(n) \|_{L^q(du)} \lesssim \frac{\min(1, |t_1-t_2|^{\varepsilon/q} \langle n \rangle^{2\varepsilon/q})}{\langle n \rangle^{2/q}}, \quad \varepsilon \in [0,1].$$

$$\mathbb{E}[\,|\,\mathbb{I}(t_1,t_2,n)|^2] \lesssim$$



$\lesssim \frac{1}{\langle n_2 \rangle^2}$.

As in Υ, the inner portion is bounded by

$$\iint \frac{(\hat{S}_{t_1-u}(n_1) - \hat{S}_{t_2-u}(n_1))(\hat{S}_{t_1-u'}(n_1) - \hat{S}_{t_2-u'}(n_1))}{|u-u'|\,\langle n_1 \rangle^{2\gamma}}\, du\, du'$$

H-L-S

$$\lesssim \frac{1}{\langle n_1 \rangle^{2\gamma}} \| \hat{S}_{t_1-u}(n_1) - \hat{S}_{t_2-u}(n_1) \|_{L^2(du)} \| \hat{S}_{t_1-u}(n_1) - \hat{S}_{t_2-u}(n_1) \|_{L^q(du)}$$

$$\frac{1}{q} = \frac{3}{2} - \gamma.$$

(By lemma) $$\lesssim \frac{1}{\langle n_1\rangle^{2\gamma+1+3-2\gamma}} \cdot |t_1-t_2|^{\varepsilon}\langle n_1\rangle^{2\varepsilon}$$

$$\lesssim |t_1-t_2|^{\varepsilon}\langle n_1\rangle^{-4+2\varepsilon}, \quad \text{for small enough } \varepsilon.$$

$$\Rightarrow \mathbb{E}\left[|I(t_1,t_2,n)|^2\right] \lesssim \sum_{\substack{|n_1|\sim|n_2| \\ n=n_1+n_2}} \frac{|t_1-t_2|^{\varepsilon}}{\langle n_1\rangle^{4-2\varepsilon}\langle n_2\rangle^{2}} \lesssim |t_1-t_2|^{\varepsilon}\langle n\rangle^{-3+2\varepsilon}.$$

Prop^{n} 3-6

$\Rightarrow$  $\in C(\mathbb{R}_+; C_x^{\beta}(\mathbb{T}^3)), \ \beta<0.$

Remark: For time differencing, the ⇑ double arrows (a time difference) only appear on branches that directly connect to the root node $(t,n).(\bullet, \ominus)$.
Hence, one could have an arbitrarily complex tree, but for time differencing purposes (i.e. the propagation of the spatial regularity), only branches near the root node matter.
Of course, estimating the spatial regularity for a fixed time, requires one to study the whole tree.

Appendix:

Moments of Complex-Valued Gaussian r.v.s

In this appendix, we give a proof of the following fact:

Let $g \sim \mathcal{N}_{\mathbb{C}}(0,1)$, $k, j \in \mathbb{Z}_{\geq 0}$. Then
$$\mathbb{E}[g^k \bar{g}^j] = k!\,\delta_{kj}. \quad \text{--- } (*)$$

Since $g \sim \mathcal{N}_{\mathbb{C}}(0,1)$, $\mathrm{Re}\,g, \mathrm{Im}\,g \sim \mathcal{N}_{\mathbb{R}}(0, 1/2)$ and are independent.

$$\Rightarrow 2X := 2[(\mathrm{Re}\,g)^2 + (\mathrm{Im}\,g)^2]$$
$$\sim \chi^2(2) \rightarrow \text{Chi-Squared of degree 2}$$

$$\Rightarrow \mathbb{E}[e^{tX}] = \frac{1}{1-t} = \sum_{j=0}^{\infty} t^j, \quad 0<t<1. \text{ --- } (1)$$

(Alternatively, we can directly compute
$$\mathbb{E}[e^{tX}] \underset{\substack{\downarrow \\ \text{independence}}}{=} \mathbb{E}[e^{t(\mathrm{Re}\,g)^2}]\,\mathbb{E}[e^{t(\mathrm{Im}\,g)^2}] = \left(\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{tx^2} e^{-x^2} dx\right)^2$$
$$= \frac{1}{1-t} \quad (0<t<1).)$$

On the other hand,
$$\mathbb{E}[e^{tX}] = \mathbb{E}[e^{t|g|^2}] = \sum_{j=0}^{\infty} \frac{t^j}{j!}\mathbb{E}[|g|^{2j}] \text{ --- } (2)$$

Comparing coefficients of (1) and (2) implies the "diagonal" case of $(*)$, namely, $j=k$.

Now suppose $k \neq j$.

2

By transformation rules for random variables, we can write

$$g = Re^{i\Theta},$$

where

- $R \sim \text{Rayleigh}(1/\sqrt{2})$ $\left(R = \sqrt{(\text{Re}\, g)^2 + (\text{Im}\, g)^2}\right)$
- $\Theta \sim \mathcal{U}(0, 2\pi)$
- R and Θ are independent.

Then

$$\mathbb{E}[g^k \bar{g}^j] = \mathbb{E}[R^{k+j} e^{i(k-j)\Theta}]$$

$$\text{(Indep.)} = \mathbb{E}[R^{k+j}]\, \underbrace{\mathbb{E}[e^{i(k-j)\Theta}]}_{=0 \text{ as } k \neq j}.$$