MIGSAA Extended Project

Dynamics near the ground state energy for the Cubic Nonlinear Klein-Gordon equation in three dimensions

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Abstract

In this report, we detail some of the results obtained by Nakanishi, Schlag in their theory of the global behaviour of solutions to the radial cubic non-linear Klein-Gordon equation in three dimensions in the energy space $H^1 \times L^2$. We describe the proof of scattering for certain global solutions with energies less than that of the ground state, Q. For solutions with energy only slightly above the ground state energy, we describe the complete classification of the global in time behaviour. This is a combination of trapping by Q, scattering to 0 or finite time blow-up. The tools used involve constructing center-stable manifolds and analysing the unstable modes of the solution culminating in a crucial One-Pass Theorem.

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Chapter 1

Introduction

Our goal here is to study the long time behaviour of solutions to the focusing[1] non-linear Klein-Gordon equation (NLKG)

$$\begin{gathered}\partial_t^2 u - \Delta u + u - u^3 = 0, \\ (u, \partial_t u)|_{t=0} = (u(0), \partial_t u(0)) \in \mathcal{H} := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3).\end{gathered} \tag{1.0.1}$$

The NLKG is a Hamiltonian PDE with Hamiltonian (or energy)

$$E(u(t), \partial_t u(t)) := \int_{\mathbb{R}^3} \frac{1}{2}u^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{4}u^4 \,\mathrm{d}x, \tag{1.0.2}$$

As the energy is autonomous in time, we have the conservation

$$E(u(t), \partial_t u(t)) = E(u(0), \partial_t u(0)),$$

for all t in the lifespan of u. In the field of dispersive PDEs, we say that (1.0.1) is *sub-critical* with respect to this energy in the sense that the non-linear contribution, the u^4 term, can be controlled by the other terms by Sobolev embedding.

Amongst all solutions of (1.0.1), there is a distinguished one known as the *ground state*, denoted by Q. It is the unique time-independent, radial, positive weak solution in $H^1(\mathbb{R}^3)$ of

$$-\Delta Q + Q - Q^3 = 0.$$

It exhibits exponential decay as $|x| \to \infty$ and by elliptic regularity, it is smooth and hence a classical solution. Furthermore, it is the positive solution of the time-independent NLKG that minimizes the stationary energy

$$J(\varphi) := \int_{\mathbb{R}^3} \frac{1}{2}\varphi^2 + \frac{1}{2}|\nabla \varphi|^2 - \frac{1}{4}\varphi^4 \,\mathrm{d}x. \tag{1.0.3}$$

The ground state plays a pivotal role in the long-time dynamics of (1.0.1).

The seminal work of Payne, Sattinger [1] described the dichotomy in behaviour of solutions of (1.0.1) with energy below the ground state energy, that is, those u such that

$$E(u, \partial_t u) < J(Q).$$

[1] The corresponding theory for the defocusing case is far simpler and is classical. We will thus only consider the focusing case here.

The distinction here is dependent upon the sign of the functional

$$K_0 := \int_{\mathbb{R}^3} \varphi^2 + |\nabla\varphi|^2 - \varphi^4 \,\mathrm{d}x, \tag{1.0.4}$$

which gives rise to two disjoint regions of $\mathcal{H}$, labelled as $\mathcal{PS}_\pm$, and defined by

$$\mathcal{PS}_+ := \{(u_0, u_1) \in \mathcal{H} \,|\, E(u_0, u_1) < J(Q),\ K_0(u_0) \geq 0\} \tag{1.0.5}$$

$$\mathcal{PS}_- := \{(u_0, u_1) \in \mathcal{H} \,|\, E(u_0, u_1) < J(Q),\ K_0(u_0) < 0\} \tag{1.0.6}$$

The following theorem is the key result of the theory of Payne, Sattinger.

Theorem 1.1. *The regions $\mathcal{PS}_\pm$ are invariant under the flow of NLKG in the following sense: if $(u(0), \dot{u}(0)) \in \mathcal{PS}_\pm$, then the solution to NLKG with this initial data $(u(t), \dot{u}(t)) \in \mathcal{PS}_\pm$ for as long as the solution exists.*
Furthermore, solutions to NLKG which lie in $\mathcal{PS}_+$ exist for all times, whereas those in $\mathcal{PS}_-$ blow up in finite time in both temporal directions.

That solutions in $\mathcal{PS}_+$ lead to global evolutions is an easy consequence of the control on the L^4 norm that (1.0.4) provides and energy conservation. The blow-up result follows from a convexity argument. We devote Chapter 2 of this manuscript to detailing the improvement that Ibrahim, Masmoudi and Nakanishi [2] made to the Payne–Sattinger theory, namely the proof of scattering for data in $\mathcal{PS}_+$.

It is natrual to ask what about the behaviour of solutions having energies $E(\vec{u}) \geq J(Q)$? For the case of the wave equation, Duyckaerts, Merle [3] obtained a surprising result for the case when $E(\vec{u}) = J(Q)$. They showed that the only solutions, modulo symmetries, are the trivial ground state itself and two others: both scattering to Q in forward time, while one scatters to zero and the other blows-up in backward time. Understanding of the dynamics above the ground state was completely unknown. In the radial setting, Nakanishi and Schlag [4] tackled the problem of solutions with energy at most *slightly* above the ground state; that is solutions in the space

$$\mathcal{H}^\epsilon := \{\vec{u} \in \mathcal{H} \,:\, E(\vec{u}) < J(Q) + \epsilon^2\}, \tag{1.0.7}$$

for some $\epsilon \ll 1$. In this perturbative regime about Q, the phase space $\mathcal{H}_{\mathrm{rad}}$ splits into nine distinct regions with combinations of them corresponding to the center and center-stable/unstable manifolds.

Theorem 1.2. (Nakanishi, Schlag [4]) *The set of solutions to (1.0.1) with radial initial data $\vec{u}(0) \in \mathcal{H}_{rad}$ splits into nine non-empty sets characterised as follows:*

1. *Scattering to 0 for both $t \to \pm\infty$*
2. *Finite time blow-up on both sides of $\pm t > 0$*
3. *Scattering to 0 as $t \to \infty$ and finite time blow-up in $t < 0$*
4. *Finite time blow-up in $t > 0$ and scattering to 0 as $t \to -\infty$*
5. *Trapped by $\pm Q$ for $t \to \infty$ and scattering to 0 as $t \to -\infty$*
6. *Scattering to 0 as $t \to \infty$ and trapped by $\pm Q$ as $t \to -\infty$*
7. *Trapped by $\pm Q$ for $t \to \infty$ and finite time blow-up in $t < 0$*
8. *Finite time blow-up in $t > 0$ and trapped by $\pm Q$ as $t \to -\infty$*
9. *Trapped by $\pm Q$ as $t \to \pm\infty$,*

Here "trapped by Q" means that the solution is contained within an $\mathcal{O}(\epsilon)$ neighbourhood of $(\pm Q, 0)$ forever after some time (or before some time). The sets $(5) \cup (7) \cup (9)$ and $(6) \cup (8) \cup (9)$ are codimension one Lipschitz manifolds in $\mathcal{H}_{rad}$ corresponding to the center-stable manifold W^{cs} and center-unstable manifold W^{cu}, respectively, around $(\pm Q, 0)$. Their intersection, (9), defines the center manifold W^c.

The classification given by Theorem 1.2 is the goal of Chapters 3 and 4. The former focuses on the stable behaviour about the ground states, namely the construction of the center-stable manifold and the properties of solutions which reside on it. The arguments here are a melding of dispersive theory and dynamical systems, with a dash of spectral theory. The latter chapter however discusses the behaviour away from the ground states which is dominated by the unstable modes of the solution. The analysis is heavily based on ideas from the theory of ODEs and the results are given more in the context of the language of orbital stability. The crucial result is the One-Pass Theorem which constrains the dynamics near the ground states.

Here we will only have space to discuss the scattering result [2] in the Payne-Sattinger theory and the results slightly above the ground state for the radial NLKG [4], both of which are detailed in [5] which we follow. However, the theory initiated by Nakanishi and Schlag is robust enough to be applicable to other dispersive equations of interest. The first generalisation was extending the results we describe here for NLKG to the non-radial setting [6], which involves a subtle redefinition of (1.0.7) due to the presence of Lorentz transformations. Kreiger and the previous two authors [7] considered the 1D NLKG which presents a difficulty due to weaker dispersion. Nakanishi and Schlag [8] provided a similar theory for the 3D radial cubic NLS. More recently, Nakanishi and Roy have described the radial dynamics in the critical cases for NLKG in three and five dimensions [9] and NLS in three dimensions [10]. Similar results have also been obtained for the critical 3D wave equation [11, 12]. The philosophies of the previous works have also been usful for studying to dispersive PDE with potentials [13–15] and to generalised NLS equations [16].

Chapter 2

Scattering below the ground state

Our goal in this chapter will be to present in detail the proof of scattering below the ground state for radial solutions in $\mathcal{PS}_+$. This requires a fair amount of machinery. The simplest ingredient is the small data scattering result which is a consequence of the small data global existence theory. This result guarantees the existence of at least some nontrivial scattering solutions. We state it here without proof.

Proposition 2.1. *For any $(u(0), \partial_t u(0)) \in \mathcal{H}$, there exists a unique strong solution $u \in C([0,T) \times H^1) \cap C^1([0,T); L^2)$ to the non-linear Klein-Gordon equation (NLKG)*

$$\partial_t^2 u - \Delta u + u - u^3 = 0, \ \ (u, \partial_t u)|_{t=0} = (u(0), \partial_t u(0)) \in \mathcal{H}, \tag{2.0.1}$$

for some $T = T(\|\vec{u}(0)\|_{\mathcal{H}}) > 0$. If $\|\vec{u}(0)\|_{\mathcal{H}} \ll 1$, then the solution exists globally in time and satisfies

$$\|u\|_{L_t^3([0,\infty);\ L^6(\mathbb{R}^3))} \lesssim \|\vec{u}(0)\|_{\mathcal{H}}.$$

If $T^* > 0$ is the maximal forward time of existence, then $T^* < \infty$ implies that

$$\|u\|_{L_t^3([0,T^*);\ L^6(\mathbb{R}^3))} = \infty.$$

If $T^* = \infty$ and $\|u\|_{L_t^3([0,T^*);\ L^6(\mathbb{R}^3))} < \infty$, then u scatters in the following sense: there exists $(v_0, v_1) \in \mathcal{H}$ such that with $v(t) = S_0(t)(v_0, v_1)$, one has

$$\|u(t) - v(t)\|_{\mathcal{H}} \to 0, \ \ \text{as } t \to \infty,$$

where $S_0(t)$ is the linear (free) Klein-Gordon evolution. If u scatters, then $\|u\|_{L_t^3([0,\infty);\ L^6(\mathbb{R}^3))} < \infty$.

The next key ingredient is the profile decomposition of Bahouri-Gerard [17]. The development of this technology is largely the reason for the 36 year gap between the works of Payne-Sattinger [1] and Ibrahim-Masmoudi-Nakanishi [2]. Essentially, any sequence of solutions to NLKG in $\mathcal{H}$ has a subsequence which asymptotically splits into individually localised pieces while also conserving the energy. This phenomenon is responsible for the lack of compactness of such sequences. The final two ingredients are a perturbation lemma and a finer variational characterisation of the ground state which is given by Lemma 2.2. We will then combine these results to give the scattering proof for radial data. Scattering also occurs for non-radial initial data however some modifications are required. For details, see Section 2.4.4 in Nakanishi, Schlag [5].

Lemma 2.2. *One has*

$$\begin{aligned} J(Q) &= \inf\{J(\varphi) : K_0(\varphi) = 0,\, \varphi \in H^1\backslash\{0\}\} = \inf\{G_0(\varphi) : K_0(\varphi) \leq 0,\, \varphi \in H^1\backslash\{0\}\} \\ &= \inf\{J(\varphi) : K_2(\varphi) = 0,\, \varphi \in H^1\backslash\{0\}\} = \inf\{G_2(\varphi) : K_2(\varphi) \leq 0,\, \varphi \in H^1\backslash\{0\}\} \end{aligned} \tag{2.0.2}$$

where $G_0(\varphi) := (J - \frac{1}{4}K_0)(\varphi) = \frac{1}{4}\|\varphi\|_{H^1}^2$ *and* $G_2(\varphi) = (J - \frac{1}{3}K_2)(\varphi) = \frac{1}{6}\|\nabla\varphi\|_2^2 + \frac{1}{2}\|\varphi\|_2^2$. *The minimizers are exactly* $\pm Q(\cdot + x_0)$ *where* $x_0 \in \mathbb{R}^3$ *is arbitrary. The sets*

$$\begin{aligned} &\{\varphi \in H^1 : J(\varphi) < J(Q),\, K_j(\varphi) \geq 0\}, \\ &\{\varphi \in H^1 : J(\varphi) < J(Q),\, K_j(\varphi) < 0\} \end{aligned} \tag{2.0.3}$$

do not depend on the choice of $j = 0$ *or* $j = 2$. *Moreover, if* $J(\varphi) < J(Q)$ *and* $K_2(\varphi) < 0$, *then*

$$-K_2(\varphi) \geq 2\,(J(Q) - J(\varphi)) \tag{2.0.4}$$

and if $J(\varphi) < J(Q)$ *and* $K_2(\varphi) \geq 0$, *then*

$$K_2(\varphi) \geq c_0 \min\left(J(Q) - J(\varphi), \|\nabla\varphi\|_2^2\right) \tag{2.0.5}$$

for some absolute constant $c_0 > 0$.

2.1 Profile decomposition

The profile decomposition is intimately connected to the symmetries of the underlying equation. For radial NLKG, the notable[1] symmetries are time translations, spatial rotations and Lorentz transformations, which form a subgroup of the full Poincare group. As we seek to represent a general bounded sequence of free KG solutions by a number of *fixed* profiles, we have to take into account these symmetries. Fortunately, we can forget about the symmetries forming compact subgroups which in our case would be the spatial rotations. This is because we can pass to a limit and thus incorporate these into the fixed profiles anyway. We may also forget about the Lorentz transformations, $L(\tilde{v})$, with boost velocity $\tilde{v}$. Composing any free KG solution with $L(\tilde{v})$ will increase the kinetic energy as we send $\tilde{v} \to 1$.

Theorem 2.3 (Radial, linear profile decomposition)**.** *Let* $u_n(t) = S_0(t)\vec{u}_n(0)$ *be a sequence of free radial Klein-Gordon solutions bounded in* $\mathcal{H} := H^1 \times L^2$. *Then, possibly after replacing it with a subsequence, there exist a sequence of free solutions* v^j *bounded in* $\mathcal{H}$, *and a sequence of times* $t_n^j \in \mathbb{R}$ *such that*

1. *For every* $j < k$,
$$\lim_{n\to\infty} |t_n^j - t_n^k| = \infty. \tag{2.1.1}$$
2. *For every* $k \geq 1$,
$$u_n(t) = \sum_{j=1}^{k} v^j(t + t_n^j) + \gamma_n^k(t). \tag{2.1.2}$$

   where for every $j < k$, *the errors* γ_n^k *satisfy* $\vec{\gamma}_n^k(-t_n^j) \rightharpoonup 0$ *in* $\mathcal{H}$ *as* $n \to \infty$ *and asymptotically vanish in the sense that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma_n^k\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \textit{for all } 2 < p < 6. \tag{2.1.3}$$

[1] We do not need to consider reflections $u \to -u$.

Moreover, we have orthogonality of the free energy

$$\|\vec{u}_n\|_{\mathcal{H}}^2 = \sum_{j=1}^{k} \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{\gamma}_n^k\|_{\mathcal{H}}^2 + o_n(1) \tag{2.1.4}$$

as $n \to \infty$.

Remark 1. As the energy is conserved globally under the free KG-flow, we are free to evaluate (2.1.4) at any time t. Usually the most convenient is $t = 0$.

2. The v^j are known as limiting profiles, asymptotic profiles or free concentrating waves and we say that two sequences $\{t_n^j\}$ and $\{t_n^k\}$ are orthogonal if (2.1.1) holds.
3. Notice that the errors γ_n^k are free KG solutions which are uniformly "small" in n in the Strichartz norm, but not in the energy norm.

Proof. The proof follows in a few steps which we outline for clarity.
Step 1: Reduction for (2.1.3)
We claim that it is sufficient to estimate the error γ_n^k in $L_t^\infty L_x^p$ for any fixed $2 < p < 6$. Fix such a p and suppose that $\gamma_n^k \in L_t^\infty L_x^p$. As γ_n^k is bounded in H_x^1 uniformly in time[2] then by the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, $\gamma_n^k \in L_t^\infty L_x^6$. Now as each γ_n^k is a free solution, the Strichartz estimate of Lemma 2.46 in [5], $\gamma_n^k \in L_t^2 B_{6,2}^{\frac{1}{6}}$. Using the definition of the Besov spaces and the Littlewood-Payley square function estimate we have

$$\|\gamma_n^k\|_{L_x^6} \simeq \| \|P_N \gamma_n^k\|_{l_N^2} \|_{L_x^6} \leq \| \|P_N \gamma_n^k\|_{L_x^6} \|_{l_N^2} \leq \|2^{N/6} \|P_N \gamma_n^k\|_{L_x^6} \|_{l_N^2} = \|\gamma_n^k\|_{B_{6,2}^{\frac{1}{6}}},$$

which implies that $\gamma_n^k \in L_t^2 L_x^6$. Using interpolation of space-time Lebesgue spaces (see Lemma A.1), we see that for any $2 < p < 6$, $\gamma_n^k \in L_t^3 L_x^6$.
Step 2: Construction of the profiles and verification of orthogonality
Set $\gamma_n^0 := u_n$ and define

$$\nu^1 := \limsup_{n \to \infty} \|u_n\|_{L_t^\infty L_x^p}.$$

Suppose that $\nu^1 > 0$. Then we can find a sequence $t_n^1 \in \mathbb{R}$ such that

$$\|\gamma_n^0(-t_n^1)\|_{L_x^p} \geq \frac{1}{2}\nu^1.$$

Passing to a subsequence, we may assume that $t_n^1 \to t_\infty^1 \in [-\infty, \infty]$ as $n \to \infty$. Now since $\vec{\gamma}_n^0(-t_n^1) \in \mathcal{H}$ [3] is bounded, there exists a weakly converging subsequence, which we still label $\vec{\gamma}_n^0(-t_n^1)$, in $\mathcal{H}$. Furthermore, since u_n is radial and the embedding $H_{\text{rad}}^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact, there exists a further subsequence that converges strongly in L_x^p. We define

$$\vec{v}^1(0) := \lim_{n \to \infty} \vec{\gamma}_n^0(-t_n^1),$$

as the strong L_x^p limit. Notice that a priori we do not know if the weak H^1 limit $\gamma_n^0(-t_n^1)$ is equal to the strong L^p limit $v^1(0)$. This potential issue in fact does not occur due to Lemma A.2.

Applying this result we see that $\gamma_n^0(-t_n^1) \rightharpoonup v^1(0)$ in H^1 and hence $\vec{v}^1(0) \in \mathcal{H}$, and thus can be used as initial data for the free KG equation. We define

$$v^1(t) := S_0(t)\vec{v}^1(0) \tag{2.1.5}$$

[2] This will be apparent from Step 2 as a consequence of u_n being bounded as we define the errors iteratively $\gamma_n^k(t) := \gamma_n^{k-1} - v^k(t + t_n^k)$, with $\gamma_n^0 = u_n$.

[3] Remember this is just u_n, a free solution, that is uniformly bounded in H^1 by assumption.

as the global in-time solution forming the first profile. That $\vec{v}^1(0)$ is in fact radial follows from Lemma A.3.

Furthermore, one can show that radial initial data for the free KG equation launches radial solutions. This is obvious from the explicit solution formula

$$u(x,t) = \int_{\mathbb{R}^3} \cos(t\langle\xi\rangle)\hat{u}_0(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi + \int_{\mathbb{R}^3} \frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}\hat{u}_1(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi.$$

Therefore $v^1(t)$ is a radial free KG solution. By Sobolev embedding,

$$\frac{1}{2}\nu^1 \le \liminf_{n\to\infty} \|\gamma_n^0(-t_n^1)\|_{L_x^p} = \|v^1(0)\|_{L_x^p} \lesssim \|v^1(0)\|_{H^1}. \tag{2.1.6}$$

On the other hand, if $\nu^1 = 0$, then u_n already converges and we do not need to proceed, as this would imply that $\vec{v}^1(0) = 0$ and by uniqueness of solutions to the free KG (following from finite speed of propagation), $v^1(t) \equiv 0$. Hence we would take $v^j(t) \equiv 0$ for all j, and set $\gamma_n^l = \gamma_n^0$ for all $l > 0$.
Set

$$\gamma_n^1(t) := \gamma_n^0(t) - v^1(t + t_n^1), \tag{2.1.7}$$

and note that

$$\vec{\gamma}_n^1(-t_n^1) = \vec{\gamma}_n^0(-t_n^1) - \vec{v}^1(0) \rightharpoonup 0, \text{ in } \mathcal{H}.$$

Also this implies that $\gamma_n^1(t)$ is a radial free KG solution. At this stage we have the decomposition

$$u_n(t) = v^1(t + t_n^1) + \gamma_n^1(t).$$

We now move onto the k-th step in the iteration. Set

$$\nu^k := \limsup_{n\to\infty} \|\gamma_n^{k-1}\|_{L_t^\infty L_x^p}.$$

If $\nu^k = 0$, then by the same arguments as before, we get $v^k \equiv 0$ and take $v^j \equiv 0$ and $\gamma_n^j = \gamma_n^{k-1}$ for all $j \ge k$. So suppose that $\nu^k > 0$. Then there exists a sequence $t_n^k \in \mathbb{R}$, with $t_n^k \to t_\infty^k \in [-\infty, \infty]$ and such that

$$\|\gamma_n^{k-1}(-t_n^k)\|_{L_x^p} \ge \frac{1}{2}\nu^k.$$

Now since $\vec{\gamma}_n^{k-1}(-t_n^k) \in \mathcal{H}$ is bounded, there exists a weakly converging subsequence, in $\mathcal{H}$. Furthermore, since γ_n^k is radial and the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact, there exists a further subsequence that converges strongly in L_x^p. We define

$$\vec{v}^k(0) := \lim_{n\to\infty} \vec{\gamma}_n^{k-1}(-t_n^k),$$

as the strong L_x^p limit which is bounded in $\mathcal{H}$. Then we obtain the k-th profile v^k by the free KG flow of $\vec{v}^k(0)$, i.e.

$$v^k(t) := S_0(t)\vec{v}^k(0). \tag{2.1.8}$$

Again by the Sobolev embedding,

$$\frac{1}{2}\nu^k \lesssim \|v^k(0)\|_{H^1}.$$

We set

$$\gamma_n^k(t) := \gamma_n^{k-1}(t) - v^k(t + t_n^k), \tag{2.1.9}$$

and notice that

$$\vec{\gamma}_n^k(-t_n^k) = \vec{\gamma}_n^{k-1}(-t_n^k) - \vec{v}^k(0) \rightharpoonup 0, \quad \text{in } \mathcal{H}.$$

At this stage we have the decomposition

$$u_n(t) = \sum_{j=1}^{k} v^j(t+t_n^j) + \gamma_n^k(t).$$

We verify the orthogonality condition. Suppose otherwise that for some $j<k$, $t_n^j - t_n^k \to c \in \mathbb{R}$ as $n\to\infty$[4] . Repeating (2.1.9) a finite number of times, we obtain

$$\vec{\gamma}_n^j(t) = \vec{\gamma}_n^k(t) + \sum_{l=j+1}^{k} \vec{v}^l(t+t_n^l), \tag{2.1.10}$$

and hence

$$\vec{\gamma}_n^j(-t_n^k) = \vec{\gamma}_n^k(-t_n^k) + \vec{v}^k(0) + \sum_{l=j+1}^{k-1} \vec{v}^l(t_n^l - t_n^k). \tag{2.1.11}$$

We first claim that $\vec{\gamma}_n^j(-t_n^k) \rightharpoonup 0$ in $\mathcal{H}$. By density, it suffices to show that for all $\vec{\phi}=(\phi_1,\phi_2)\in \mathcal{S}(\mathbb{R}^3)\times\mathcal{S}(\mathbb{R}^3)$,

$$\left\langle \vec{\gamma}_n^1(-t_n^2)\,|\,\vec{\phi}\right\rangle \to 0, \quad \text{as } n\to\infty.$$

By unitarity of the free propagator, we have

$$\begin{aligned}\left\langle \vec{\gamma}_n^j(-t_n^k)\,|\,\vec{\phi}\right\rangle &= \left\langle \vec{\gamma}_n^1(-t_n^j + (t_n^j - t_n^k))\,|\,\vec{\phi}\right\rangle\\ &= \left\langle S_0(t_n^j - t_n^k)\vec{\gamma}_n^1(-t_n^j)\,|\,\vec{\phi}\right\rangle\\ &= \left\langle \vec{\gamma}_n^1(-t_n^j)\,|\,S_0(t_n^k - t_n^j)\vec{\phi}\right\rangle.\end{aligned}$$

Now $S_0(t_n^k - t_n^j)\vec{\phi} \to S_0(-c)\vec{\phi}$ strongly in L^2 by Plancherel's identity as $n\to\infty$, and by construction, $\vec{\gamma}_n^k(-t_n^j)\rightharpoonup 0$ in $\mathcal{H}$. Using Lemma A.4 we verify that $\gamma_n^k(-t_n^k)\rightharpoonup 0$ in $\mathcal{H}$ as $n\to\infty$. We also have for all $l\in\{j+1,\dots,k-1\}$, $\vec{v}^l(t_n^l - t_n^k)\rightharpoonup 0$ in $\mathcal{H}$ as $n\to\infty$. This follows by a similar argument except we make use of the pointwise decay estimate for free KG solutions as in Section 2.5 of [5]. By density, we can find $\{\vec{\psi}_m^l\}_m = \{(\psi_{m,1}^l, \psi_{m,2}^l)\}$ such that $\vec{\psi}_m^l \to \vec{v}^l(0)$ in $\mathcal{H}$. We compute, with $\vec{\phi}\in\mathcal{S}^2(\mathbb{R}^3)$,

$$\begin{aligned}\left|\left\langle \vec{v}^l(t_n^l - t_n^k)\,|\,\vec{\phi}\right\rangle\right| &= \left|\left\langle S_0(t_n^l - t_n^k)\vec{v}^1(0)\,|\,\vec{\phi}\right\rangle\right|\\ &= \left|\left\langle \vec{v}^1(0) - \vec{\psi}_m^l\,|\,S_0(t_n^k - t_n^l)\vec{\phi}\right\rangle\right| + \left|\left\langle \vec{\psi}_m^l\,|\,S_0(t_n^k - t_n^l)\vec{\phi}\right\rangle\right|\\ &\le \|S_0(t_n^k - t_n^l)\vec{\phi}\|_{\mathcal{H}}\|\vec{v}^1(0) - \vec{\psi}_m^l\|_{\mathcal{H}} + \|S_0(t_n^k - t_n^l)\vec{\phi}\|_{W^{1,\infty}\times L^\infty}\|\vec{\psi}_m^l\|_{W^{1,1}\times L^1}\\ &\lesssim \|\vec{\phi}\|_{\mathcal{H}}\|\vec{v}^1(0) - \vec{\psi}_m^l\|_{\mathcal{H}} + \frac{1}{|t_n^k - t_n^l|^{3/2}}\|\vec{\psi}_m^l\|_{W^{1,1}\times L^1}.\end{aligned}$$

Taking $n\to\infty$ and then $m\to\infty$ implies $\vec{v}^l(t_n^l - t_n^k)\rightharpoonup 0$ in $\mathcal{H}$. Thus taking the weak limit in $\mathcal{H}$ of (2.1.11) as $n\to\infty$, we find $\vec{v}^k(0)=0$ which is a contradiction. Proceeding onwards, if we only have a finite number of profiles then we restrict the sequence indexing to that of the final subsequence. If we have an infinite number of profiles, we extract the diagonal subsequence.

Step 3: Energy splitting (2.1.4)

As the free energy is conserved by the free KG flow, it suffices to verify (2.1.4) for $t=0$. We have

$$\|\vec{u}_n(0)\|_{\mathcal{H}}^2 = \left\|\sum_{j=1}^{k} \vec{v}^j(t_n^j) + \vec{\gamma}_n^k(0)\right\|_{\mathcal{H}}^2.$$

[4] If there were in fact more than one j for which we had convergence in $\mathbb{R}$, for example, if we had $j_1<j_2$, then we need only apply the argument to the largest of the two, j_2. All other times diverge against t_n^k.

Expanding the scalar product in $\mathcal{H}$, we find that the cross terms are of two forms ($j \neq l$):

$$\left\langle \vec{v}^j(t_n^j) \,|\, \vec{v}^l(t_n^l) \right\rangle = \left\langle \vec{v}^j(0) \,|\, S_0(t_n^l - t_n^j)\vec{v}^l(0) \right\rangle, \tag{2.1.12}$$

$$\left\langle \vec{v}^j(t_n^j) \,|\, \vec{\gamma}^k(0) \right\rangle = \left\langle \vec{v}^j(0) \,|\, \vec{\gamma}^k(-t_n^j) \right\rangle. \tag{2.1.13}$$

The first of these vanishes by using the pointwise decay estimate as we did above, while the second one vanishes by putting $t = -t_n^k$ into (2.1.10) and using the pointwise decay estimate to show the vanishing of the $\vec{v}^l$ terms. We also have replaced $\vec{v}^j(0)$ by a Schwartz approximation. Therefore these cross terms behave like $o_k(1)$ as $n \to \infty$.
Step 4: Asymptotic vanishing of errors (2.1.3) We denote by a_n^k the sequence of all the cross-terms that were obtained by expanding out $\|\vec{u}_n(0)\|_{\mathcal{H}}^2$. This sequence satisfies, for fixed k, $\lim_{n\to\infty} a_n^k = 0$. Using (2.1.4), we have

$$\begin{aligned}
\limsup_{n\to\infty} \|\vec{u}_n\|_{\mathcal{H}}^2 &= \limsup_{n\to\infty} \left(\sum_{j=1}^k \|\vec{v}^j\|_{\mathcal{H}}^2 + \|\vec{\gamma}_n^k\|_{\mathcal{H}}^2 + a_n^k \right) \\
&= \sum_{j=1}^k \|\vec{v}^j\|_{\mathcal{H}}^2 + \limsup_{n\to\infty} \left(\|\vec{\gamma}_n^k\|_{\mathcal{H}}^2 + a_n^k \right) \\
&\geq \sum_{j=1}^k \|\vec{v}^j\|_{H^1}^2
\end{aligned}$$

Now by construction, for each $j \leq k$, $\|\vec{v}^j(0)\|_{H^1} \lesssim \nu^k$, and hence

$$\limsup_{n\to\infty} \|\vec{u}_n\|_{\mathcal{H}}^2 \geq \sum_{j=1}^k \|\vec{v}^j\|_{H^1}^2 \gtrsim \sum_{j=1}^k (\nu^j)^2,$$

with the bound above being uniform in k. Therefore the series above converges implying $\nu^j \to 0$ as $j \to \infty$ and thus

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma_n^k\|_{L_t^\infty L_x^p(\mathbb{R}\times\mathbb{R}^3)} = 0. \tag{2.1.14}$$

□

2.2 Nonlinear profile decomposition

Suppose $\{\vec{u}_n(t)\}_n$ are a sequence of non-linear radial KG solutions. We want to associate to this sequence a non-linear profile decomposition similar to what we established for sequences of linear radial solutions in the previous section. This is indeed possible.

Theorem 2.4. *Let $\{\vec{u}_n(t)\}_n$ be a sequence of radial NLKG solutions bounded in $\mathcal{H}$. Then, possibly after replacing by a subsequence, there exists a sequence of NLKG solutions $\{\vec{U}^j\}_j$, bounded in $\mathcal{H}$ and a sequence of times $t_n^j \in \mathbb{R}$ such that (2.1.1) holds and we may write*

$$u_n(t) = \sum_{j=1}^k U^j(t + t_n^j) + \gamma_n^k(t) + \eta_n^k(t), \tag{2.2.1}$$

where the errors $\{\gamma_n^k(t)\}_n$ are as in Theorem 2.3 and

$$\|\vec{\eta}_n^k(0)\|_{\mathcal{H}} \to \infty \text{ as } n \to \infty. \tag{2.2.2}$$

Proof. We let $\{\vec{w}_n(t)\}_n$ be a sequence of linear solutions which are launched from the initial data $\{\vec{u}_n(0)\}_n$, that is, the same initial data as for the non-linear evolution. We can apply the linear profile decomposition to $\{\vec{w}_n(t)\}_n$ to obtain

$$\vec{w}_n(t) = \sum_{j=1}^{n} v^j(t + t_n^j) + \vec{\gamma}_n^k(t), \tag{2.2.3}$$

where the $\{\vec{v}^j\}_j$ are the linear profiles and $\vec{\gamma}_n^k$ are the errors. Putting $t = 0$, we get a decomposition for the initial data

$$\vec{u}_n(0) = \sum_{j=1}^{n} v^j(t_n^j) + \vec{\gamma}_n^k(0). \tag{2.2.4}$$

We would like to use the sequence $\{\vec{v}^j(t_\infty^j)\}$ as initial data for the non-linear evolution, namely we solve

$$(\Box + 1)U^j = (U^j)^3, \tag{2.2.5}$$

$$\vec{U}^j(t_\infty^j) = \vec{v}^j(t_\infty^j). \tag{2.2.6}$$

We must be careful here since we cannot directly appeal to the local well-posedness theory to claim existence to the above problem. This is because at most one t_∞^j will actually be finite, all others will be $\pm\infty$. We split the analysis into two cases: when $t_n^j \to t_\infty^j \in \mathbb{R}$ and when $t_n^j \to t_\infty^j = \pm\infty$.

$\boldsymbol{t_n^j \to t_\infty^j} \in \mathbb{R}$: In this case we can appeal to the local well-posedness theory to obtain an interval I about t_∞^j such that U^j exists locally. We now verify the important approximation property

$$\|\vec{U}^j(t_n^j) - \vec{v}^j(t_n^j)\|_{\mathcal{H}} \to 0 \tag{2.2.7}$$

as $n \to \infty$. By the triangle inequality,

$$\begin{aligned}\|\vec{U}^j(t_n^j) - \vec{v}^j(t_n^j)\|_{\mathcal{H}} \leq& \|\vec{U}^j(t_n^j) - \vec{U}^j(t_\infty^j)\|_{\mathcal{H}} + \|\vec{U}^j(t_\infty^j) - \vec{v}^j(t_\infty^j)\|_{\mathcal{H}} \\ &+ \|\vec{v}^j(t_\infty^j) - \vec{v}^j(t_n^j)\|_{\mathcal{H}} \to 0.\end{aligned}$$

The first term tends to zero as $\vec{U}^j$ is continuous in time, the second vanishes identically being the initial data for U^j and the third term vanishes by continuity of $\vec{v}^j$ in time.

$\boldsymbol{t_n^j \to t_\infty^j = \pm\infty}$: We construct the U^j by using a fixed point argument about time $t = \pm\infty$. By the same argument, we only need consider the case when $t_\infty^j = \infty$. We define the solution operator

$$(\Gamma U^j)(t) := S_0(t)v^j + \int_t^\infty \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}(U^j(t'))^3\, dt'.$$

Fix $t \in [T, \infty)$ where T will be chosen so large so that

$$\|S_0(t)v^j\|_{L_t^3([T,\infty);L_x^6)} < \epsilon_0$$

where ϵ_0 is a sufficiently small constant so that the fixed point argument can be applied. Such an ϵ_0 and T exist as the well-posedness theory guarantees that the v^j are free solutions which scatter so their $L_T^3L_x^6 := L_t^3([T,\infty); L_x^6)$ norms can be made arbitrarily small on $[T,\infty)$ by using the Monotone Convergence theorem. We will run the fixed point argument within a ball

$$B_\eta := \{f \in L_t^3([T,\infty); L_x^6) \,:\, \|f\|_{L_t^3([T,\infty);L_x^6)} \leq \eta\} \subset L_t^3([T,\infty); L_x^6).$$

We show first that $\Gamma : B_\eta \to B_\eta$. From the definition of Γ, we have

$$\begin{aligned}\|\Gamma U^j\|_{L^3_T L^6_x} &\le \epsilon_0 + C_{\text{strich}} \|U^j\|^2_{L^3_T L^6_x} \|U^j\|_{L^3_T L^6_x} \\ &\le \epsilon_0 + C_{\text{strich}} \eta^2 \eta.\end{aligned}$$

Choosing η such that $C_{\text{strich}}\eta^2 < 1/2$ we get

$$\|\Gamma U^j\|_{L^3_T L^6_x} \le \epsilon_0 + \frac{1}{2}\eta < \eta,$$

for small enough ϵ_0. We also have the difference estimate

$$\begin{aligned}\|\Gamma U^j - \Gamma V^j\|_{L^3_T L^6_x} &\le C(\|U^j\|^2_{L^3_T L^6_x} + \|V^j\|^2_{L^3_T L^6_x})\|U^j - V^j\|_{L^3_T L^6_x} \\ &\le 2C\eta^2 \|U^j - V^j\|_{L^3_T L^6_x} \\ &\le \frac{1}{2}\|U^j - V^j\|_{L^3_T L^6_x},\end{aligned}$$

by choosing η potentially smaller than before, $\eta^2 < \min(1/4C, 1/2C_{\text{stric}})$. Thus by the contraction mapping theorem, each U^j exists locally around $t = \infty$. For the approximation property, we have

$$\begin{aligned}\|\vec{U}(t^j_n) - \vec{V}^j(t^j_n)\|_{\mathcal{H}} &= \left\| \int_{t^j_n}^\infty \frac{\sin((t^j_n - t')\langle\nabla\rangle)}{\langle\nabla\rangle} (U^j(t'))^3 \, dt' \right\|_{\mathcal{H}} \\ &\le \int_{t^j_n}^\infty \|U^j(t')\|^3_{L^6_x} \, dt' \\ &= \|U^j\|^3_{L^3_t([t^j_n,\infty);L^6_x} \to 0,\end{aligned}$$

as $n \to \infty$ by the Monotone Convergence Theorem. □

2.3 The Perturbation Lemma

Suppose we have a sequence $\{u_n\}_n$ of local NLKG solutions and consider its nonlinear profile decomposition

$$u_n(t) = \sum_{j=1}^k U^j_n(t) + \gamma^k_n(t) + \eta^k_n(t).$$

Suppose that each nonlinear profile U^j_n is a global solutions with finite $L^3_t L^6_x$ norm. By the orthogonality (2.1.1), we can show that, for large n,

$$\left(\sum_j U^j_n\right)^3 = \sum_j \left(U^j_n\right)^3 + o_n(1),$$

where the error vanishes in $L^1_t L^2_x$ as $n \to \infty$ and arises precisely because NLKG does not respect superposition of solutions. So u_n is composed of a global, scattering 'almost NLKG solution' $\sum_j U^j_n$ and a sum of error terms γ^k_n, η^k_n. The Perturbation Lemma then implies that for large n, u_n will be global in time and is uniformly bounded, in n, in $L^3_t L^6_x$, implying scattering. This is where the Perturbation Lemma comes into play for the scattering proof.

Lemma 2.5. *There are continous functions $\epsilon_0, C_0 : (0,\infty) \to (0,\infty)$ such that the following holds: Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded), $u, v \in C(I; H^1) \cap C^1(I; L^2)$ satisfying for some $B > 0$,*

$$\|v\|_{L^3_t(I;L^6_x)} \le B, \tag{2.3.1}$$

$$\|\text{eq}(u)\|_{L^1_t(I;L^2_x)} + \|\text{eq}(v)\|_{L^1_t(I;L^2_x)} + \|w_0\|_{L^3_t(I;L^6_x)} \le \epsilon \le \epsilon_0(B), \tag{2.3.2}$$

where $\mathrm{eq}(u) := \Box u + u - u^3$, *with* $\Box := \partial_t^2 - \Delta$, *is to be understood in the Duhamel sense, and* $\vec{w}_0(t) := S_0(t-t_0)(\vec{u}-\vec{v})(t_0)$ *for some arbitrary but fixed* $t_0 \in I$. *Then*

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L_t^\infty(I;\mathcal{H})} + \|u - v\|_{L_t^3(I;L_x^6)} \leq C_0(B)\epsilon, \tag{2.3.3}$$

and in particular

$$\|u\|_{L_t^3(I;L_x^6)} < \infty. \tag{2.3.4}$$

Proof. We cannot conclude boundedness by a single bootstrap argument as the Strichartz norm of v is not necessarily small. This issue motivates us to partition the interval I so that, on each segment, v will have a small enough Strichartz norm so that the argument will close. Using the fixed $t_0 \in I$ as an anchor and with some absolute and small $\delta_0 > 0$, we partition the right half of I as follows:

$$t_0 < t_1 < \cdots < t_n \leq \infty, \ \ I_j = (t_j, t_{j+1}), \ \ I \cap (t_0, \infty) = (t_0, t_n) \tag{2.3.5}$$

$$\|v\|_{Z(I_j)} \leq \delta_0, \ (j = 0, \dots, n-1) \ \ n \leq C(B, \delta_0), \tag{2.3.6}$$

where for convenience we have set $Z(I) := L^3(I; L^6)$ for any interval I. Estimates on $I \cap (-\infty, t_0)$ are the same by time reversal symmetry and are thus omitted. To be more concrete, we choose the times t_j recursively by setting

$$t_j = \sup\left\{ t > t_{j-1} \ : \ \|v\|_{Z((t_{j-1},t)} \leq \frac{1}{2}\delta_0 \right\}.$$

Then, putting T as the right endpoint of I, we have

$$\int_0^T \|v(t)\|_{L^6}^3 \, dt = \sum_{j=0}^{n-1} \int_{t_{j-1}}^{t_j} \|v(t)\|_{L^6}^3 \, dt \leq \sum_{j=0}^{n-1} \left(\frac{\delta_0}{2}\right)^3 = n\left(\frac{\delta_0}{2}\right)^3.$$

Hence

$$\|v\|_{Z(I)} \leq n^{1/3}\left(\frac{\delta_0}{2}\right).$$

We can now choose $n = n(\delta_0, B)$ such that, say,

$$n^{1/3}\left(\frac{\delta_0}{2}\right) = 2B,$$

which implies that

$$n(\delta_0, B) = \left\lceil \left(\frac{4B}{\delta_0}\right)^3 \right\rceil.$$

The important point here is that if we need to reduce δ_0 in order to get the bootstrap machine started, then we pay the price by increasing the number of sets in our partition. Likewise, if the Strichartz bound on v given by B is large, then again the number of partitions must be large. This little heuristic computation makes the dependencies of n explicit.
We set

$$w := u - v, \ e := (\Box + 1)(u - v) - u^3 + v^3 = \mathrm{eq}(u) - \mathrm{eq}(v),$$

and $\vec{w}_j(t) := S_0(t - t_j)\vec{w}(t_j)$ for each $0 \leq j < n$. We construct iteratively, on each I_j, w such that it is the unique fixed point of the operator

$$\Gamma_j w := w_j(t) + \int_{t_j}^t \frac{\sin((t-s)\langle\nabla\rangle)}{\langle\nabla\rangle}(e + (v+w)^3 - v^3)(s)\mathrm{d}s.$$

Fixed points w will then solve, in the strong sense, $(\Box + 1)w = e + (w+v)^3 - v^3$.

Step 1) We construct the first such w living on the interval I_0. By Strichartz and Hölder inequalities,

$$\begin{aligned}\|\Gamma_0 w\|_{Z_0} &\leq \|w_0\|_{Z_{\mathbb{R}}} + \left\| \int_{t_j}^{t} \frac{\sin((t-s)\langle\nabla\rangle)}{\langle\nabla\rangle}(e+(v+w)^3-v^3)(s)\mathrm{d}s \right\|_{Z_0} \\ &\leq C_1\epsilon + C_1\|e+(w+v)^3-v^3\|_{L^1_t(I_0;L^2_x)} \\ &\leq C_1\epsilon + C_1\|e\|_{L^1_tL^2_x} + C_1\|w\|^3_{L^3_tL^6_x} + C_1\|\|v\|^2_{L^6_x}\|w\|_{L^6_x}\|_{L^1_t} \\ &\leq C\epsilon + C\left(\|v\|^2_{Z_0}+\|w\|^2_{Z_0}\right)\|w\|_{Z_0}.\end{aligned}$$

Using bound (2.3.6), we get

$$\|\Gamma_0 w\|_{Z_0} \leq C\epsilon + \left(\frac{1}{8}+C\|w\|^2_{Z_0}\right)\|w\|_{Z_0},$$

where we have chosen δ_0 so small so that $C\delta_0^2 < \frac{1}{8}$. We now choose ϵ so small so that

$$2^{2(3)}C^3\epsilon^2 < \frac{1}{2}, \tag{2.3.7}$$

and hence

$$\|\Gamma_0 w\|_{Z_0} \leq C\epsilon + \frac{C\epsilon}{2} + \frac{C\epsilon}{2} \leq 2C\epsilon.$$

Thus Γ_0 maps the ball $B_{2C\epsilon} \subset \mathcal{H}$ to itself. Similarly we obtain the difference estimate

$$\|\Gamma_0 w_1 - \Gamma_0 w_2\|_{Z_0} \leq \left(\frac{1}{8}+C\|w_1\|^2_{Z_0}+C\|w_2\|^2_{Z_0}\right)\|w_1-w_2\|_{Z_0} \leq \left(\frac{1}{8}+\frac{1}{2}\right)\|w_1-w_2\|_{Z_0}.$$

This verifies that Γ_0 is a contraction on $B_{2C\epsilon}$ and hence $w(t) = (\Gamma_0 w)(t)$ for all $t \in I_0$. As we now know that w exists on I_0 we would like to obtain some uniform control on it over I_0. Notice that in the construction above, we essentially obtained the bound

$$\|w\|_{Z_0} \leq \|w-w_0\|_{Z_0} + \|w_0\|_{Z(\mathbb{R})} \leq C\epsilon + \left(\frac{1}{8}+C\|w\|^2_{Z_0}\right)\|w\|_{Z_0}.$$

To recap, at Step 1 we *obtain estimates* on the following quantities

$$\|w\|_{Z_0}, \quad \|w_1\|_{Z(\mathbb{R})}$$

using the following *previously known* quantity

$$\|w_0\|_{Z(I)}(\leq \epsilon).$$

The estimates we get at each step for the quantities above will be the same. We show by a continuity argument that in fact

$$\|w\|_{Z_0} \leq 2C\epsilon, \quad \|w\|_{Z(\mathbb{R})} \leq 2C\epsilon.$$

Continuity argument: Since $X(t) := \|w\|_{L^3((t_0,t),L^6)}$ is continuous[5] with $\lim_{t\to t_0} X(t) = 0$, there exists $\delta t > 0$ small such that

$$X(t_0+\delta t) \leq 4C\epsilon.$$

[5] By the dominated convergence theorem (using $\mathbb{1}$ insertion), the function $X(t) := \|\cdot\|_{L^3((\tilde{t},t),L^6)}$ is continuous and $\lim_{t\to\tilde{t}} X(t) = 0$.

By choosing ϵ sufficiently small, our estimate above will imply that we have the strictly better bound

$$X(t_0 + \delta t) \leq 2C\epsilon.$$

By a process of continuation, we therefore conclude that

$$X(t) \leq 2C\epsilon, \quad \text{for all } t \in I_0.$$

Our estimate says

$$X(t) \leq C\epsilon + \left(\frac{1}{8} + CX(t)^2\right) X(t).$$

Plugging in our bootstrap hypothesis $X(t_0 + \delta t) \leq 4C\epsilon$ (writing $t = t_0 + \delta t$), we get

$$X(t) \leq C\epsilon + \frac{4C\epsilon}{8} + C(4C\epsilon)(4C\epsilon)^2 \leq C\epsilon + \frac{1}{2}C\epsilon + C\epsilon(2^{2(3)}C^3\epsilon^2).$$

Here we choose ϵ small enough so that

$$2^{2(3)}C^3\epsilon^2 < \frac{1}{2},$$

which is consistent with (2.3.7). Then we get

$$X(t) \leq C\epsilon + \frac{1}{2}C\epsilon + \frac{1}{2}C\epsilon = 2C\epsilon,$$

and hence

$$\|w\|_{Z_0} \leq 2C\epsilon.$$

Noting that the same estimate for w_1 on $Z(\mathbb{R})$ holds, we also obtain $\|w_1\|_{Z(\mathbb{R})} \leq 2C\epsilon$.
Step j) At step j, we *obtain estimates* on the following quantities

$$\|w\|_{Z_j}, \quad \|w_{j+1}\|_{Z(\mathbb{R})}$$

using the following *previously known* quantity

$$\|w_j\|_{Z(I)} \leq 2^j C\epsilon.$$

Using that w exists so far up to time t_j, we easily solve the fixed point problem (2.3) using essentially the same estimates as in Step 1 and noting that the same choice of ϵ_0 of (2.3.7) will be able to deal with the exponential growth coming from $\|w_j\|_{Z(\mathbb{R})}$. This growth is of no real concern as our time slicing has only a finite number of partitions and we only require boundedness at the end of the day. As before then derive the estimate

$$\|w\|_{Z_j} \leq \|w - w_j\|_{Z_j} + \|w_j\|_{Z(\mathbb{R})} \leq 2^j C\epsilon + \left(\frac{1}{8} + C\|w\|_{Z_j}^2\right) \|w\|_{Z_j}.$$

Continuity argument: We make the bootstrap hypothesis

$$X(t) := \|w\|_{Z_{[t_j,t)}} \leq 2^{j+2}C\epsilon,$$

and we will obtain the better bound

$$X(t) \leq 2^{j+1}C\epsilon.$$

From our basic estimate, we have

$$\begin{aligned} X(t) &\leq 2^j C\epsilon + \frac{2^{j+2}C\epsilon}{8} + C(2^{j+2}C\epsilon)(2^{j+2}C\epsilon)^2 \\ &\leq 2^j C\epsilon + 2^{j-1}C\epsilon + 2^j C\epsilon(2^{2(j+3)}C^3\epsilon^2). \end{aligned}$$

Choosing ϵ so that
$$2^{2(j+3)}C^3\epsilon^2 < \frac{1}{2},$$
we get $X(t) \leq 2^{j+1}C\epsilon$ and thus by continuity, $X(t_{j+1}) \leq 2^{j+1}C\epsilon$.
In order to ensure we can proceed through to the n-th step and all previous ones, we choose $\epsilon_0(n)$ so small so that
$$2^{2(n+3)}C^3\epsilon_0^2(n) < \frac{1}{2}.$$
Then for all $\epsilon \leq \epsilon_0(n)$ the preceeding arguments follow through and we can therefore bound w on I by
$$\|w\|_{Z(I)} = \sum_{j=0}^{n-1} \|w\|_{Z_j} \leq \sum_{j=0}^{n-1} 2^{j+1}C\epsilon \simeq 2^{n+1}C\epsilon < \infty. \tag{2.3.8}$$
This allows us to conclude the uniform bound
$$\|u\|_{Z(I)} \leq \|u-v\|_{Z(I)} + \|v\|_{Z(I)} \leq \|w\|_{Z(I)} + B < \infty. \tag{2.3.9}$$
Using essentially the same arguments of partitioning the time domain and bootstrapping on each time interval we also obtain the $L_t^\infty(I;\mathcal{H})$ uniform bound (2.3.3). □

2.4 Scattering in the radial case

With all the pieces in play, we are now prepared to describe the proof of scattering for solutions in $\mathcal{PS}_+$.

Theorem 2.6. *[2] All solutions $u(t) \in \mathcal{PS}_+$ scatter as $t \to \pm\infty$ and $\|u\|_{L_t^3 L_x^6} < \infty$. Moreover, there exists a function $N : (0, J(Q)) \to (0,\infty)$ such that for all solutions in $\mathcal{PS}_+$,*
$$\|u\|_{L_t^3 L_x^6} < N(E(\vec{u})). \tag{2.4.1}$$

The proof follows the Kenig-Merle method and is indirect. We give an outline of it here.
For small energies, the local well-posedness theory implies that there is small ball in $\mathcal{PS}_+$ about zero for which solutions exists globally in time, scatter and have small Strichartz norm $\|\cdot\|_{L_t^3 L_x^6}$. If the conclusion of Theorem 2.6 failed to hold, then there must exist a *minimal* energy $0 < E_* < J(Q)$, for which we can find a sequence of data $\{(u_n^0, \dot{u}_n^0)\}_{n\in\mathbb{N}} \subset \mathcal{PS}_+$ with corresponding global solutions $u_n(t)$ for which
$$E(\vec{u}_n) \uparrow E_*, \tag{2.4.2}$$
and for which the Strichartz norm becomes unbounded for large n so that
$$\|u_n\|_{L_t^3 L_x^6} \to \infty. \tag{2.4.3}$$
We would then like to pass to the limit to deduce the existence of a *critical element* $u_* \in \mathcal{PS}_+$ which satisfies
$$E(\vec{u}_*) = E_*, \quad \|u_*\|_{L_t^3 L_x^6} = \infty. \tag{2.4.4}$$
A lack of compactness prevents us from doing so. This arises from the occurrence of two possibilities: Either the waves u_n wander off to infinity (that is, are arbitrarily translated) in space-time or they split into individual waves which become separated in space-time as $n \to \infty$ and for which the energy decouples. The former can be handled by careful translations. The latter suggests applying a nonlinear profile decomposition to our sequence u_n which represents

it is a sum of weakly interacting 'profiles' U^j which asymptotically diverge. If we assume that there exists at least two non-vanishing profiles, say U^1, U^2, we can show that

$$E(\vec{U}^j) < E_*,$$

and hence by the minimality of the energy, each profile is a global solution which scatters. At this point we have a decomposition of the sequence u_n into a sum of profiles U^j which are global, scattering NLKG solutions and an error term γ_n^k. The perturbation lemma then applies giving

$$\limsup_{n\to\infty} \|u_n\|_{L_t^3 L_x^6} < \infty,$$

which is a contradiction, from which we conclude that there is only one profile which is in fact our critical element.
This critical element has the further property that at least one of its trajectories

$$\mathcal{K}_{\pm} = \{(u_*(\cdot + x(t), t), \dot{u}_*(\cdot + x(t), t)) \,|0 < \pm t < \infty\}, \tag{2.4.5}$$

is precompact in $\mathcal{H}$, for some path $x(t)$ in $\mathbb{R}^3$. A *rigidity* argument, relying on virial identities, then shows that any critical element with a precompact trajectory must necessarily vanish and hence have zero energy. This is a contradiction from which we obtain Theorem 2.6.
As we have developed the radial theory, we will first prove Theorem 2.6 under the additional assumption of radial symmetry.

Proof of Theorem 2.6 in the radial case.
Step 1: Extracting a critical element
We suppose, in order to obtain a contradiction, that the theorem fails. This implies the existence of a sequence of global solutions $\{u_n\}$, bounded in $\mathcal{H}$, and a minimal energy E_* satisfying (2.4.2) and (2.4.3). We apply the radial linear profile decomposition of Theorem 2.3 to $\vec{u}_n(0)$ to obtain free profiles v^j and times t_n^j as in (2.1.2). We associate to each pair $\{(v^j, t_n^j)\}$, a nonlinear profile U^j, as in Theorem 2.4, which solves NKLG locally around $t = t_\infty^j$, and satisfies

$$\lim_{n\to\infty} \|\vec{v}^j(t_n^j) - \vec{U}^j(t_n^j)\|_{\mathcal{H}} = 0. \tag{2.4.6}$$

This gives rise to the useful heuristic of "swapping U^j's for v^j's" because we have, for large enough n,

$$\|U^j(t_n^j)\|_{\mathcal{H}} = \|v^j(t_n^j)\|_{\mathcal{H}} + o_n(1). \tag{2.4.7}$$

Combining this with the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, we see we can also swap in the L^4 norm, that is

$$\|U^j(t_n^j)\|_{L^4} = \|v^j(t_n^j)\|_{L^4} + o_n(1). \tag{2.4.8}$$

Locally around $t = 0$ we have the nonlinear profile decomposition

$$u_n(t) = \sum_{j=1}^{k} U^j(t + t_n^j) + \gamma_n^k(t) + \eta_n^k(t), \tag{2.4.9}$$

where we recall the free wave errors γ_n^k and the energy error η_n^k satisfy

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\gamma_n^k\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \text{for all } \ 2 < p < 6, \tag{2.4.10}$$

and

$$\lim_{n\to\infty} \|\eta_n^k(0)\|_{\mathcal{H}} = 0. \tag{2.4.11}$$

As the energy contains an L^4 norm, we seek to expand the L^4 norm of $u_n(0)$ in terms of the profiles v^j. We get

$$\|u_n(0)\|_{L^4}^4 = \Big\|\sum_{j=1}^{k} v^j(t_n^j) + \gamma_n^k(0)\Big\|_{L^4}^4 = \sum_{j=1}^{k} \|v^j(t_n^j)\|_{L^4}^4 + \|\gamma_n^k(0)\|_{L^4}^4 + (\text{cross-terms}).$$

The cross-terms are of two types: products of four profiles or products of profiles with $\gamma_n^k(0)$ terms. Using Hölder, we can bound each cross-term as a product of their individual L^4 norms. For the latter case, we can take k so large so that by (2.4.10), $\|\gamma_n^k(0)\|_{L^4}$ can be taken arbitrarily small while for the former ones, we take n so large and using the dispersive decay estimate and the fact that all, except for possibly one j, $|t_n^j| \to \infty$, to show they can be taken arbitrarily small. Using (2.4.8), we have

$$\|u_n(0)\|_{L^4}^4 = \sum_{j=1}^{k} \|U^j(t_n^j)\|_{L^4}^4 + o(1), \quad \textit{as} \ \ k, n \to \infty. \tag{2.4.12}$$

Now by (2.4.2), (2.4.12), (2.4.11) and energy conservation, we have

$$\begin{aligned} E_* + o(1) > E(\vec{u}_n(t)) = E(\vec{u}_n(0)) &= \sum_{j=1}^{k} E(\vec{U}^j(t_n^j)) + E(\vec{\gamma}_n^k(0)) + E(\vec{\eta}_n^k(0)) \\ &= \sum_{j=1}^{k} E(\vec{U}^j(t)) + E(\vec{\gamma}_n^k(0)) + o(1). \end{aligned} \tag{2.4.13}$$

Also

$$\begin{aligned} K_0(u_n(0)) &= \Bigg\|\sum_{j=1}^{k} v^j(t_n^j) + \gamma_n^k(0)\Bigg\|_{H^1}^2 - \sum_{j=1}^{k} \|v^j(t_n^j)\|_{L^4}^4 - \|\gamma_n^k(0)\|_{L^4}^4 + o(1) \\ &= \sum_{j=1}^{k} \left(\|U^j(t_n^j)\|_{H^1}^2 - \|U^j(t_n^j)\|_{L^4}^4\right) + \Big(\|\gamma_n^k(0)\|_{H^1}^2 - \|\gamma_n^k(0)\|_{L^4}^4\Big) + o(1) \qquad (2.4.14) \\ &= \sum_{j=1}^{k} K(U^j(t_n^j)) + K(\gamma_n^k(0)) + o(1), \qquad (2.4.15) \end{aligned}$$

as $k, n \to \infty$. Similarly, we obtain

$$\begin{aligned} J(Q) > E_* + o(1) > E(\vec{u}_n(0)) &> E(\vec{u}_n(0)) - \frac{1}{4}K_0(u_n(0)) \\ &\geq G_0(u_n(0)) + \frac{1}{4}\|\dot{u}_n(0)\|_{L^2}^2 = \frac{1}{2}E_0(\vec{u}_n(0)) \\ &= \sum_{j=1}^{k} \frac{1}{2}E_0(\vec{U}^j(t_n^j)) + \frac{1}{2}E_0(\vec{\gamma}_n^k(0)) + o(1) \\ &\geq \sum_{j=1}^{k} G_0(\vec{U}^j(t_n^j)) + G_0(\vec{\gamma}_n^k(0)) + o(1). \end{aligned}$$

Here E_0 is the free energy and we have used that $\vec{u}_n(0) \in \mathcal{PS}_+$ implies that $K_0(u_n(0)) \geq 0$. Since $G_0(\varphi) \geq 0$, it follows from Proposition 2.2 that $K_0(U^j(t_n^j)) \geq 0$ and $K_0(\gamma_n^k(0)) \geq 0$. Now by energy conservation,

$$E(u(t)) = E(u(0)) = K_0(u(0)) + \frac{1}{4}\|u(0)\|_{H^1}^2 + \frac{1}{2}\|\dot{u}(0)\|_{L^2}^2 \geq K_0(u(0)),$$

and hence $E(\vec{U}^j(t)) \geq 0$ and $E(\vec{\gamma}_n^k(t)) \geq 0$.
We define statements (A) and (B) by

$$(A) \text{ There exists at least two non-vanishing profiles } v^j, \text{ say } v^1, v^2. \tag{2.4.16}$$

$$(B) \limsup_{k\to\infty} \limsup_{n\to\infty} \|\vec{\gamma}_n^k\|_{\mathcal{H}} > 0. \tag{2.4.17}$$

We have four possibilities: (A, B), (A', B), (A, B'), (A', B') with the prime denoting the negation of the statement. We will now show that assuming the truth of at least one of the statements A or B leads to a contradiction.
Suppose that (A) is true. Then, as $n \to \infty$,

$$\begin{aligned} E(\vec{U}^j(t_n^j)) &\geq G_0(U^j(t_n^j) + \frac{1}{4}\|\dot{U}^j(t_n^j)\|_{L^2}^2 = \frac{1}{4}\|U^j(t_n^j)\|_{H^1}^2 + \frac{1}{4}\|\dot{U}^j(t_n^j)\|_{L^2}^2 \\ &= \frac{1}{4}\|v^j(t_n^j)\|_{H^1}^2 + \frac{1}{4}\|\dot{v}^j(t_n^j)\|_{L^2}^2 + o(1) \\ &= \frac{1}{2}E_0(\vec{v}^j(t_n^j)) + o(1) = \frac{1}{2}E_0(\vec{v}^j(0)) + o(1) > 0. \end{aligned}$$

As each nonlinear profile solves NLKG, this implies $E(\vec{U}^1) = E(\vec{U}^j(t_n^j)) > 0$, $E(\vec{U}^2) > 0$. By (2.4.13) we also have $E(\vec{U}^1), E(\vec{U}^2) < E_*$. Now if we assume (B) is true, then by definition of the energy and by (2.4.10), there exists a $\delta_0 > 0$ such that for large n and large k, $E(\vec{\gamma}_n^k(0)) > \delta_0 > 0$. Now (2.4.13) implies that $0 < E(\vec{U}^j) < E_*$. Therefore assuming either (A) or (B) leads to the same conclusion. By the minimality of E_*, each U^j must be a global in time solution that scatters with

$$\|U^j\|_{L_t^3 L_x^6} < \infty. \tag{2.4.18}$$

We now seek to apply the perturbation lemma on $I = \mathbb{R}$, with $u = u_n$ and

$$v(t) := \sum_{j=1}^{k} U^j(t + t_n^j). \tag{2.4.19}$$

By definition, $\mathrm{eq}(u) = 0$. As for $\mathrm{eq}(v)$, we have

$$\|\mathrm{eq}(v)\|_{L_t^1 L_x^2} \to 0, \tag{2.4.20}$$

as $n \to \infty$. For convenience, we define $f : x \mapsto x^3$. We write

$$\begin{aligned} \mathrm{eq}(v) &= (\Box + 1)v - f(v) \\ &= \sum_{j=1}^{k} f(U^j(t + t_n^j)) - f\left(\sum_{j=1}^{k} U^j(t + t_n^j)\right). \end{aligned}$$

The key here is that the difference on the right hand side consists of terms which are products of at least two different profiles, say $j \neq j'$. We then expect from the orthogonality of the times $\{t_n^j\}_j$ that each term should become arbitrarily small for large n. In order to prove this is the case, we introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for all $|t| \geq 2$, and set, for $R > 0$,

$$v_R(t) := \sum_{j=1}^{k} \chi\left(\frac{t + t_n^j}{R}\right) U^j(t + t_n^j) =: \sum_{j=1}^{k} U_R^j(t + t_n^j). \tag{2.4.21}$$

The point of using a cut-off is so that we can uniformly control the supports of all the terms; this would not be possible using a separate smooth approximation for each U^j. For n sufficiently large, each cut-off profile $U_R^j(t + t_n^j)$ is at most a distance $2R$ from any other such profile. Therefore

$$U_R^j(t + t_n^j) U_R^l(t + t_n^l) = 0, \tag{2.4.22}$$

for n sufficiently large and $j \neq l$. A further important property is that

$$\|U^j - U^j_R\|_{L^3_t L^6_x} = \|(1-\chi_R)U^j\|_{L^3_t L^6_x} \to 0 \tag{2.4.23}$$

as $R \to 0^+$. Writing

$$\mathrm{eq}(v) = \sum_{j=1}^{k} \left[f(U^j) - f(U^j_R) \right] + f(v_R) - f(v) + \sum_{j=1}^{k} f(U^j_R) - f(v_R),$$

we get

$$\begin{aligned} \|\mathrm{eq}(v)\|_{L^1_t L^2_x} \leq & \sum_{j=1}^{k} \|f(U^j) - f(U^j_R)\|_{L^1_t L^2_x} + \|f(v_R) - f(v)\|_{L^1_t L^2_x} \\ & + \left\| \sum_{j=1}^{k} f(U^j_R) - f(v_R) \right\|_{L^1_t L^2_x}. \end{aligned}$$

For the first term on the right hand side, we get

$$\|f(U^j) - f(U^j_R)\|_{L^2_x} \lesssim \left(\|U^j\|^2_{L^6_x} + \|U^j_R\|^2_{L^6_x} \right) \|U^j - U^j_R\|_{L^6_x}.$$

Integrating over time, using Hölder, (2.4.18) and (2.4.23) this term vanishes as $n \to \infty$. The second term can be estimated similarly with the same result. Finally, by taking n sufficiently large, the third term is identically zero by (2.4.22). This verifies (2.4.20).

We also need to verify that $v(t)$ can be uniformly bounded in k in the Strichartz norm, that is,

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^{k} U^j(t + t^j_n) \right\|_{L^3_t L^6_x} \tag{2.4.24}$$

By (2.1.4), we can find a j_0 such that for all $k > j_0$,

$$\limsup_{n\to\infty} \left\| \sum_{j=j_0}^{k} U^j(t^j_n) \right\|^2_{\mathcal{H}} \leq \epsilon_0^2, \tag{2.4.25}$$

for some fixed ϵ_0. Viewing $\vec{U}^j(t^j_n)$ as initial data for the NLKG with ϵ_0 chosen sufficiently small so that the small data result of the local well-posedness theory applies, we obtain

$$\|U^j(\cdot)\|_{L^3_t L^6_x} \lesssim \|\vec{U}^j(t^j_n)\|_{\mathcal{H}}, \quad \text{for all } \ j \geq j_0.$$

Hence

$$\begin{aligned} \limsup_{n\to\infty} \left\| \sum_{j=j_0}^{k} U^j(t + t^j_n) \right\|^3_{L^3_t L^6_x} = \sum_{j=j_0}^{k} \|U^j(\cdot)\|^3_{L^3_t L^6_x} & \leq C \limsup_{n\to\infty} \sum_{j=j_0}^{k} \|U^j(\cdot)\|^3_{\mathcal{H}} \\ & \leq C \limsup_{n\to\infty} \left(\sum_{j=j_0}^{k} \|U^j(\cdot)\|^2_{\mathcal{H}} \right)^{3/2} \\ & \leq C \limsup_{n\to\infty} \|\vec{u}_n(0)\|^3_{\mathcal{H}}. \end{aligned}$$

This now implies (2.4.24) as the term over $1 \leq j < j_0$ is easily bounded by the finiteness of the interval. We verify the final assumption for applying the perturbation lemma which is the smallness, for large enough n, of

$$\|S_0(t)(\vec{u}_n - \vec{v})(0)\|_{L^3_t L^6_x}.$$

This follows easily by the properties of $\vec{\gamma}_n^k$ and $\eta_n^k(0)$ and a Strichartz inequality since

$$\begin{aligned}\|S_0(t)(\vec{u}_n - \vec{v})(0)\|_{L_t^3L_x^6} &= \|S_0(t)(\vec{\gamma}_n^k + \eta_n^k)(0)\|_{L_t^3L_x^6}\\ &\le \|\gamma_n^k\|_{L_t^3L_x^6} + \|S_0(t)\vec{\eta}_n^k(0)\|_{L_t^3L_x^6}\\ &\le \|\gamma_n^k\|_{L_t^3L_x^6} + C\|\vec{\eta}_n^k(0)\|_{\mathcal{H}} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Taking k and n sufficiently large we apply the perturbation lemma to (2.4.9) to conclude

$$\limsup_{n\to\infty} \|u_n\|_{L_t^3L_x^6} < \infty,$$

which contradicts (2.4.3). This leaves us with only the case (A', B') and hence there exists only one non-vanishing profile, say v^1 and

$$\lim_{n\to\infty} \|\vec{\gamma}_n^2\|_{\mathcal{H}} = 0. \tag{2.4.26}$$

This now implies that $E(\vec{U}^1) = E_*$ and $K(U^1) \ge 0$ so $U^1 \in \mathcal{PS}_+$. Now by definition of E_*,

$$\|U^1\|_{L_t^3L_x^6} = \infty. \tag{2.4.27}$$

Therefore U^1 is the critical element we have been searching for and we write $u_* := U^1$.

Step 2: Precompactness of $\mathcal{K}_\pm$

Without loss of generality, we show that $\mathcal{K}_+$ is precompact in $\mathcal{H}$ with $x(t) \equiv 0$ (c.f. (2.4.5)) and suppose that

$$\|u_*\|_{L_t^3([0,\infty);L_x^6)} = \infty. \tag{2.4.28}$$

In order to obtain a contradiction, we suppose otherwise so that there exists $\delta > 0$ so that for some sequence $t_n \to \infty$ we have

$$\|\vec{u}_*(t_n) - \vec{u}_*(t_m)\|_{\mathcal{H}} > \delta, \quad \text{for all } \ n > m. \tag{2.4.29}$$

We now apply the arguments from Step 1 to $U^1(t_n)$, using (2.4.27) and the minimality of E_* to obtain

$$\vec{u}_*(t_n) = \vec{V}(\tau_n) + \vec{\gamma}_n(0) \tag{2.4.30}$$

where $\vec{V}, \vec{\gamma}_n$ are free KG solutions, τ_n is a sequence in $\mathbb{R}$ and $\|\vec{\gamma}_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. Suppose that $\tau_n \to \tau_\infty \in \mathbb{R}$. Using (2.4.30), we have

$$\|\vec{u}_*(t_n) - \vec{u}_*(t_m)\|_{\mathcal{H}} \le \|\vec{V}(\tau_n) - \vec{V}(\tau_m)\|_{\mathcal{H}} + \|\vec{\gamma}_n(0)\|_{\mathcal{H}} + \|\vec{\gamma}_m(0)\|_{\mathcal{H}} \to 0$$

as $m, n \to \infty$ which contradicts (2.4.29). Suppose that $\tau_n \to \infty$. As

$$\|V(\cdot + \tau_n)\|_{L_t^3([0,\infty);L_x^6)} \to 0,$$

as $n \to \infty$ by a change of variables $t' = t + \tau_n$. Then using (2.4.30) as initial data for the NLKG, uniqueness of solutions and the small data scattering we get $\|u_*(\cdot + t_n)\|_{L_t^3([0,\infty);L_x^6)} < \infty$ for large n which contradicts (2.4.28). Finally, suppose that $\tau_n \to -\infty$. Then

$$\|V(\cdot + \tau_n)\|_{L_t^3((-\infty,0];L_x^6)} \to 0,$$

as $n \to \infty$. This gives that for all large n, there exists some fixed constant B such that $\|u_*(\cdot + t_n)\|_{L_t^3((-\infty,0];L_x^6)} < B < \infty$. As $t_n \to \infty$, we get a contradiction to (2.4.28) by taking the limit in n. Therefore $\mathcal{K}_+$ is precompact in $\mathcal{H}$.

Step 3: Rigidity argument: The key part of the rigidity argument is the following virial identity:

$$\frac{d}{dt}\langle \chi_R \dot{u}_* | A u_* \rangle_{L^2} = -K_2(u_*) + \mathcal{O}\left(\int_{|x|>R} |\dot{u}_*|^2 + |\nabla u_*|^2 + |u_*|^2\, dx\right). \tag{2.4.31}$$

Here $A = \frac{1}{2}(x.\nabla + \nabla.x)$ and $\chi_R(x) := \chi(x/R)$ is a smooth, radial cut-off function satisfying $\chi = 1$ on $|x| \leq 1$ and $\chi = 0$ on $|x| \geq 2$. By the compactness of $\overline{\mathcal{K}}_+$, the $\mathcal{O}(\cdot)$ term in (2.4.31) is uniformly small (tightness).
We use (2.4.31) to show that the compactness of $\overline{\mathcal{K}}_+$ leads to a contradiction unless $u_* \equiv 0$. However this itself is a contradiction as $0 < E(\vec{u}_*) = E_*$, whence Theorem 2.6 follows. If we integrate both sides of (2.4.31) from 0 to a time t_0, the left hand side can be bounded by $\mathcal{O}(R)$ by energy conservation and since $K \geq 0$ implies that the free energy is uniformly bounded for all time. If the right hand side before integration is bounded by $-\delta_0 < 0$ for some fixed δ_0, then by taking t_0 large, we can be obtain a contradiction to the inequality.

A potential snag we may encounter in this procedure as that there is no reason a priori why $K_2(u_*(t))$ must be uniformly bounded above by $-\delta_0$. From (2.0.5), we have

$$-K_2(u_*) \leq c_0 \min(J(Q) - J(u_*), \|\nabla u_*\|_2^2) \leq -\delta_1 \|\nabla u_*\|_2^2,$$

for some $\delta_1 > 0$. The above inequality will be useless if there was a time t_1 such that $u_*(t_1) = 0$, as it cannot be concluded from this that $u_* \equiv 0$ as $\dot{u}(t_1)$ may be large. A way around this problem uses the following estimate: for every $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\|u_*(t)\|_2^2 \leq C(\epsilon)\|\nabla u_*(t)\|_2^2 + \epsilon\|\dot{u}_*(t)\|_2^2, \quad \forall\, t \geq 0. \tag{2.4.32}$$

To prove this, assume that (2.4.32) did not hold. Then there exists an $\epsilon > 0$ and a sequence t_n such that

$$\|u_*(t_n)\|_2^2 > n\|\nabla u_*(t_n)\|_2^2 + \epsilon\|\dot{u}_*(t_n)\|_2^2, \quad \forall\, n \geq 1. \tag{2.4.33}$$

As u_* is uniformly bounded in L^2, the above implies that $\nabla u_*(t_n) \to 0$ in L^2. By precompactness of $\mathcal{K}_+$, there exists a subsequence, which we label $u_*(t_n)$, that converges strongly in H^1_x. This implies that $u_*(t_n) \to 0$ in H^1_x. This can be obtained by identifying limits in larger spaces. Hence $\|\dot{u}_*(t_n)\|_2^2 \to 0$ which implies that $E_0(u_*) \to 0$. However, $E(\vec{u}_*(t_n)) \leq E_0(u_*(t_n))$ so $E(\vec{u}_*(t_n)) \to 0$ as $n \to \infty$. This says that eventually $E(\vec{u}_*(t_n))$ decreases but since u_* solves the NLKG, its energy is conserved and hence $E(\vec{u}_*) = 0$, which implies that $u_* \equiv 0$ which is a contradiction as $E(\vec{u}_*) = E_* > 0$.
Using (2.4.32) with $\epsilon = \frac{1}{2}$, we find

$$\begin{aligned}\frac{d}{dt}\langle u_* | \dot{u}_* \rangle &= \|\dot{u}_*\|_2^2 + \langle u_* | \Delta u_* - u_* + u_*^3 \rangle \\ &= \|\dot{u}_*\|_2^2 - \|\nabla u_*\|_2^2 - \|u_*\|_2^2 + \|u_*\|_4^4 \\ &\geq \frac{1}{2}\|\dot{u}_*\|_2^2 - C(1/2)\|\nabla u_*\|_2^2,\end{aligned}$$

and using this and (2.4.32) with $\epsilon = \frac{1}{4}$ we also get

$$\frac{1}{4}\|\dot{u}_*\|_2^2 + \|u_*\|_{H^1}^2 \leq C\|\nabla u_*\|_2^2 + \frac{d}{dt}\langle u_* | \dot{u}_* \rangle,$$

where $C := 1 + C(1/2) + C(1/4)$. As $K(u_*) \geq 0$, then the left hand side of the above is $\simeq E(\vec{u}_*)$. Integrating both sides of the above from 0 to $t_0 > 0$ gives

$$t_0 E(\vec{u}_*) \lesssim \int_0^{t_0} \|\nabla u_*(t)\|_2^2 \, dt. \tag{2.4.34}$$

Now we choose R so large so that the integral term on the right-hand side of (2.4.31) can be bounded by $\delta_2 E(\vec{u}_*)$ for some $\delta_2 \ll \delta_1$. Integrating (2.4.31) over 0 to t_0 yields

$$\langle \chi_R \dot{u}_* | A u_* \rangle_{L^2} \big|_0^{t_0} \leq -\delta_1 \int_0^{t_0} \|\nabla u_*(t)\|_2^2 \, dt + C t_0 \delta_2 E(\vec{u}_*).$$

Not that the left-hand side is uniformly bounded in time by $\mathcal{O}(RE(\vec{u}_*))$. Using (2.4.34) we obtain a contradiction by taking $t_0 \to \infty$.

□

Chapter 3

Invariant manifolds by The Lyapunov-Perron Method

In this chapter, we begin the analysis of NLKG solutions with energies

$$E(\vec{u}) < J(Q) + \epsilon^2, \tag{3.0.1}$$

where $\epsilon \ll 1$ and to be determined. In the regime where $E(\vec{u}) < J(Q)$ there where only two possible dynamics: scattering to zero or finite time blow-up. Slightly above the ground state, there is an additional behaviour: trapping by the ground state, that is the solution eventually remains entirely within a small ball about one of the ground states $(\pm Q, 0)$. In fact, more is known in this case as one can construct center-stable manifolds locally about the ground states which can be used to derive further properties of trapped solutions. The construction is given in Theorem 3.4 and is the key result of this chapter.

We follow the arguments of [5] where the construction is via the Lyapunov-Perron method. An additional construction is provided in [5] known as the Bates-Jones approach [18]. We do not consider this construction here. The methods are quite independent and each has its advantages and disadvantages. The Lyapunov-Perron method requires a full knowledge of the spectral properties of a certain operator in order to derive Strichartz estimates for the perturbed evolution about Q. The Bates-Jones approach does not require such heavy spectral information however it does not obtain as much as information as the Lyapunov-Perron approach does; such as a scattering statement, and is also inflexible with respect to other powers of the non-linearity in the Klein-Gordon equation.

3.1 Linearisation and spectral properties

Writing $u = Q + v$, we have

$$\partial_t^2 v + L_+ v = 3Qv^2 + v^3 =: N(v), \tag{3.1.1}$$

where

$$L_+ := -\Delta + 1 - 3Q^2 \tag{3.1.2}$$

is the linearised operator about Q. Such a decomposition leads to expansions of the energy and K_0, namely

$$E(Q + v, \partial_t v) = J(Q) + \frac{1}{2}\langle L_+ v | v\rangle + \frac{1}{2}\|\partial_t v\|_{L^2}^2 + \mathcal{O}(\|v\|_{H^1}^3), \tag{3.1.3}$$

$$K_0(Q + v) = -2\left\langle Q^3 | v\right\rangle + \mathcal{O}(\|v\|_{H^1}^2). \tag{3.1.4}$$

Note that $\langle\cdot|\cdot\rangle$ is the L^2 inner product. With $\rho > 0$, the L^2-normalised eigenfunction of L_+ whose existence is stated later in Lemma 3.2, we write

$$v(t,x) = \lambda(t)\rho(x) + \gamma(t,x), \tag{3.1.5}$$

where $\gamma \in P_\rho^\perp(H^1)$ in the sense that $\gamma \in H^1$ and is orthogonal to ρ in the L^2 inner product. We define the projection operators $P_\rho := \rho\langle\rho|$ and $P_\rho^\perp := 1 - P_\rho$, which are projections using the L^2 inner product. Inserting this further decomposition into (3.1.1) and projecting onto and off ρ, we obtain the system

$$\begin{cases} \ddot{\lambda} - k^2\lambda = N_\rho(v) =: \langle N(v)|\rho\rangle\,, \\ \partial_t^2\gamma + L_+\gamma = P_\rho^\perp N(v) =: N_c(v), \quad \gamma \in P_\rho^\perp(H^1). \end{cases} \tag{3.1.6}$$

Generally the second equation (3.1.6) should have the operator L_+ replaced with $\omega^2 := P_\rho^\perp L_+$, however a short calculation shows that on $P_\rho^\perp(H^1)$, $\omega^2 = L_+$ so the additional orthogonal projection here is unnecessary.

Using (3.1.3) and (3.1.4), we obtain the further decompositions in terms of (λ, γ)

$$E(Q+v, \partial_t v) = J(Q) + \frac{1}{2}(\dot{\lambda}^2 - k^2\lambda^2) + \frac{1}{2}\langle L_+\gamma|\gamma\rangle + \frac{1}{2}\|\partial_t\gamma\|_{L^2}^2 + \mathcal{O}(\|v\|_{H^1}^3), \tag{3.1.7}$$

$$K_0(Q+v) = -2\left\langle Q^3|\lambda\rho + \gamma\right\rangle + \mathcal{O}(\|v\|_{H^1}^2). \tag{3.1.8}$$

We now state a useful result that says that for $\gamma \in P_\rho^\perp(H^1)$, the perturbation by $-3Q^2$ in L_+ is largely unimportant.

Lemma 3.1. *For any $\gamma \in P_\rho^\perp(H^1)$, we have*

$$\langle L_+\gamma|\gamma\rangle \simeq \|\gamma\|_{H^1}^2. \tag{3.1.9}$$

Proof. Notice that the upper bound $\langle L_+\gamma|\gamma\rangle \lesssim \|\gamma\|_{H^1}^2$ is trivial upon writing out L_+ and integrating by parts. For the other direction, we first show that if $f \perp Q^3$, then there exists a constant $c_0 > 0$ such that

$$\langle L_+ f|f\rangle \geq c_0\|f\|_{L^2}^2. \tag{3.1.10}$$

To show this suppose otherwise, so that there exists $f \perp Q^3$ and $\langle L_+f|f\rangle < c_0\|f\|_{L^2}^2$. Setting $v = \epsilon Q + \delta f$ for small $\epsilon, \delta \in \mathbb{R}$, we find using (3.1.4) and $G_0(u) = (1/4)\|u\|_{H^1}^2$ that

$$K_0(Q+v) \leq -2\epsilon\|Q\|_4^4 + c_0\delta^2, \quad G_0(Q+v) < J(Q) - \frac{1}{2}\epsilon\|Q\|_{L^4}^4 + \frac{3}{4}\delta^2 c_0.$$

Choosing $\epsilon \sim \delta^3$, we get $K_0(Q+v) \leq 0$ and $G_0(Q+v) < J(Q)$ which contradicts (2.0.2). Now from the Min-Max Principle (see [19]),

$$\mu_2(L_+) := \sup_\phi \inf_{\psi\perp\phi} \frac{\langle L_+\psi|\psi\rangle}{\|\psi\|_{L^2}^2} \geq \inf_{\psi\perp Q^3} \frac{\langle L_+\psi|\psi\rangle}{\|\psi\|_{L^2}^2} \geq c_0 > 0.$$

As the supremum is attained for ρ, we find

$$\langle L_+\gamma|\gamma\rangle \geq c_0\|\gamma\|_{L^2}^2 \tag{3.1.11}$$

for all $\gamma \in P_\rho^\perp(H^1)$. Explicitly,

$$\langle L_+\gamma|\gamma\rangle = \|\gamma\|_{H^1}^2 - 3\left\langle Q^2\gamma|\gamma\right\rangle.$$

Combining this and (3.1.11), implies that for any $\theta \in [0,1]$,

$$\begin{aligned} \langle L_+\gamma|\gamma\rangle &\geq (1-\theta)c_0\|\gamma\|_{L^2}^2 + \theta\|\gamma\|_{H^1}^2 - 3\theta\left\langle Q^2\gamma|\gamma\right\rangle \\ &\geq (c_0 - c_0\theta - 3\|Q\|_{L^\infty}^2\theta)\|\gamma\|_{L^2}^2 + \theta\|\gamma\|_{H^1}^2. \end{aligned}$$

Choosing $\theta \in (0,1)$ such that the first term vanishes, we obtain the desired lower bound. □

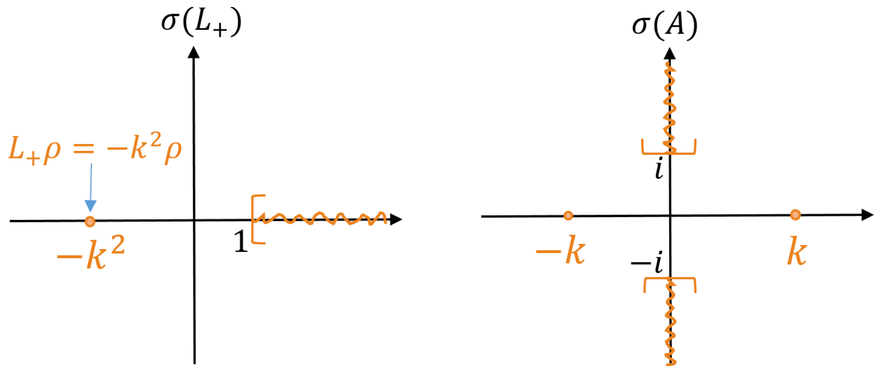

Figure 3.1: Plots of the spectrum of L_+ and A.

For constructing the center manifold for NLKG solutions satisfying (3.0.1), it is convenient to rewrite (3.1.1) as a first order system of PDEs. We easily verify that (3.1.1) can be written equivalently as the system

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} + \begin{pmatrix} 0 \\ N(v) \end{pmatrix}. \tag{3.1.12}$$

As in the theory of finite dimensional ODE, it is crucial to examine the spectral properties of the operator A which also requires those of L_+.

Lemma 3.2. (Spectral properties of L_+ and A) *Consider the unbounded operators $L_+ : D(L_+) = H^2_{\mathrm{rad}}(\mathbb{R}^3) \subset L^2_{\mathrm{rad}}(\mathbb{R}^3) \to L^2_{\mathrm{rad}}(\mathbb{R}^3)$ and $A : D(A) = H^2_{\mathrm{rad}}(\mathbb{R}^3) \times L^2_{\mathrm{rad}}(\mathbb{R}^3) \to L^2_{\mathrm{rad}}(\mathbb{R}^3) \times L^2_{rad}(\mathbb{R}^3)$. We have:*

(i) *L_+ is self-adjoint and bounded from below,*
(ii) *L_+ has only one negative eigenvalue $-k^2$, with $k > 0$, which is non-degenerate and has no eigenvalue at 0 or in the continuous spectrum $[1, \infty)$,*
(iii) *L_+ satisfies the 'Gap Property': L_+ has no eigenvalues in $(0, 1]$ and has no resonance at the threshold 1,*
(iv) *$\sigma(A) = \{z \in \mathcal{C} : z^2 \in -\sigma(L_+)\} = \{\pm k\} \cup i[1, \infty) \cup i(-\infty, -1]$.*

Proof. The proofs of these statements can be found in [5] and have been omitted here in order to focus on the the latter sections in this chapter. Properties (i), (ii) and (iv) are standard arguments. However (iii) is not. For this property, Demanet and Schlag [20] obtained a numerical proof and later an analytical proof was given by the second author and collaborators [21]. The gap property will be used to derive Strichartz estimates for L_+ which will be used in the construction of the center-stable manifold. □

The motivation for the existence of a center-stable manifold about $(\pm Q, 0)$ is now clear from Lemma 3.2. The portion of the spectrum of L_+ along the imaginary axis (see Figure 3.1) is responsible for the non-hyperbolic nature of the equilibrium Q. We then attribute the stable/unstable behaviours of solutions to NLKG just above the ground state to the single negative/positive eigenvalues of A.

3.2 The center-stable manifold

In this section, we will detail the construction of the center-stable manifold for NLKG by using the Lyapunov-Perron method. The manifold exists locally around the ground state $(Q, 0)$ in $\mathcal{H}$

and is ‘tangent’ to the *linear stable* subspace

$$\mathcal{T} := \{(v_0, v_1) \in \mathcal{H} \,|\, \langle kv_0 + v_1|\rho\rangle = 0\,\} \tag{3.2.1}$$

at $(Q, 0)$. The motivation for this subspace arises from considering the linear version of (3.1.12) (setting $N(v) \equiv 0$). The linearised operator A has eigenfunctions $e_\pm := (\rho, \pm k\rho)$ with corresponding eigenvalues $\pm k$ respectively. As we expect the direction e_+ to be responsible for the exponentially growing modes, then in order to have stable solutions, we must have initial data $(v_0, v_1) \in \mathcal{H}$ satisfying

$$P_+ \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = 0, \tag{3.2.2}$$

where $P_\pm$ are the Riesz projections onto $\sigma(A) \setminus \{\pm k\}$ defined by

$$P_\pm \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} := \frac{1}{2k} \left\langle \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \middle| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \rho \\ \pm k\rho \end{pmatrix} \right\rangle e_\pm = \frac{1}{2k} \left\langle \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \middle| \begin{pmatrix} \pm k\rho \\ \rho \end{pmatrix} \right\rangle e_\pm. \tag{3.2.3}$$

The condition (3.2.2) is thus exactly the condition as stated in (3.2.1) and we also see that $\mathcal{T} = \ker P_+$. We construct the center-stable manifold $\mathcal{M}$ as a graph over $\mathcal{T} \cap B_\nu(Q, 0)$ where $B_\nu(Q, 0)$ is a ball of sufficiently small radius ν centred about $(Q, 0)$ in $\mathcal{H}$.

To isolate the exponentially growing modes, it is more convenient to work with the variables (λ, γ) rather than with v. We thus uniquely associate to each $(v_0, v_1) \in \mathcal{H}$, the quantities $(\lambda_0, \lambda_1, \gamma_0, \gamma_1) \in \mathbb{R}^2 \times P_\rho^\perp(H^1) \times P_\rho^\perp(L^2)$ through

$$v_0 = \lambda_0\rho + \gamma_0, \quad v_1 = \lambda_1\rho + \gamma_1,$$

where $\gamma_0, \gamma_1 \perp \rho$ in L^2. The linear stability condition (3.2.2) becomes $k\lambda_0 + \lambda_1 = 0$ and the linear stable subspace (3.2.1) is

$$\mathcal{T} = \{(\lambda_0, \lambda_1, \gamma_0, \gamma_1) \in \mathbb{R}^2 \times P_\rho^\perp(H^1) \times P_\rho^\perp(L^2) \,|\, k\lambda_0 + \lambda_1 = 0\,\}. \tag{3.2.4}$$

We now seek to determine a suitable stability condition as above but for the nonlinear case. By considering the linear versions of (3.1.6), we see that the unstable behaviour will be entirely due to the λ component. Writing the Duhamel formula for the λ equation in (3.1.6) and collecting the exponetial powers we obtain

$$\lambda(t) = \frac{e^{kt}}{2k} \left[k\lambda(0) + \dot\lambda(0) + \int_0^\infty e^{-ks} N_\rho(v)\, ds\right] + \frac{e^{-kt}}{2k} \left[k\lambda(0) - \dot\lambda(0)\right] - \frac{1}{2k} \int_0^\infty e^{-k|t-s|} N_\rho(v)\, ds.$$

Provided that $N_\rho(v) \in L_t^1(0, \infty)$, then $\lambda \in L_t^\infty(0, \infty)$ if and only if

$$k\lambda(0) + \dot\lambda(0) = -\int_0^\infty e^{-ks} N_\rho(v)(s)\, ds. \tag{3.2.5}$$

The condition (3.2.5) is, as expected, a higher order correction to (3.2.2). Inserting this into the Duhamel formula implies that such λ and γ must satisfy

$$\lambda(t) = e^{-kt} \left[\lambda(0) + \frac{1}{2k} \int_0^\infty e^{-ks} N_\rho(v)(s)\, ds\right] - \frac{1}{2k} \int_0^\infty e^{-k|t-s|} N_\rho(v)(s)\, ds, \tag{3.2.6}$$

$$\gamma(t) = \cos(\omega t)\gamma(0) + \frac{\sin(\omega t)}{\omega}\dot\gamma(0) + \int_0^t \frac{\sin(\omega(t-s))}{\omega} N_c(v)(s)\, ds, \tag{3.2.7}$$

where we recall that $\omega := \sqrt{L_+}$ on $P_\rho^\perp(H^1)$. Obtaining solutions to (3.1.1) that do not grow exponentially is thus equivalent to obtaining solutions to the system (3.2.6)-(3.2.7). That (3.2.5) is still satisfied, in fact at any time $t_0 \in [0, \infty)$, by solutions to (3.2.6)-(3.2.7) is a direct computation but we state it as a lemma for ease of reference.

Lemma 3.3. *Let $(\lambda(t), \gamma(t))$ be global in time solution to (3.2.6)-(3.2.7). Then, for any $t_0 \in [0, \infty)$, we have*

$$k\lambda(t_0) + \dot{\lambda}(t_0) = -e^{kt_0} \int_{t_0}^{\infty} e^{-ks} N_\rho(v)(s)\, ds. \tag{3.2.8}$$

Before giving the statement of the Centre-Stable Manifold Theorem, it is instructive to describe the construction of points on $\mathcal{M}$ and give meaning to the statement made earlier that $\mathcal{M}$ is obtained 'as a graph over $\mathcal{T} \cap B_\nu(Q, 0)$.' We begin with any point $(\lambda_0, \lambda_1, \gamma_0, \gamma_1) \in \mathcal{T} \cap B_\nu(Q, 0)$. Notice that $\mathcal{T}$ is really only parameterized by the triple $(\lambda_0, \gamma_0, \gamma_1)$ as λ_1 is obtained from the linear stability condition $\lambda_1 = -k\lambda_0$. We use $(\lambda_0, \gamma_0, \gamma_1)$ as an initial data set for (3.2.6)-(3.2.7) and construct a global in time solution $v(t) = \lambda(t)\rho + \gamma(t)$. Evaluating at $t = 0$ gives $(\lambda(0), \dot{\lambda}(0), \gamma(0), \dot{\gamma}(0))$, with the point being that $\lambda(0) = \lambda_0$, $\gamma(0) = \gamma_0$, $\dot{\gamma}(0) = \gamma_1$ but λ_1 will not necessarily equal $\dot{\lambda}(0)$ as the latter will satify (3.2.5). The quadruple $(\lambda(0), \dot{\lambda}(0), \gamma(0), \dot{\gamma}(0))$ is then said to be a corresponding point on $\mathcal{M}$. In order to go back from $\mathcal{M}$ to $\mathcal{T} \cap B_\nu(Q, 0)$, we let $(v(0), \partial_t v(0)) \in \mathcal{M}$ which gives rise to a global solution to (3.2.6)-(3.2.7). By Lemma (3.3), (3.2.5) is satisfied. We have that indeed

$$\begin{pmatrix} \tilde{v}(0) \\ \partial_t \tilde{v}(0) \end{pmatrix} := (\mathrm{Id} - P_+) \begin{pmatrix} v(0) \\ \partial_t v(0) \end{pmatrix} \in \mathcal{T}.$$

This follows as

$$P_+ \begin{pmatrix} \tilde{v}(0) \\ \partial_t \tilde{v}(0) \end{pmatrix} = P_+(\mathrm{Id} - P_+) \begin{pmatrix} v(0) \\ \partial_t v(0) \end{pmatrix} = (P_+ - P_+^2) \begin{pmatrix} v(0) \\ \partial_t v(0) \end{pmatrix} = 0.$$

We therefore have a well-defined map $\phi : \mathcal{T} \cap B_\nu(Q, 0) \to \mathcal{H}$ such that

$$\phi(\lambda_0, \lambda_1, \gamma_0, \gamma_1) = (\lambda(0), \dot{\lambda(0)}, \gamma(0), \dot{\gamma}(0)), \tag{3.2.9}$$

which fosters the following definition for the center-stable manifold

$$\mathcal{M} := \{\phi(\lambda_0, \lambda_1, \gamma_0, \gamma_1)\, |(\lambda_0, \lambda_1, \gamma_0, \gamma_1) \in \mathcal{T} \cap B_\nu(Q, 0) \text{ and } \dot{\lambda}(0) \text{ satisfies (3.2.5)}\}. \tag{3.2.10}$$

The Center-Stable Manifold Theorem says that indeed such a manifold exists and it has some useful properties.

Theorem 3.4 (Center-Stable Manifold)**.** *Assume L_+ satisfies the gap property (see Lemma 3.2, (iii)). Then there exists a $\nu > 0$ small and a smooth graph $\mathcal{M} \subset B_\nu(Q, 0) \subset \mathcal{H}$, as defined by (3.2.10), such that the following hold.*

(i) $(Q, 0) \in \mathcal{M}$ and $\mathcal{M}$ is tangent to $\mathcal{T}$ at $(Q, 0)$ in the following sense,

$$\sup_{(Q+v_0, v_1^*) = \phi(Q+v_0, v_1) \in \partial B_\delta(Q,0) \cap \mathcal{M}} |\langle kv_0 + v_1^* | \rho \rangle| \lesssim \delta^2, \quad \forall\, 0 < \delta < \nu. \tag{3.2.11}$$

(ii) For all $(u_0, u_1) := (Q + v_0, v_1) \in \mathcal{M}$ there exists a unique global solution to NLKG of the form $u(t) = Q + \lambda(t)\rho + \gamma(t)$ with the following properties:
   (a) $\|(v, \partial_t v)\|_{L_t^\infty((0,\infty);\mathcal{H})} + \|v\|_{L_t^3((0,\infty);L_x^6(\mathbb{R}^3))} \lesssim \nu$.
   (b) v scatters to a linear solution v_{lin} which satisfies $(\partial_t^2 + L_+)v_{lin} = 0$, that is, there exists a unique free KG solution γ_∞ such that

$$|\lambda(t)| + |\dot{\lambda}(t)| + \|\vec{\gamma}(t) - \vec{\gamma}_\infty(t)\|_{\mathcal{H}} \to 0, \text{ as } t \to \infty. \tag{3.2.12}$$

   (c) (Energy splitting) $E(\vec{u}) = J(Q) + \frac{1}{2}\|\vec{\gamma}_\infty\|_{\mathcal{H}}^2$.

(iii) Any solution $u(t)$ to NLKG that remains in $B_\nu(Q, 0)$ for all $t \geq 0$ necessarily lies entirely on $\mathcal{M}$.

(iv) $\mathcal{M}$ is invariant under the flow of NLKG.

Proof. We begin with constructing solutions (λ, γ).
Construction of (λ, γ)
We will construct the pair (λ, γ) as a fixed point in space $X = \{(0, \infty) \ni t \mapsto (\lambda(t), \gamma(t)) \in \mathbb{R} \times P_\rho^\perp(H^1)\}$ with norm

$$\|(\lambda, \gamma)\|_X := \|\lambda\|_{L^1 \cap L^\infty(0,\infty)} + \|\gamma\|_{S(0,\infty)},$$

where S denotes the Strichartz space $S := L_t^2 L_x^6 \cap L_t^\infty H_x^1$. Let $\Lambda(\lambda, \gamma)$ denote the $\mathbb{R}$-valued right hand side of (3.2.6) and $\Gamma(\lambda, \gamma)$ denote the $H_x^1(\mathbb{R}^3)$ right hand side of (3.2.7). We obtain (λ, γ) as a fixed point of the operator $\Lambda \times \Gamma$ on X. By Fubini, we find

$$\|\Lambda(\lambda, \gamma)\|_{L_t^1 \cap L_t^\infty(0,\infty)} \leq \left(1 + \frac{1}{k}\right) |\lambda(0)| + \frac{1}{k}\left(1 + \frac{3}{2k}\right) \|N_\rho(v)\|_{L_t^1(0,\infty)}. \tag{3.2.13}$$

By Cauchy-Schwarz, $\|N_\rho(v)\|_{L_t^1(0,\infty)} \leq \|N(v)\|_{L_t^1 L_x^2(0,\infty)}$ which implies

$$\|\Lambda(\lambda, \gamma)\|_{L_t^1 \cap L_t^\infty(0,\infty)} \lesssim |\lambda(0)| + \|N(v)\|_{L_t^1((0,\infty); L_x^2)}.$$

As for $\Gamma(\lambda, \gamma)$ we make use of the Strichartz estimate for ω given by (3.9). Using that $\|N_c(v)\|_{L_t^1((0,\infty); L_x^2)} \leq \|N(v)\|_{L_t^1((0,\infty); L_x^2)}$ we obtain

$$\|(\Lambda \times \Gamma)(\lambda, \gamma)\|_X \lesssim |\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|N(v)\|_{L_t^1((0,\infty); L_x^2)}. \tag{3.2.14}$$

We recall that $N(v) := 3Qv^2 + v^3$, and hence by Hölder,

$$\|N(v)\|_{L_t^1((0,\infty); L_x^2)} \leq 3\|Q\|_{L_x^6} \|v\|^2_{L_t^2 L_x^6} + \|v\|^3_{L_t^3 L_x^6}.$$

Now the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ implies that $S \hookrightarrow L_t^2 L_x^6 \cap L_t^\infty L_x^6 \hookrightarrow L_t^q L_x^6$ for all $2 \leq q \leq \infty$. Using this we easily obtain, for the same range of q,

$$\|v\|_{L_t^q L_x^6} \lesssim \|(\lambda, \gamma)\|_X, \tag{3.2.15}$$

and hence

$$\|N(v)\|_{L_t^1((0,\infty); L_x^2)} \lesssim \|(\lambda, \gamma)\|_X^2 + \|(\lambda, \gamma)\|_X^3. \tag{3.2.16}$$

With this we finally get

$$\|(\Lambda \times \Gamma)(\lambda, \gamma)\|_X \leq C\left(|\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|(\lambda, \gamma)\|_X^2 + \|(\lambda, \gamma)\|_X^3\right). \tag{3.2.17}$$

Choosing ν so small so that $\nu + \nu^2 \leq 1/C$ and such that if

$$|\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}} \leq \nu/(2C), \tag{3.2.18}$$

then $\Lambda \times \Gamma$ maps the ball $\{(\lambda, \gamma) \in X \mid \|(\lambda, \gamma)\|_X \leq \nu\}$ into itself. We also obtain the contraction estimate

$$\|(\Lambda \times \Gamma)(\lambda_1, \gamma_1) - (\Lambda \times \Gamma)(\lambda_2, \gamma_2)\|_X \leq \frac{1}{2}\|(\lambda_1, \gamma_1) - (\lambda_2, \gamma_2)\|_X,$$

by decreasing ν if necessary. The ν in the statement of theorem is really $\nu := \nu/(2C)$, so that we have for any initial data $(\lambda(0), \vec{\gamma}(0))$ satisfying $|\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}} \leq \nu$, a unique global in time fixed point (λ, γ) such that $\|(\lambda, \gamma)\|_X \leq 2C\nu$. We then see that $u(t) = Q + \lambda(t)\rho + \gamma(t)$ satisfies NLKG and by Lemma 3.3, (3.2.5) is satisfied. Furthermore, we deduce that $\|u(t) - Q\|_{H_x^1} \leq \|(\lambda, \gamma)\|_X \leq 2C\nu$ and $\|\partial_t u(t)\|_{L_x^2} \leq |\dot{\lambda}(t)| + \|\partial_t \gamma(t)\|_{L_x^2}$. We can obtain a better bound for $\partial_t u$ by differentiating in time (3.2.6) and (3.2.7). In fact, using (3.2.17), we find

$$|\dot{\lambda}(t)| \lesssim |\lambda(0)| + \|(\lambda, \gamma)\|_X^2 \|(\lambda, \gamma)\|_X^3 \lesssim \nu,$$

and using Lemma 3.1, we find similarly

$$\|\partial_t\gamma(t)\|_{L^2_x} \lesssim \|\vec{\gamma}(0)\|_{H^1_x} + \|(\lambda,\gamma)\|_X^2\|(\lambda,\gamma)\|_X^3 \lesssim \nu. \tag{3.2.19}$$

Therefore

$$\|\vec{u}(t) - (Q,0)\|_{\mathcal{H}} = \|\vec{v}\|_{\mathcal{H}} \lesssim |\lambda(0)| + \|\vec{\gamma}(0)\|_{\mathcal{H}}, \quad \forall\, t \geq 0. \tag{3.2.20}$$

As the dependency of $(\lambda(t),\gamma(t))$ on $(\lambda(0),\vec{\gamma}(0))$ is real analytic,

$$\lambda_1^* = \lambda_1^*(\lambda(0),\vec{\gamma}(0)) = -k\lambda(0) - \int_0^\infty e^{-ks}N_\rho(\lambda(s)\rho+\gamma(s))\,ds$$

is a smooth function in terms of $(\lambda(0),\vec{\gamma}(0))$ and hence the mapping ϕ defined by (3.2.9) is smooth and so $\mathcal{M}$ is a smooth graph over $\mathcal{T}\cap B_\nu(Q,0)$ as claimed.

(i) Tangency

Let $(Q+v_0,v_1^*) \in \partial B_\delta(Q,0)\cap\mathcal{M}$. Using (3.2.5) and (3.2.16), we have

$$|\langle kv_0 + v_1^*|\rho\rangle| = |k\lambda_0 + \lambda_1^*| \leq \|N_\rho(v)\|_{L^1_t((0,\infty);L^2_x)} \lesssim (|\lambda_0| + \|(\gamma_0,\gamma_1)\|_{\mathcal{H}})^2,$$

with an absolute constant. Now from (3.2.20), we get $|\lambda_0| + \|\gamma_0\|_{H^1_x} \gtrsim \delta$. For the upper bound, we use

$$|\lambda_0| = \|\lambda_0\rho\|_{L^2_x} = \|P_\rho v_0\|_{L^2_x} \leq \|v_0\|_{L^2_x} \leq \|v_0\|_{H^1_x},$$
$$\|\gamma_0\|_{H^1_x} \leq \|v_0\|_{H^1_x} + \|\lambda_0\rho\|_{H^1_x} \lesssim \|v_0\|_{H^1_x}.$$

In a similar manner, we obtain $\|v_1^*\|_{L^2_x} \simeq \|\gamma_1\|_{L^2_x}$ and hence

$$|\lambda_0| + \|(\gamma_0,\gamma_1)\|_{\mathcal{H}} \simeq \delta,$$

completing the proof of tangency.

(ii) (a) Let $(u_0,u_1)\in\mathcal{M}$. As before, we construct a global solution $u(t) = Q + \lambda(t)\rho + \gamma(t)$. Then $v(t) := u(t) - Q = \lambda(t)\rho + \gamma(t)$ satisfies

$$\begin{aligned}\|(v,\partial_t v)\|_{L^\infty_t((0,\infty);\mathcal{H})} &\lesssim \|\lambda\|_{L^\infty_t(0,\infty)} + \|\gamma\|_{L^\infty_t((0,\infty);H^1)} + \|\partial_t\gamma\|_{L^\infty_t((0,\infty);L^2_x)}\\ &\lesssim \|(\lambda,\gamma)\|_X + \nu \lesssim \nu,\\ \|v\|_{L^q_t((0,\infty);L^6_x)} &\lesssim \|(\lambda,\gamma)\|_X \lesssim \nu, \quad \forall\, q\in[2,\infty),\end{aligned}$$

where we have used (3.2.19) and (3.2.15), thus verifying (a).

(ii) (b) We first verify that $|\lambda(t)|+|\dot{\lambda}(t)| \to 0$ as $t\to\infty$. By (3.2.8) and that $N_\rho(v)\in L^1_t((0,\infty))$, it suffices to obtain the decay for $|\lambda(t)|$. Using (3.2.6), we may further reduce to showing that

$$\int_0^t e^{-k(t-s)}|N_\rho(v)(s)|\,ds + \int_t^\infty e^{-k(s-t)}|N_\rho(v)(s)|\,ds = \int_0^\infty e^{-k|t-s|}|N_\rho(v)(s)|\,ds \to 0, \tag{3.2.21}$$

as $t\to\infty$. For the second integral on the left we have

$$\int_t^\infty e^{-k(s-t)}|N_\rho(v)(s)|\,ds \leq \int_t^\infty e^{-k(t-t)}|N_\rho(v)(s)|\,ds = \|N_\rho(v)\|_{L^1_t(t,\infty)} \to 0, \quad t\to\infty,$$

as $N_\rho(v)\in L^1_t((0,\infty))$. For the first integral, we split the integration domain and estimate each piece separately, that is,

$$\begin{aligned}\int_0^t e^{-k(t-s)}|N_\rho(v)(s)|\,ds &= \int_0^{t/2} e^{-k(t-s)}|N_\rho(v)(s)|\,ds + \int_{t/2}^t e^{-k(t-s)}|N_\rho(v)(s)|\,ds\\ &\leq e^{-kt/2}\|N_\rho(v)\|_{L^1_t(0,\infty)} + \|N_\rho(v)\|_{L^1_t(t/2,\infty)}\\ &\to 0, \quad t\to\infty,\end{aligned}$$

which verifies (3.2.21).
We now construct the unique scattering element γ_∞, which solves the free equation $(\partial_t^2+\omega^2)\gamma_\infty = 0$ and satisfies

$$\|\vec{\gamma} - \vec{\gamma}_\infty\|_{\mathcal{H}} \to 0, \quad \text{as } t \to \infty.$$

From the definition of a free solution,

$$\gamma_\infty(t) = \cos(\omega t)\gamma_\infty(0) + \frac{\sin(\omega t)}{\omega}\partial_t\gamma_\infty(0).$$

Rewriting (3.2.7) in the form

$$\begin{aligned}\gamma(t) = \cos(\omega t)\left[\gamma(0) - \int_0^\infty \frac{\sin(\omega s)}{\omega}N_c(v)ds\right] \\ + \frac{\sin(\omega t)}{\omega}\left[\partial_t\gamma(0) + \int_0^\infty \cos(\omega s)N_c(v)ds\right] + \int_t^\infty \frac{\sin(\omega(t-s))}{\omega}N_c(v)\,ds, \quad (3.2.22)\end{aligned}$$

motivates us to define the scattering initial data by

$$\begin{aligned}\gamma_\infty(0) &:= \gamma(0) - \int_0^\infty \frac{\sin(\omega s)}{\omega}N_c(v)ds, \\ \partial_t\gamma_\infty(0) &:= \partial_t\gamma(0) + \int_0^\infty \cos(\omega s)N_c(v)ds,\end{aligned}$$

and hence

$$\gamma(t) - \gamma_\infty(t) = \int_t^\infty \frac{\sin(\omega(t-s))}{\omega}N_c(v)\,ds.$$

Therefore

$$\|\gamma(t) - \gamma_\infty(t)\|_{H^1} \le \int_t^\infty \|N_c(v)(s)\|_{L_x^2}ds \le \|N(v)\|_{L_t^1((t,\infty);L_x^2)} \to 0,$$

as $t \to \infty$. Similarly, we also have

$$\|\partial_t\gamma(t) - \partial_t\gamma_\infty(t)\|_{L_x^2} \to 0 \quad \text{as } t \to \infty.$$

It is easy to see that $v_\infty(t) := 0 \cdot \rho + \gamma_\infty(t)$ is a free solution for $(\partial_t^2 + L_+)$ and that $u_\infty(t) = Q + v_\infty(t)$.
We now consider the uniqueness of the radiation γ_∞. Fix initial data in $\mathcal{T}$ and suppose we have two scattering solutions $v_\infty(t)$ and $\tilde{v}_\infty(t)$ emanating from the same initial data, and such that they are free solutions for the operator $(\partial_t^2 + L_+)$ and satisfy

$$\|\vec{v}(t) - \vec{v}_\infty(t)\|_{\mathcal{H}}, \|\vec{v}(t) - \vec{\tilde{v}}_\infty(t)\|_{\mathcal{H}} \to 0, \quad \text{as } t \to \infty.$$

It is clear that

$$\|\vec{v}_\infty(t) - \vec{\tilde{v}}_\infty(t)\|_{\mathcal{H}} \to 0, \quad \text{as } t \to \infty.$$

The projections $\gamma_\infty(t) := P_\rho^\perp v_\infty(t)$ and $\tilde{\gamma}_\infty(t) := P_\rho^\perp \tilde{v}_\infty(t)$ are seen to solve $(\partial_t^2 + \omega^2)\gamma = 0$, which implies that the corresponding energy

$$E_{\text{lin}}(\gamma) := \frac{1}{2}\int_{\mathbb{R}^3} (\partial_t\gamma)^2 + (\omega\gamma)^2\,dx$$

is conserved for γ_∞ and $\tilde{\gamma}_\infty$. The difference $\gamma_d(t) := \gamma_\infty(t) - \tilde{\gamma}_\infty(t)$ is also a free solution with constant energy $E_{\text{lin}}(\gamma_d)$. Now (3.1.7) and Lemma 3.1 imply

$$E_{\text{lin}(t)}(\gamma_d) \simeq \frac{1}{2}\|\partial\gamma_d(t)\|_{L_x^2}^2 + \frac{1}{2}\|\gamma_d(t)\|_{H^1}^2,$$

and hence $E_{\text{lin}}(\gamma_d(t)) \to 0$ as $t \to \infty$, which implies $E_{\text{lin}}(\gamma_d) = 0$. Therefore $\gamma_\infty(t) = \tilde{\gamma}_\infty(t)$ for all $t \geq 0$. As for the projections onto ρ, they satisfy the linear ODE $(\partial_t^2 - k^2)\lambda = 0$ and hence $\lambda_d(t) = c_1 e^{kt} + c_2 e^{-kt}$ for some constants c_1, c_2. Using that $|\lambda_d(t)|, |\dot{\lambda}_d(t)| \to 0$ as $t \to \infty$, implies $c_1 = c_2 = 0$ and hence $\lambda_\infty \equiv \tilde{\lambda}_\infty$. This shows the scattering element v_∞ is unique.

Energy splitting: We show that

$$E(u) - J(Q) - \frac{1}{2}\|\vec{\gamma}_\infty\|_{\mathcal{H}}^2 \to 0,$$

as $t \to \infty$ because the energy splitting (c) will then follow by energy conservation. We have

$$\lambda(t), \dot{\lambda}(t), \|\vec{\gamma}(t) - \vec{\gamma}_\infty(t)\|_{\mathcal{H}} \to 0,$$

as $t \to \infty$. Furthermore, by the dispersive estimate and interpolation $\|\gamma_\infty\|_{L_x^p} \to 0$ as $t \to \infty$ for any $2 < p \leq \infty$. The energy splitting is now straightforward upon taking the limit as $t \to \infty$ of (3.1.7) and using all the decay and convergence properties as listed above.

(iv) Invariance property: Let $(Q + v_0, v_1^*) \in \mathcal{M} \subset B_\nu(Q, 0)$. Projecting onto $\mathcal{T}$ we obtain an initial data set $(\lambda_0, \gamma_0, \gamma_1)$ satisfying

$$|\lambda_0| + \|\gamma_0\|_{H_x^1} + \|\gamma_1\|_{L_x^2} \simeq \|(v_0, v_1^*)\|_{\mathcal{H}} \leq \nu.$$

As in the first part of this proof, we construct from this data global solutions $(\lambda(t), \gamma(t))$ satisfying (3.2.6) and (3.2.7) with corresponding global NLKG solution $u(t) = Q + \lambda(t)\rho + \gamma(t)$. Now we fix a $t_0 \in (0, \infty)$ and taking $(\lambda(t_0), \gamma(t_0), \partial_t\gamma(t_0))$, which automatically satisfies

$$|\lambda(t_0) + \|\gamma(t_0)\|_{H_x^1} + \|\partial_t\gamma(t_0)\|_{L_x^2} \leq \|(\lambda, \gamma)\|_X \leq \nu,$$

we construct global solutions $(\tilde{\lambda}^{t_0}(t), \tilde{\gamma}^{t_0}(t))$ which begin at $t = 0$. Now we notice that if $(\lambda(t), \gamma(t))$ solves the pair (3.2.6) and (3.2.7), then so does the translate $(\lambda(t + t_0), \gamma(t + t_0))$. From this we conclude $\tilde{\lambda}^{t_0}(t) = \lambda(t + t_0)$ and $\tilde{\gamma}^{t_0}(t) = \gamma(t + t_0)$ for all $t \geq 0$. To show that the translate $(u(t_0), \partial_t u(t_0))$ remains on $\mathcal{M}$, we need to verify (3.2.5). We have

$$\dot{\lambda}(t_0) = \dot{\tilde{\lambda}}^{t_0}(0) = \lambda_1^*(\lambda(t_0), \gamma(t_0), \partial_t\gamma(t_0)) =: \lambda_1^*(t_0),$$

and

$$\lambda_1^*(t_0) + k\lambda(t_0) = -\int_0^\infty e^{-ks} N_\rho(\tilde{\lambda}^{t_0}(s)\rho + \tilde{\gamma}^{t_0}(s))ds = -e^{-kt_0}\int_{t_0}^\infty e^{-ks} N_\rho(\lambda(s)\rho + \gamma(s))ds.$$

This would conclude the invariance proof, however we have made a crucial assumption here that any global NLKG solution living in $B_\nu(Q, 0)$ in fact has projections $(\lambda, \gamma) \in X$. In order to verify this claim, we seek to show that $\lambda \in L^1(0, \infty)$ and $\gamma \in L_t^2((0, \infty); L_x^6)$. As $\|\vec{u} - (Q, 0)\|_{L_t^\infty([0,\infty);\mathcal{H})} < \nu$, we have

$$\|\lambda\|_{L_t^\infty([0,\infty))} + \|\dot{\lambda}\|_{L_t^\infty([0,\infty))} + \|\vec{\gamma}\|_{L_t^\infty([0,\infty);\mathcal{H})} \lesssim \nu. \tag{3.2.23}$$

For any $T > 0$, we have

$$\|\lambda\|_{L_t^1([0,T))} \leq T\|\lambda\|_{L_t^\infty([0,\infty))}, \quad \|\gamma\|_{L_t^2([0,T);L_x^6)} \leq T^{1/2}\|\gamma\|_{L_t^\infty([0,\infty);H_x^1)}.$$

Of course these will not suffice; we must obtain uniform in T bounds. Consider the norm

$$\|(\lambda, \gamma)\|_{X_T} := \|\lambda\|_{L_t^1([0,T))} + \|\lambda\|_{L_t^\infty([0,\infty))} + \|\gamma\|_{L_t^2([0,T);L_x^6)} + \|\gamma\|_{L_t^\infty([0,\infty);H_x^1)}.$$

From (3.2.6), we compute

$$\|\lambda\|_{L_t^1([0,T))} \lesssim \nu + \|N_\rho(v)\|_{L_t^1([0,T))} + \|N_\rho(v)\|_{L_t^\infty([0,\infty))}.$$

The last term is $\mathcal{O}(\nu^2)$ by (3.2.23) while for the other term, one can show that

$$\|N_\rho(v)\|_{L^1_t([0,T))} \leq \|N(v)\|_{L^1_t([0,T);L^2_x)} \lesssim \nu^2 + \|(\lambda,\gamma)\|^2_{X_T},$$

which implies $\|\lambda\|_{L^1_t([0,T))} \lesssim \nu + \|(\lambda,\gamma)\|^2_{X_T}$. Similarly,

$$\|\gamma\|_{L^2_t([0,T);L^6_x)} \lesssim \nu + \|(\lambda,\gamma)\|^2_{X_T},$$

and hence

$$\|(\lambda,\gamma)\|_{X_T} \lesssim \nu + \|(\lambda,\gamma)\|^2_{X_T}.$$

A continuity argument then implies $\|(\lambda,\gamma)\|_{X_T} \lesssim \nu$ and hence

$$\|\lambda\|_{L^1_t([0,T))} \lesssim \nu, \quad \|\gamma\|_{L^2_t([0,T);L^6_x)} \lesssim \nu, \quad \text{for any } T > 0.$$

Fatou's lemma now gives

$$\|\lambda\|_{L^1_t([0,\infty))} \leq \liminf_{T\to\infty} \|\lambda \mathbb{1}_{[0,T)}\|_{L^\infty_t([0,\infty))} \lesssim \nu.$$

The same holds for γ and thus $(\lambda,\gamma) \in X$. This completes the proof of invariance and the proof of Theorem 3.4. □

3.3 The stable and unstable manifolds

Corollary 3.5. *Let ν be as given in Theorem 3.4. Then there exists a smooth, one dimensional manifold $W^s \subset B_\nu(Q,0)$ such that*

(i) If $(u_0,u_1) \in B_\nu(Q,0)$ with $\vec{u}(t) \to (Q,0)$ in $\mathcal{H}$ as $t \to \infty$, then $\vec{u}(t) \in W^s$ for all $t \geq 0$,
(ii) W^s is tangent to the line

$$\mathcal{T}^s := \{(Q,0) + \lambda(\rho, -k\rho) \,|\, \lambda \in \mathbb{R}\}. \tag{3.3.1}$$

Moreover, $W^s \setminus \{(Q,0)\} = W^s_+ \cup W^s_-$, $W^s_+ \cap W^s_- = \emptyset$, where $W^s_\pm$ each consist of a single solution trajectory which approaches $(Q,0)$ exponentially fast and any two solutions starting on one of $W^s_\pm$ differ only by a time translation.

We also have the unstable manifold W^u which is tangent to the line

$$\mathcal{T}^u := \{(Q,0) + \lambda(\rho, k\rho) \,|\, \lambda \in \mathbb{R}\},$$

and thus transverse to $\mathcal{T}$. This definition will be made clearer in the coming proof.

Proof. Notice firstly that for any $(u_0,u_1) \in B_\nu(Q,0)$ with $\vec{u}(t) \to (Q,0)$ in $\mathcal{H}$ as $t \to \infty$ stays in $B_\nu(Q,0)$ for all sufficiently large $t \geq 0$. This implies that $u(t)$ does not have any exponentially growing modes and so must satisfy (3.2.8). By the invariance statement in Theorem 3.4, $(u(t), \partial_t u(t)) \in \mathcal{M}$ for all $t \geq 0$. We also have that $v(t) := u(t) - Q$ scatters to some $\gamma_\infty(t)$. By the convergence to the ground state and energy conservation, $E(\vec{u}) = J(Q)$ and hence $\vec{\gamma}_\infty \equiv 0$. Since fixing $(\gamma_\infty(0), \partial_t\gamma_\infty(0)) = (0,0)$ is equivalent to fixing (γ_0,γ_1), we see that any solution to NLKG that tends to $(Q,0)$ in $\mathcal{H}$ forward in time is only parametrised by $\lambda(0)$, and this dependence is smooth. We then set

$$W^s := \{(Q + v_0, v_1^*) \in \mathcal{M} \,|\, \vec{u}(t) \to (Q,0), \; t \to \infty\}.$$

The invariance under time translations follows from the invariance property of $\mathcal{M}$. As W^s is solely parametrized by $\lambda(0)$ it is natural to define the subspaces

$$W^s_+ := \{(Q + v_0, v_1^*) \in W^s \,|\, \lambda_0 := \langle v_0|\rho\rangle > 0\},$$
$$W^s_- := \{(Q + v_0, v_1^*) \in W^s \,|\, \lambda_0 := \langle v_0|\rho\rangle < 0\}.$$

Notice that $\lambda_0 = 0$ corresponds to $u = Q$. For the tangency property (3.3.1), Theorem 3.4 implies that W^s is tangent to $\mathcal{T}$ at $(Q, 0)$. The subsets

$$\begin{aligned}\mathcal{T}^s &:= \{(Q + v_0, v_1) \in \mathcal{T} \,|\, (Q + v_0, v_1^*) \in W^s\}\\ &= \{(Q + \lambda_0\rho + \gamma_0(\lambda_0), -k\lambda_0\rho + \gamma_1(\lambda_0)) \,|\, \lambda_0 \in \mathbb{R}\},\\ L^s &:= \{(Q + \lambda_0\rho, -k\lambda_0\rho) \,|\, \lambda_0 \in \mathbb{R}\},\end{aligned}$$

both belong to $\mathcal{T}$ and it is clear that $\mathcal{T}^s$ is tangent to W^s and to L^s at $(Q, 0)$. This implies the tangency of L^s to W^s at $(Q, 0)$.
To see that W^s_+ consists of a single solution trajectory, consider two points $(Q + v_0, v_1^*), (Q + \tilde{v}_0, \tilde{v}_1^*) \in W^s_+$. The case for W_- is analogous. We may suppose that $\lambda_0 > \tilde{\lambda}_0 > 0$. We construct, using the fixed point argument, NLKG solutions of the form $u = Q + \lambda\rho + \gamma$ and $\tilde{u} = Q + \tilde{\lambda}\rho + \tilde{\gamma}$. As $\lambda(t)$ is continuous, $\lambda(0) = \lambda_0$ and, by Theorem 3.4, $\lambda(t) \to 0$ as $t \to \infty$, there exists $t_0 > 0$ such that $\lambda(t_0) = \tilde{\lambda}_0$. The translate $u(t_0 + t)$ is determined uniquely by the data $\lambda(t_0)$ and hence $u(t_0 + t) = \tilde{u}(t)$ verifying the claim for W^s_+.
Finally to deduce the exponential decay, we can take any $(Q + v_0, v_1^*) \in W^s_+$ and solve (3.2.6) and (3.2.7) by a fixed point argument in the space

$$\|(\lambda, \gamma)\|_Y := \sup_{t>0} e^{kt} \left[|\lambda(t)| + \|(\gamma(t), \partial_t\gamma(t)\|_{\mathcal{H}}\right].$$

As such a solution decays exponentially, by virtue of being in the space Y, it also belongs to the space X from Theorem 3.4, and thus coincides with the solution constructed in X. □

3.4 Wave operators

The goal of this section is to justify the Strichartz estimates for L_+ that were used in the proof of Theorem 3.4; this is the result of Corollary 3.9. The key idea is to somehow relate the linear group $e^{it\sqrt{L_+}}$ to the linear KG group $e^{it\sqrt{-\Delta+}}$ so as to be able to make use of the Strichartz estimates for the latter group. A useful relationship is obtained through the theory of *wave operators*. This theory is fundamental in the study of scattering for dispersive PDE. We only briefly discuss it here. The essential pieces are the aforementioned relation, namely(3.4.3), and a theorem of Yajima [22] (Theorem 3.8).

Proposition 3.6. *Define $\tilde{H} := -\Delta + V$ and $\tilde{H}_0 := -\Delta$ and suppose $V \in L^2(\mathbb{R}^3)$. Then the* wave operator

$$\tilde{W} := s\text{-}\lim_{t\to\infty} e^{-itH} e^{itH_0}, \tag{3.4.1}$$

is a bounded operator on $L^2(\mathbb{R}^3)$ to itself, where $s\text{-}\lim_{t\to\infty}$ denotes the strong limit in $L^2(\mathbb{R}^3)$.

Proof. Let $s, t \in \mathbb{R}$ and suppose that $s < t$. Then for any $f \in \mathcal{S}(\mathbb{R}^3)$, we can write

$$e^{-it\tilde{H}} e^{it\tilde{H}_0} f - e^{-is\tilde{H}} e^{is\tilde{H}_0} f = \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau} e^{-i\tau\tilde{H}} e^{i\tau\tilde{H}_0} f \,\mathrm{d}\tau = -i\int_s^t e^{-i\tau\tilde{H}} V e^{i\tau\tilde{H}_0} f \,\mathrm{d}\tau.$$

Taking the L^2 norm of both sides, using the unitarity of $e^{-i\tau\tilde{H}}$ and the assumption $V \in L^2(\mathbb{R}^3)$, we obtain

$$\|e^{-it\tilde{H}}e^{it\tilde{H}_0}f - e^{-is\tilde{H}}e^{is\tilde{H}_0}f\|_{L^2} \leq \|V\|_{L^2}\int_s^t \|e^{i\tau\tilde{H}_0}f\|_{L^\infty}\mathrm{d}\tau.$$

Recall the pointwise decay estimate for the Schrödinger evolution, $\|e^{i\tau\tilde{H}_0}f\|_{L^\infty} \lesssim \langle\tau\rangle^{-3/2} \in L^1((0,\infty))$ and let $\{t_n\}_{n\in\mathbb{N}}$ be some sequence of times such that $t_n \to \infty$ as $n \to \infty$ and put $t = t_n$ and $s = t_m$, $m < n$. Then, by monotone convergence, the right hand side tends to zero as $n \to \infty$ and thus the sequence of bounded operators $\{e^{-it_n\tilde{H}}e^{it_n\tilde{H}_0}\}_{n\in\mathbb{N}}$ is Cauchy and hence converges. Whence (3.4.1) is well defined on $\mathcal{S}(\mathbb{R}^3)$, which is dense in $L^2(\mathbb{R}^3)$ so $\tilde{W}$ extends uniquely to a bounded operator on $L^2(\mathbb{R}^3)$. □

Proposition 3.7. (Properties of wave operators) *The wave operators W satisfy the following properties:*

(i) *W is an isometry on $L^2(\mathbb{R}^3)$ to $\mathrm{Ran}(W)$,*
(ii) *WW^* is an orthogonal projection onto $\mathrm{Ran}(W)$ and $W^*W = \mathrm{Id}$,*
(iii) *Let $H := -\Delta + 1 + V$ and $\tilde{H}_0 := -\Delta + 1$ where $V \in L^2(\mathbb{R}^3)$. Then the wave operator W, associated to (H, H_0), exists and is bounded on $L^2(\mathbb{R}^3)$ and furthermore $W = \tilde{W}$, in the sense that $Wf = \tilde{W}f$ for all $f \in L^2(\mathbb{R}^3)$,*
(iv) *For 'suitable' V, $WW^* = P_c(H) = \mathrm{Id} - P_{pp}(H)$, where P_{pp} is the projection onto the space spanned by the eigenfunctions of H.*
(v) *For any bounded continuous f, we have the identity*

$$f(H)P_c(H) = Wf(H_0)W^*. \tag{3.4.2}$$

In particular, $V = -3Q^2$ is 'suitable' and so $WW^* = P_\rho^\perp$ and

$$f(L_+)P_\rho^\perp = Wf(-\Delta+1)W^*, \tag{3.4.3}$$

for any such f as above.

Proof. (i) Observe that $\|e^{-itH}e^{itH_0}f\|_{L^2} = \|f\|_{L^2}$ for all $f \in L^2$ and for all t.
(ii) Applying (i) to the sum of functions $f + g$ we obtain $\langle Wf, Wg\rangle = \langle f, g\rangle$, which gives $W^*W = \mathrm{Id}$. Thus $(WW^*)^* = WW^*$ and $(WW^*)^2 = WW^*$, verifying the projection claim.
(iii) Using that $e^{-it}e^{it} = 1$, we can construct the wave operator W as we did in Proposition (3.6). Then

$$\tilde{W} = \text{s-}\lim_{t\to\infty} e^{-it(\tilde{H}+1)}e^{it(\tilde{H}_0+1)} = \text{s-}\lim_{t\to\infty} e^{-itH}e^{itH_0} = W.$$

(iv) A proof and some of many possible conditions can be found in Reed and Simon [19].
(v) We deduce (3.4.2) from the property

$$e^{isH}W = We^{isH_0}, \quad \Rightarrow E(\cdot)W = WE_0(\cdot),$$

where E and E_0 are the spectral projections of H and H_0, respectively. The proof of this is straightforward after making the observation

$$\text{s-}\lim_{t\to\infty} e^{-itH}e^{itH_0} = \text{s-}\lim_{t\to\infty} e^{-i(t+s)H}e^{i(t+s)H_0}.$$

Using this property, (iv) and the spectral theorem, we have

$$\begin{aligned} f(H)P_c(H) = \int f(\lambda)E(d\lambda)P_c(H) &= \int f(\lambda)E(d\lambda)WW^* \\ &= \int f(\lambda)WE_0(d\lambda)W^* \\ &= W\left(\int f(\lambda)E_0(d\lambda)\right)W^* = Wf(H_0)W^*. \end{aligned}$$

For $H = L_+$, we know that L_+ has only one eigenfunction ρ, so $P_{pp}(L_+) = P_\rho$. Thus (3.4.3) follows from (3.4.2).

□

We refer to the *threshold* of the spectrum of an operator as the edge of the continuous spectrum. A *regular threshold* can be characterised as

$$\sup_{\mathrm{Im} z>0} \|\langle x\rangle^{-\sigma}(H - z^2)^{-1}P_c(H)\langle x\rangle^{-\sigma}\|_{L^2\to L^2} < \infty,$$

where $\sigma > 0$ is sufficiently large. For an explanation of the relevance of this condition for dispersive estimates we refer to Section 3.4 in [5].

Theorem 3.8. (Yajima [22] *Let V be real-valued and $|V(x)| \lesssim \langle x\rangle^{-\nu}$ where $\nu > 5$. Assume furthermore that the threshold for $\tilde{H} = -\Delta + V$ is regular. Then the wave operator W from* (3.4.1) *is bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$.*

It is beyond the scope of this report to detail the proof of Yajima's theorem. We simply seek to apply it to $\tilde{H} = -\Delta + V$ where $V = -3Q^2$, which is real-valued. Then by property (iv) of Proposition (3.7), we will deduce L^p boundedness for W the wave operator associated to $H = L_+$. The exponential decay for Q ensures the decay condition for V is satisfied, while the gap property for L_+ ensures that the spectral assumptions are satisfied.

Corollary 3.9. *Any solution of*

$$\partial_t^2\gamma + L_+\gamma = F, \quad \vec{\gamma}(0) = (\gamma(0), \partial_t\gamma(0))$$

in $[0,T]\times\mathbb{R}^3_x$ satisfies the estimate

$$\|\gamma\|_{St(0,T)} \lesssim \|\vec{\gamma}(0)\|_{\mathcal{H}} + \|F\|_{L^1_tL^2_x}, \tag{3.4.4}$$

where

$$St(0,T) := (L^2_tL^6_x \cap L^\infty_tH^1_x)([0,T]\times\mathbb{R}^3).$$

Proof. We recall the Duhamel formula

$$\gamma(t) = \cos(t\sqrt{L_+})\gamma(0) + \frac{\sin(t\sqrt{L_+})}{\sqrt{L_+}}\partial_t\gamma(0) + \int_0^t \frac{\sin((t-s)\sqrt{L_+})}{\sqrt{L_+}}P_\rho^\perp N(v)(s)\,ds.$$

It suffices to estimate for the operator $e^{it\sqrt{L_+}}$. Consider firstly the linear estimate

$$\|e^{it\sqrt{L_+}}f\|_{St(0,T)} \lesssim \|f\|_{H^1}.$$

Using (3.4.3) we write

$$\|e^{it\sqrt{L_+}}f\|_{St(0,T)} = \|We^{it\sqrt{-\Delta+1}}W^*f\|_{St(0,T)}.$$

By the L^p boundedness of the wave operator W and the Strichartz estimates for the ordinary Klein-Gordon group $e^{it\sqrt{-\Delta+1}}$, we obtain

$$\|We^{it\sqrt{-\Delta+1}}W^*f\|_{St(0,T)} \lesssim \|e^{it\sqrt{-\Delta+1}}W^*f\|_{St(0,T)} \lesssim \|W^*f\|_{H^1} \lesssim \|f\|_{H^1}.$$

In deriving the last inequality, we have used the easy fact that $\langle\nabla\rangle$ commutes with W. For the middle term, we use the boundedness of the multiplier corresponding to $\sqrt{-\Delta+1}/\sqrt{L_+}$. For the non-linear term, we make use of the non-linear Strichartz estimates for $e^{it\sqrt{-\Delta+1}}$. □

Chapter 4

Above the ground state

In the previous chapter, we studied the stable dynamics close to the ground states $(\pm Q, 0)$. Theorem 3.4 introduced us to the new type of behaviour in this regime (3.0.1) which is trapping by the ground state. What occurs for non-trapped trajectories? As the energy is not too far from that of the ground state, we might reasonably expect to be able to mimic the Payne-Sattinger theory giving a scattering, blow-up dichotomy. This would require us to control the sign of K_0 eventually, which a priori is not obvious. The key ingredients are the following three facts:

1. The sign of K_0 can only change if you enter a small 2ϵ-ball about $(\pm Q, 0)$ (Proposition 4.9)
2. Solutions not trapped by the 2ϵ-balls are ejected to a much greater distance (Ejection Lemma 4.4)
3. Upon exit from the 2ϵ-ball, the solution cannot re-enter (One-Pass Theorem 4.12).

Combining these three results leads to the full characterisation of the behaviour slightly above the ground state.

4.1 Nonlinear distance function

In this subsection, we define the nonlinear distance function which is designed to measure a notion of distance from on of the fixed ground states $(\pm Q, 0)$. With $u = \sigma(Q + v)$, with $v = \lambda\rho + \gamma$ and $\sigma = \pm$, we define the linearized energy as

$$\begin{aligned} \|\vec{v}\|_E^2 &:= \frac{1}{2}\left[k^2\langle v|\rho\rangle^2 + \|\omega P_\rho^\perp v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2\right] \\ &= \frac{1}{2}\left[k^2|\lambda|^2 + |\dot{\lambda}|^2 + \|\omega\gamma\|_{L^2}^2 + \|\partial_t\gamma\|_{L^2}^2\right]. \end{aligned} \tag{4.1.1}$$

The name is motivated by the fact that we have the exact decomposition of the energy

$$E(\vec{u}) - J(Q) + k^2\lambda^2 = \|\vec{v}\|_E^2 - C(v), \quad C(v) := \langle Q|v^3\rangle + \|v\|_4^4/4. \tag{4.1.2}$$

Lemma 4.1. *We have the equivalence*

$$\|\vec{v}\|_E^2 \simeq \|\vec{v}\|_{\mathcal{H}}^2 = \|v\|_{H^1}^2 + \|\dot{v}\|_{L^2}^2. \tag{4.1.3}$$

Proof. By Lemma 3.1, we have

$$\|\vec{v}\|_E^2 \simeq \frac{1}{2}\left[k^2|\lambda|^2 + |\dot{\lambda}|^2 + \|\gamma\|_{H^1}^2 + \|\partial_t\gamma\|_{L^2}^2\right].$$

It suffices to show that $\|v\|_{H^1}^2 \simeq |\lambda|^2 + \|\gamma\|_{H^1}^2$. Using $L_+\rho = -k^2\rho$, we find

$$\|v\|_{H^1}^2 = \|\rho\|_{H^1}^2|\lambda|^2 + \|\gamma\|_{H^1}^2 + 2\lambda\langle 3Q^2\rho|\gamma\rangle_{L^2}.$$

As an upper bound here is trivial we focus on the lower bound. We have

$$\|v\|_{H^1}^2 \geq \|\rho\|_{H^1}^2|\lambda|^2 + \|\gamma\|_{H^1}^2 - |\lambda|^2\|3Q^2\rho\|_{L^2}^2 - \|\gamma\|_{L^2}^2,$$
$$\|v\|_{H^1}^2 \geq \|v\|_{L^2}^2 = |\lambda|^2 + \|\gamma\|_{H^1}^2.$$

Combining these, we obtain

$$\|v\|_{H^1}^2 \geq (1-\theta+\theta(\|\rho\|_{H^1}^2 - \|3Q^2\rho\|_{L^2}^2))|\lambda|^2 + \theta\|\nabla\gamma\|_{L^2}^2 + (1-\theta)\|\gamma\|_{L^2}^2,$$

for $\theta \in [0,1]$. If $\|\rho\|_{H^1}^2 - \|3Q^2\rho\|_{L^2}^2 \geq 0$ we are done, else we may choose $\theta \in (0,1)$ such that the first term vanishes. This furnishes the desired lower bound. □

For small enough $\|\vec{v}\|_{H^1}$, we expect to be able to control the non-linearity $C(v)$. A precise statement of this is that there exists a fixed $0 < \delta_E \ll 1$ such that when

$$\|\vec{v}\|_E \leq 4\delta_E, \quad \text{then} \quad |C(v)| \leq \|\vec{v}\|_E^2/2. \tag{4.1.4}$$

To see this, we use Sobolev embedding and (4.1.2) to obtain, for some constant C depending on norms of Q,

$$|C(v)| \leq (C(Q)\delta_E + C_0\delta_E^2)\|\vec{v}\|_E^2 \leq \|\vec{v}\|_E^2/2,$$

by choosing δ_E sufficiently small. We can now define the *nonlinear distance function*

$$d_\sigma(\vec{u}) := \sqrt{\|\vec{v}\|_E^2 - \chi(\|\vec{v}\|_E/2\delta_E)C(v)}, \tag{4.1.5}$$

where χ is a smooth cut-off on $\mathbb{R}$ such that $\chi(r) \equiv 1$ when $|r| \leq 1$ and $\chi(r) \equiv 0$ for $|r| \geq 2$.

Lemma 4.2. *The nonlinear distance function satisfies the following properties:*

$$\|\vec{v}\|_E/2 \leq d_\sigma(\vec{u}) \leq 2\|\vec{v}\|_E \tag{4.1.6}$$
$$d_\sigma(\vec{u}) = \|\vec{v}\|_E + \mathcal{O}(\|\vec{v}\|_E^2), \tag{4.1.7}$$
$$d_\sigma(\vec{u}) \leq \delta_E \implies d_\sigma^2(\vec{u}) = E(\vec{u}) - J(Q) + k^2\lambda^2. \tag{4.1.8}$$

Proof. The proof of (4.1.6) follows easily by splitting into the cases $\|\vec{v}\|_E > \delta_E$ and so forth. For (4.1.7), we notice that in deriving (4.1.4), we could have made the finer estimate of $|C(v)| \lesssim \|\vec{v}\|_E^3$. Then

$$\frac{d_\sigma(\vec{u}) - \|\vec{v}\|_E}{\|\vec{v}\|_E^2} = \frac{\sqrt{1-\chi(\|\vec{v}\|_E/2\delta_E)C(v)/(\|\vec{v}\|_E^2)} - 1}{\|\vec{v}\|_E} \sim \|\vec{v}\|_E + \mathcal{O}(\|\vec{v}\|_E^2),$$

when $0 \neq \|\vec{v}\|_E \ll 1$. Finally to obtain (4.1.8), we notice that $d_\sigma(\vec{u}) \leq \delta_E$ implies $\|\vec{v}\|_E \leq 2\delta_E$ in which case $d_\sigma(\vec{u}) = \|\vec{v}\|_E^2 - C(v)$. □

In order to differentiate between the two possible ground states $(\pm Q, 0)$, we define the

$$d_Q(\vec{u}) := \min_{\sigma\in\{\pm\}} d_\sigma(\vec{u}) \simeq \min_{\pm} \|\vec{u} \pm (Q,0)\|_{\mathcal{H}}.$$

When $d_Q(\vec{u}) \leq 2\delta_E$, then (4.1.6) and (4.1.3) imply that $\|\vec{v}\|_{\mathcal{H}} \lesssim 4\delta_E$. As $\|\vec{v}\|_{\mathcal{H}}$ is either one of $\|\vec{u} \pm (Q,0)\|_{\mathcal{H}}$, so

$$\min_{\pm} \|\vec{u} \pm (Q,0)\|_{\mathcal{H}} \lesssim 4\delta_E.$$

However, $\|(\vec{u}+(Q,0)) - (\vec{u}-(Q,0))\|_{\mathcal{H}} = 2\|(Q,0)\|_{\mathcal{H}}$ so having chosen δ_E sufficiently small, we can make a unique choice for σ, allowing us to write $d_Q(\vec{u}) = d_\sigma(\vec{u})$.

4.2 Ejection Lemma

In order to study the blow-up behaviour contained within the mode λ it is more natural to also consider the quantities

$$\lambda_+ := \frac{1}{2}\left(\lambda + \frac{1}{k}\dot{\lambda}\right), \quad \lambda_- := \frac{1}{2}\left(\lambda - \frac{1}{k}\dot{\lambda}\right). \tag{4.2.1}$$

In the case of the linear version of (3.1.6) (setting $N_\rho(v) \equiv 0$), we have that when $\lambda_\pm = 0$, $\lambda(t) = \lambda(0)e^{\mp kt}$, indicating that λ_- should correspond to the stable mode and λ_+ to the unstable mode. Inverting (4.2.1) gives

$$\lambda = \lambda_+ + \lambda_-, \quad \dot{\lambda} = k(\lambda_+ - \lambda_-), \tag{4.2.2}$$

and this shows that the λ ODE in (3.1.6) becomes the system

$$\dot{\lambda}_+ = -k\lambda_- - \frac{1}{2k}N_\rho(v), \quad \dot{\lambda}_- = k\lambda_+ + \frac{1}{2k}N_\rho(v). \tag{4.2.3}$$

Lemma 4.3. *For any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u}) < J(Q) + d_Q^2(\vec{u})/2, \quad d_Q(\vec{u}) \le \delta_E, \tag{4.2.4}$$

then $d_Q(\vec{u}) \simeq |\lambda|$ and λ has a fixed sign within each connected component of the region given by (4.2.4).

Proof. By the assumption, (4.1.8) implies

$$d_q(\vec{u}) = E(\vec{u}) - J(Q) + k^2\lambda^2 < d_Q^2(\vec{u}) + k^2\lambda^2,$$

and hence $k^2\lambda^2/16 \le \|\vec{v}\|_E^2/8 \le d_Q^2(\vec{u})/2 < k^2\lambda^2$. Fix a connected component of (4.2.4) and suppose that the sign of λ does change. Then, by the intermediate value theorem, there is some time $t = T$ for which $\lambda(T) = 0$, but this is a contradiction for then $d_Q(\vec{u}) = 0$ giving $u = \pm Q$ while $E(\vec{u}) < J(Q)$. □

The following result, which is known as the Ejection Lemma, essentially formalizes our expectation that the unstable dynamics are caused by the exponentially growing mode of λ which is in turn due to the fact that the ground state is unstable.

Lemma 4.4. (Ejection Lemma) *There exists a constant $0 < \delta_X \ll \delta_E$ with the following property: Let $u(t)$ be a local solution to NLKG on its maximal time of existence $[0, T]$ satisfying*

$$0 < R := d_Q(\vec{u}(0)) \le \delta_X, \quad E(\vec{u}) < J(Q) + R^2/2 \tag{4.2.5}$$

and alternatively one of

$$\begin{cases} \text{for some } t_0 \in (0,T), \ \ d_Q(\vec{u}(t)) \ge R \ \ \text{for all } t \in (0, t_0), \\ \frac{d}{dt}d_Q(\vec{u}(t))|_{t=0} \ge 0, \end{cases} \tag{4.2.6}$$

Then $d_Q(\vec{u}(t))$ monotonically increases until reaching δ_X, while the following hold:

$$\begin{aligned} &d_Q(\vec{u}(t)) \simeq -\mathfrak{s}\lambda(t) \simeq -\mathfrak{s}\lambda_+(t) \simeq Re^{kt}, \\ &|\lambda_-(t)| + \|\vec{\gamma}(t)\|_E \lesssim R + d_Q^2(\vec{u}(t)), \\ &\min_{j=0,2} \mathfrak{s}K_j(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(0)), \end{aligned} \tag{4.2.7}$$

where $\mathfrak{s} \in \{\pm 1\}$ is a fixed sign and $C_* > 1$ is an absolute constant.

Proof. Lemma (4.3) gives $d_Q(\vec{u}) \simeq |\lambda|$ as long as $R \leq d_Q(\vec{u}) \leq \delta_E$. By (4.1.8), (3.1.6) and energy conservation we obtain

$$\partial_t d_Q(\vec{u}) = 2k^2\lambda\dot{\lambda}, \quad \partial_t^2 d_Q(\vec{u}) = 2k^4|\lambda|^2 + 2k^2|\lambda|^2 + 2k^2\lambda N_\rho(v). \tag{4.2.8}$$

Now $|\dot{\lambda}|^2 \lesssim |\lambda|^2$ as long as $d_Q(\vec{u}) \simeq |\lambda| \ll 1$, and $|N_\rho(v)| \lesssim \|v\|_{H^1}^2$ so that

$$\partial_t^2 d_Q^2(\vec{u}) \simeq d_Q^2(\vec{u}). \tag{4.2.9}$$

Choosing $\delta_X \leq \delta_E$ to be small enough so that we can ensure $\|v\|_{H^1}^3 \lesssim \|v\|_{H^1}^2$ and $d_Q(\vec{u}) \ll 1$, then we have that $d_Q(\vec{u}(t)) \geq R$ monotonically increases until hitting δ_X exponentially as in (4.2.7). Note that u must hit δ_X in finite time, since otherwise once it nears the end of its maximal time of local existence, its H^1 norm must blow-up guaranteeing that it will hit δ_X anyway. As we have remained within a connected component of the region in Lemma (4.3), $d_Q(\vec{u}) \simeq \mathfrak{s}\lambda$. Furthermore, (4.2.9) implies $\partial_t d_Q(\vec{u}(t)) \geq 0$ and hence λ and $\dot{\lambda}$ have the same sign. Now from (4.2.1), we have $\lambda \simeq \lambda_+$. Next we integrate (4.2.3) for λ_+ and use that $|\lambda_+| \simeq |\lambda| \ll 1$ to obtain

$$|\lambda_+(t) - \lambda_+(0)e^{kt}| \lesssim \int_0^t e^{k(t-s)}|\lambda_+(s)|^2 ds \lesssim e^{kt}.$$

Hence we conclude that $d_Q(\vec{u}(t)) \simeq |\lambda_+(t)| \simeq Re^{kt}$. A similar argument gives the claimed bound for λ_-. For the bound on γ, we take (4.1.2) and add and subtract $C(\lambda\rho)$, differentiate in time and use energy conservation to arrive at

$$|\partial_t(\|\vec{\gamma}\|_E^2 + C(\lambda\rho) - C(v))| = \left|\partial_t\left(-\frac{1}{2}k^2\lambda^2 + \frac{1}{2}\dot{\lambda}^2 - C(\lambda\rho)\right)\right|.$$

For the right hand side, we make use of the equation for λ, λ-dominance and the easy to verify fact that $\partial_t C(\lambda(t)\rho) = N_\rho(\lambda\rho)\dot{\lambda}$, to obtain

$$|\partial_t\left(-\frac{1}{2}k^2\lambda^2 + \frac{1}{2}\dot{\lambda}^2 - C(\lambda\rho)\right)| \lesssim |N_\rho(v) - N_\rho(\lambda\rho)||\lambda|.$$

Since $\|\vec{\gamma}\|_E \lesssim |\lambda| \ll 1$, Sobolev embedding and the exponential decay of Q, yield $|N_\rho(v) - N_\rho(\lambda\rho)| \lesssim \|\gamma\|_{H^1}|\lambda|$. Inserting this bound into (4.2) and using (4.1.4), we infer

$$\|\vec{\gamma}\|_{L_t^\infty E(0,T)}^2 \lesssim R^2 + \vec{\gamma}\|_{L_t^\infty E(0,T)} R^2 e^{2kT},$$

which can be worked to give the desired estimate on γ. For the estimates on K_0 and K_2 we recall that we have

$$K_0(u) = -k^2\lambda\langle Q|\rho\rangle - \langle 2Q^3|\gamma\rangle + \mathcal{O}(\|v\|_{H^1}^2),$$
$$K_2(u) = -(k^2/2 + 2)\lambda\langle Q|\rho\rangle - \langle 2Q + Q^3|\gamma\rangle + \mathcal{O}(\|v\|_{H^1}^2).$$

Multiplying both sides by $\mathfrak{s}$, using $d_Q(\vec{u}) \simeq -\mathfrak{s}\lambda$ and that $Q, \rho > 0$, we obtain the desired estimates, where C_* will depend on norms of Q and ρ, but is otherwise fixed. □

The following corollary gives some conditions for which ejection is ensured. The conditions agree with our intuition; the unstable dynamics should dominate if it initially dominates. The proof of both statements just require one to show that (4.2.5) and (4.2.6) hold.

Corollary 4.5. *Suppose that $\vec{u}(0) \in \mathcal{H}$ satisfies one of the following:*

(i) $\vec{u}(0)$ *satisfies* (4.2.5) *and* $|\lambda_+(0)| \geq |\lambda_-(0)|$,
(ii) $\|\lambda_-(0)| + \|\vec{\gamma}(0)\|_{H^1\times L^2} \ll |\lambda_+(0)| \ll \delta_X$.

Then the ejection lemma applies.

The next two results are powerful consequences of the Ejection Lemma. The first rules out the existence of circulating trajectories outside the $2\epsilon-$ball. This result then implies (2) on our list above, that non-trapped trajectories are ejected to δ_X.

Lemma 4.6. *There does not exist an NLKG solution with $E(\vec{u}) < J(Q) + \epsilon^2$ and the following properties: u exists for all $t \geq 0$ and $2\epsilon < d_Q(\vec{u}) < \delta_X$ for all $t \geq 0$.*

Proof. Suppose, in order to obtain a contradiction, that such a solution does exist. Set $\delta_0 := \inf_{t\geq 0} d_Q(\vec{u}(t))$. If the infimum is attained, there exists some T_0 such that $d_Q(\vec{u}(T_0)) = \delta_0$. Then for all $t \geq T_0$, $d_Q(\vec{u}(t)) \geq \delta_1$ and $E(\vec{u}) < J(Q) + \epsilon^2 < J(Q) + \delta_0^1/2$. Then the Ejection Lemma applies sarting from $t = T_0$ and $d_Q(\vec{u})$ hits δ_X in finite time, a contradiction. On the other hand, suppose δ_0 is not attained. Now $d_Q(\vec{u}(t))$ cannot attain any local minimum as otherwise the Ejection Lemma would apply from such a time. Therefore $d_Q(\vec{u}(t))$ is monotonically decreasing and since it cannot hit the 2ϵ-ball, $\partial_t d_Q(\vec{u}(t)) \to 0$ as $t \to \infty$. From (4.2.8), $\partial_t^2 d_Q(\vec{u}(t)) \gtrsim \epsilon^2$ as $|\lambda| \simeq d_Q(\vec{u}) > 2\epsilon$. Combining these two estimates yields a contradiction. □

Definition 4.7. We say a trajectory $\vec{u}(t)$ is *trapped* by an R-ball if it exists for all $t \geq 0$, and if $d_Q(\vec{u}(t)) \geq R$ for all $t \geq T$, where $T > 0$ is finite. We say that $\vec{u}(t)$, defined on $[0, T)$ is *ejected* from the δ_X-ball if there exists a time interval $[t_0, t_1] \subset [0, T)$ so that $d_Q(\vec{u}(t))$ is strictly increasing on $[t_0, t_1]$ and satisfies

$$\begin{aligned} E(\vec{u}) &< J(Q) + \frac{1}{2} d_Q^2(\vec{u}(t)), \\ d_Q(\vec{u}(t_0)) &= \frac{\delta_X}{10}, \quad d_Q(\vec{u}(t_1)) = \delta_X, \\ d_Q(\vec{u}(t)) &\simeq d_Q(\vec{u}(0)) e^{k(t-t_0)}, \quad \forall t \in (t_0, t_1), \\ K_j(u(t)) &\simeq -\mathrm{sign}(\lambda(t)) d_Q(\vec{u}(t)) \quad \forall t \in (t_0, t_1) \text{ and } j = 0, 2. \end{aligned} \tag{4.2.10}$$

Corollary 4.8. *Suppose that some solution to the NLKG satisfies*

$$d_Q(\vec{u}(0)) \ll \delta_X, \quad \textit{and} \quad E(\vec{u}) < J(Q) + \epsilon^2, \quad \epsilon \ll \delta_X.$$

and is not trapped *by the 2ϵ-balls around $(\pm Q, 0)$ as measured relative to the d_Q-metric. Then $\vec{u}$ is ejected from the δ_X-ball.*

Proof. Lemma 4.6 implies that $\vec{u}(t)$ does not circulate between the 2ϵ and δ_X balls. Therefore $d_Q(\vec{u})$ either hits 2ϵ or δ_X at some finite time. Suppose it hits the 2ϵ-ball at some time $T_1 < \infty$. Since it is not trapped, there exists a $T_3 > T_1$ such that $d_Q(\vec{u}(T_3)) > 2\epsilon$, and hence, by continuity, there is a $T_1 < T_2 < T_3$ such that $\partial_t|_{t=T_2} d_Q(\vec{u}(T_3)) \geq 0$. Applying the ejection lemma at time $t = T_2$ implies we are ejected to δ_X. Now suppose it hits the δ_X ball. It suffices to assume $d_Q(\vec{u}(t))$ that achieves its infimum otherwise it would enter the 2ϵ-ball and the preceeding argument will imply ejection to δ_X. At its infimum,(4.2.5) holds and hence applying the Ejection Lemma gives that we hit δ_X and, meanwhile, after a finite amount of time, we will satisfy all of (4.2.10), giving ejection. □

4.3 Variational characterisation

Far from $(\pm Q, 0)$ we can control the behaviour by the sign of the functionals K_0, K_2. The following proposition shows that the signs of K_0, K_2 cannot change outside a small neighbourhood about $(\pm Q, 0)$. This is an analogue of the similar behaviour described by the Payne-Sattinger theory for energies less than $J(Q)$.

Proposition 4.9. *For any $\delta > 0$, there exists $\epsilon_0(\delta), \kappa_1(\delta) > 0$ and an absolute constant $\kappa_0 > 0$ such that for any $\vec{u} \in \mathcal{H}$ satisfying*

$$E(\vec{u}) < J(Q) + \epsilon_0^2(\delta), \quad d_Q(\vec{u}) \geq \delta, \tag{4.3.1}$$

we have

$$\text{sign}K_0(u) = \text{sign}K_2(u).$$

More precisely, one has either

$$K_0(u) \leq -\kappa_1(\delta) \quad \text{and} \quad K_2(u) \leq -\kappa_1(\delta), \tag{4.3.2}$$

or

$$K_0(u) \geq \min(\kappa_1(\delta), \kappa_0\|u\|_{H^1}^2) \quad \text{and} \quad K_2(u) \geq \min(\kappa_1(\delta), \kappa_0\|\nabla u\|_{L^2}^2). \tag{4.3.3}$$

Proof. We prove (4.3.2) and (4.3.3) by separating the cases for $j = 0$ and $j = 2$. For instance, we pick $j = 0$ and fix $\delta > 0$. Then, in order to obtain a contradiction to (4.3.2) and (4.3.3), we can find a subsequence $\vec{u}_n \in \mathcal{H}$ satisfying (4.3.1) with $\epsilon_0 = 1/n$, and with $\kappa_1 = 1/n$ we have

$$-\frac{1}{n} \leq K_2(u_n) \leq \min\left(\frac{1}{n}, \kappa_0\|u\|_{L^2}^2\right).$$

Then $K_0(u_n) \to 0$ and hence, for large enough n, $0 \leq \|u_n\|_{H^1}^2 \sim G_0(u_n) \lesssim J(Q) + \epsilon^2$ which implies $\|u_n\|_{H^1}^2$ is bounded. Extracting subsequences, we have u_n converges to a limit u_∞ weakly in H^1 and strongly in L^4. By weak lower semi-continuity of L^p norms and as $K_j(u_n) \to 0$, we get $\|u_\infty\|_{H^1}^2 \leq \|u_\infty\|_{L^4}^4$ which implies $K(u_\infty) \leq 0$. By the strong L^4 convergence, we also obtain $G_0(u_\infty) \leq J(Q)$ and $J(u_\infty) \leq J(Q)$. Proceeding as in Lemma 2.9 in [5], we conclude that u_n converges strongly in H^1 to 0 or $\pm Q$.
Suppose $u_n \to \pm Q$. Then by (4.3.1),

$$\delta^2 \leq \liminf_{n\to\infty} d_Q^2(\vec{u}_n) \leq C \liminf_{n\to\infty} \|v_n\|_{L^2}^2 + \liminf_{n\to\infty} \|\dot{u}_n\|_{L^2}^2 = 0,$$

which is a contradiction. If instead $u_n \to 0$ in H^1, then by the Gagliardo-Nirenberg inequality, we get, for sufficiently large n,

$$K_0(u_n) \geq \|u_n\|_{H^1}^2(1 - C\|u_n\|_{H^1}^2) \geq \kappa_0\|u_n\|_{H^1}^2$$

for a fixed $\kappa_0 > 0$, which is a contradiction. Therefore (4.3.2) and (4.3.3) are verified for K_0.
We now proceed to show that the signs of K_0 and K_2 agree. The proof is a variation on the proof of (2.0.3) in Lemma 2.2. We set

$$\mathcal{K}_j^+ := \{\vec{u} \in \mathcal{H} \,|\, E(\vec{u}) < J(Q) + \epsilon_0^2,\ d_Q(\vec{u}) > \delta,\ K_j(u) \geq 0\},$$
$$\mathcal{K}_j^- := \{\vec{u} \in \mathcal{H} \,|\, E(\vec{u}) < J(Q) + \epsilon_0^2,\ d_Q(\vec{u}) > \delta,\ K_j(u) < 0\},$$

for $j = 0, 2$. It is clear that $\mathcal{K}_j^-$ is open while for $\mathcal{K}_j^+$, (4.3.2) allows us to replace $K_j(u) \geq 0$ by $K_j(u) > -\kappa_1(\delta)$ and it is now clear that $\mathcal{K}_j^+$ is also open. Furthermore, $\mathcal{K}_j^+ \cap \mathcal{K}_j^- = \emptyset$, $\mathcal{K}_0^+ \cup \mathcal{K}_0^- = \mathcal{K}_2^+ \cup \mathcal{K}_2^-$ and $0 \in \mathcal{K}_0^+, \mathcal{K}_2^+$ as we must write $0 = Q + (-Q)$. In order to show that $\mathcal{K}_0^+ = \mathcal{K}_2^+ = \{\vec{u} \in \mathcal{H} \,|\, E(\vec{u}) < J(Q) + \epsilon_0^2,\ d_Q(\vec{u}) > \delta\}$, it suffices to show that $\mathcal{K}_j^+$, for both $j = 0, 2$, are path-connected. This follows from showing that there exists a path from any point to 0 in these sets.
Fix $j = 0, 2$. Consider the case when $\delta < d_Q(\vec{u}) < 2\delta \ll \delta_E$. Lemma 4.3 implies $d_Q(\vec{u}) \simeq |\lambda|$. From the decompositions (4.1.2) and (4.1.8), we see that if we make the transformation $\vec{u}^\nu := (Q + (1+\nu)\lambda\rho + \gamma, \dot{\lambda}\rho + \dot{\gamma})$, for $\nu > 0$, then the $|\lambda|$ increases implying that $d_Q(\vec{u}^\nu)$ increases while $E(\vec{u}^\nu)$ decreases. In particular, taking *nu* sufficiently large, we can ensure that

$$E(\vec{u}^\nu) - J(Q) \ll -d_Q^2(\vec{u}^\nu) + \mathcal{O}(\delta^2) + o(d_Q^2(\vec{u}^\nu)) \ll -\delta^2. \tag{4.3.4}$$

To arrive at this, we for instance use that $\|\gamma\|_{H^1}^2 \lesssim \|\vec{v}\|_E^2 \simeq d_Q^2(\vec{u}) \simeq \delta^2$, and that $C(v)$ will involve terms of lower order. It is clear that this transformation remains in $\mathcal{K}_j^+$ as $E(\vec{u})$ decreases, $d_Q(\vec{u})$ increases and (4.3.3) implies that K_j remains non-negative. We have thus moved any point in $\mathcal{K}_j^+ \cap \{d_Q(\vec{u}) \leq 2\delta\}$ into

$$\{\vec{u} \in \mathcal{H} \,|\, E(\vec{u}) - J(Q)) \ll -\delta^2, \;\; K_j(u) \geq 0\}$$

will remaining in $\mathcal{K}_j^+$. This set is indeed a subset of $\mathcal{K}_j^+$ as in obtaining (4.3.4) we have transformed so that $d_Q(\vec{u}) \gg \delta$. Now we can use the transformation

$$\vec{u}^\nu := (u_j^\nu, \nu \partial_t u),$$

with ν decreasing from 1 to 0, which will send $\vec{u}$ to 0. For the details, see the proof of Lemma 2.21 in [5].

In the case where $d_Q(\vec{u}) \geq 2\delta$, we apply the scaling transformation as above and we either hit 0 or we hit the sphere $d_Q(\vec{u}) = 2\delta$. This follows since we must write, for example, $u_0^\nu = \nu u = Q + (\nu - 1)Q + \nu v$ when considering $d_Q(\vec{u}^\nu)$. In the latter case we contract to 0 in the manner described for the other region. □

Lemma 4.3 implies that the sign of λ is fixed close to Q while Proposition 4.9 implies the sign of K_0 is fixed far from Q. We can combine these results to show that these signs coincide in the overlap region and the continuity of both sign functions are preserved.

Lemma 4.10. *Let $\delta_S := \delta_X/(2C_*) > 0$ and for any $\delta \in (0, \delta_S]$, define*

$$\mathcal{H}_{(\delta)} := \{\vec{u} \in \mathcal{H} \,|\, E(\vec{u}) < J(Q) + \min(d_Q^2(\vec{u})/2, \epsilon_0^2(\delta)\}, \tag{4.3.5}$$

with $\epsilon_0(\delta)$ given as in Lemma (4.4).Then there exists a unique continuous function $\mathfrak{S} : \mathcal{H}_{(\delta)} \to \{\pm 1\}$ satisfying

$$\begin{cases} \vec{u} \in \mathcal{H}_{(\delta)}, \;\; d_Q(\vec{u}) \leq \delta_E \implies \mathfrak{S}(\vec{u}) = -\operatorname{sign}\lambda \\ \vec{u} \in \mathcal{H}_{(\delta)}, \;\; d_Q(\vec{u}) \geq \delta \implies \mathfrak{S}(\vec{u}) = \operatorname{sign} K_0(u) = \operatorname{sign} K_2(u), \end{cases} \tag{4.3.6}$$

with the convention sign$\,0 = +1$.

Proof. We will give a sketch of the proof. By Lemma 4.3, sign λ is a continuous function when $d_Q(\vec{u}) \leq \delta_E$, while when $d_Q(\vec{u}) \geq \delta$, Proposition 4.9 implies that $\operatorname{sign} K_0 = \operatorname{sign} K_2$ is also a continuous function. It thus suffices to show that $\operatorname{sign} K_0$ agrees with $-\operatorname{sign}\lambda$ at $d_Q(\vec{u}) = \delta_S$. For this purpose, one can show that it suffices to consider $\vec{u}(0) \in \mathcal{H}$ that satisfies the ejection conditions and $d_Q(\vec{u}(0)) = \delta_S$. Using this as initial data for the NLKG evolution, we obtain a solution which is ejected to δ_X. At the time of ejection, $-\operatorname{sign}\lambda(u(t)) = \operatorname{sign} K_0(u(t))$. By the constancy of $\operatorname{sign} K_0$ above the δ-ball, we conclude these signs must have been equal at $t = 0$. □

Lemma 4.11. *There exists a constant $M_* \simeq J(Q)^{1/2}$ such that for all $\vec{u} \in \mathcal{H}_{(\delta)}$ with $\mathfrak{S}(\vec{u}) = +1$, we have $\|\vec{u}\|_{\mathcal{H}} \leq M_*$.*

Proof. If $d_Q(\vec{u}) \leq \delta_S \ll 1$, then recalling (4.1.3) and (4.1.6), we have

$$\|\vec{u}\|_{\mathcal{H}} \leq \|Q\|_{H^1} + \|\vec{v}\|_{\mathcal{H}} \lesssim J(Q)^{1/2} + d_Q(\vec{u})^{1/2} \lesssim J(Q)^{1/2}.$$

On the other hand, if $d_Q(\vec{u}) \geq \delta_S$, then by assumption we have $K_0(u) \geq 0$, and hence

$$\|\vec{u}\|_{\mathcal{H}}^2 \leq 4E(\vec{u}) - K_0(u) \leq 4E(\vec{u}) \leq 4\left(J(Q) + \epsilon_0^2(\delta_S)\right) \lesssim J(Q).$$

□

The point of this result is that solutions with $\mathfrak{S}(\vec{u}) = +1$ are automatically globally well defined, simply by iterating the local theory. We in fact show that such solutions in fact scatter to zero in the next chapter. As for the opposite sign $\mathfrak{S}(\vec{u}) = -1$, such solutions experience finite time blow-up; paralleling the Payne-Sattinger theory.

4.4 The One-Pass Theorem

Our goal is show that $\operatorname{sign} K_0(u(t))$ stabilizes, that is there exists a $T > 0$ such that for all $t > T$, $\operatorname{sign} K_0(u(t)) = \operatorname{sign} K_2(u(t))$ is constant. As the sign can only change upon entering the 2ϵ-ball, then it may be possible a solution exists which oscillates in and out of this ball forever and thus $\operatorname{sign} K_0(u(t))$ can never be eventually constant. If we could limit the number of times a solution could enter and exit the ball, the sign of K_0 would stabilize. This is in fact the result of The One-Pass Theorem.

We make the following choice of small constants $\epsilon_*, \delta_*, R_*, \mu > 0$

$$\begin{gathered} \delta_* \le \delta_S, \quad \delta_* \ll \delta_X, \quad \epsilon_* \le \epsilon_0(\delta_*), \\ \epsilon_* \ll R_* \ll \min(\delta_*, \kappa_1(\delta_*)^{1/2}, \kappa_0^{1/2}\mu, J(Q)^{1/2}), \\ \mu < \mu_0(M_*), \quad \mu^{1/6} \ll J(Q)^{1/2}. \end{gathered} \tag{4.4.1}$$

Theorem 4.12. (One-Pass Theorem) *Let $\epsilon_*, R_* > 0$ be as given in . If a solution u of NLKG on an interval I satisfies for some $\epsilon \in (0, \epsilon_*]$, $R \in (2\epsilon, R_*]$, and $\tau_1 < \tau_2 \in I$,*

$$E(\vec{u}) < J(Q) + \epsilon^2, \quad d_Q(\vec{u}(\tau_1)) < R = d_Q(\vec{u}(\tau_2)),$$

then for all $t \in (\tau_2, \infty) \cap I$, we have $d_Q(\vec{u}(t)) \ge R$.

Remark We may continue to apply the One-Pass theorem after $t = \tau_1$ to conclude the stronger statement that the solution may no longer return to the distance R_*, where R_* is an absolute constant independent of the solution.

The proof of the One-Pass Theorem follows in a similar vein to that of the virial argument which appeared in the scattering proof in Chapter 2. The idea will be to derive a localised virial identity of the form

$$\dot{V}_w(t) = -K_2(u) + \text{Error},$$

for some $V_w(t)$ to be defined. We then integrate this over some time interval and obtain a contradiction using *uniform away from zero* bounds obtained from those given by the Ejection Lemma and Proposition (4.9). However, (4.3.3) does not yield a uniform bound as it could be that u vanishes at some time. The estimate (2.4.32) in Chapter 2 will not work here as that required $E(\vec{u}) < J(Q)$. In fact, it is clear from the argument there that all that is required is some control on the time average of the L^2 norm of the gradient, see (2.4.34). Such a bound useful for the One-Pass Theorem is furnished from the following result.

Lemma 4.13. *For any $M > 0$ there exists $\mu_0(M) > 0$ with the following properties: Let u be a finite energy solution to NLKG on $[0, 2]$ satisfying*

$$\|\vec{u}\|_{L_t^\infty([0,2];\mathcal{H})} \le M, \quad \int_0^2 \|\nabla u(t)\|_{L_x^2}^2 dt \le \mu^2, \tag{4.4.2}$$

for some $\mu \in (0, \mu_0]$. Then u extends to a global solution and scatters to 0 as $t \to \pm\infty$. Furthermore, $\|u\|_{L_t^3 L_x^6(\mathbb{R}\times\mathbb{R}^3)} \ll \mu^{1/6}$.

For the proof of this result, consult Lemma 4.13 in [5]. The point is that any such u satisfying (4.4.2) will be well approximated by its corresponding linear solution which is already guaranteed to scatter.

Proof of One-Pass Theorem: Suppose, in order to obtain a contradiction, that there exists a solution u on its maximal existence interval I, some $\epsilon \in (0, \epsilon_*]$, $R \in (2\epsilon, R_*]$ and $\tau_1 < \tau_2 < \tau_3 \in I$ such that

$$E(\vec{u}) < J(Q) + \epsilon^2, \quad d_Q(\vec{u}(\tau_1)) < R < d_Q(\vec{u}(\tau_2)) > R > d_Q(\vec{u}(\tau_3)).$$

By continuity, there exist $T_1 \in (\tau_1, \tau_2)$ and $T_2 \in (\tau_2, \tau_3)$ such that

$$d_Q(\vec{u}(T_1)) = R = d_Q(\vec{u}(T_2)) \leq d_Q(\vec{u}(t)), \quad \text{for all} \quad t \in (T_1, T_2).$$

The key here is that $0 < T_1, T_2 < \infty$.

Step 1: Localised virial identity We define the diamond-like space-time cut-off function by

$$w(t,x) = \begin{cases} \chi(x/(t - T_1 + S)) & t < (T_1 + T_2)/2, \\ \chi(x/(T_2 - t + S)) & t > (T_1 + T_2)/2, \end{cases}$$

where χ is a smooth cut-off function on $\mathbb{R}^3$ satisfying $\chi \equiv 1$ on $|x| \leq 1$ and $\chi \equiv 0$ on $|x| \geq 2$, and $S \gg 1$ is a constant to be determined. We define the virial quantity to be

$$V_w(t) := \langle w\partial_t u \,|\, (1/2)(x \cdot \nabla + \nabla \cdot x)u \rangle_{L^2},$$

and with some effort, we obtain

$$\dot{V}_w(t) = -K_2(u(t)) + \mathcal{O}(E_{\text{ext}}(t)). \tag{4.4.3}$$

Here $E_{\text{ext}}(t)$ is the exterior free energy defined by

$$E_{\text{ext}}(t) := \int_{X(t)} e^0(u)dx, \quad e^0(u) := ((\partial_t u)^2 + |\nabla u|^2 + |u|^2)/2$$

$$x \in X(t) \quad \Longleftrightarrow \begin{cases} |x| > t - T_1 + S & T_1 < t < (T_1 + T_2)/2, \\ |x| > T_2 - t + S & (T_1 + T_2)/2 < t < T_2. \end{cases}$$

When $t = T_j$, $j = 1, 2$, $X(T_j) = \{|x| > S\}$ so writing $u = Q + \lambda\rho + \gamma$, and using λ dominance, we find that

$$E_{\text{ext}}(T_j) \lesssim e^{-2S} + R^2.$$

The exponetial term arises from the exponential decay of Q. Choosing $S \gg |\log R| \gg 1$ yields $E_{\text{ext}}(T_j) \lesssim R^2$. By the finite speed of propagation for NLKG, one can show that

$$\sup_{t \in [T_1, T_2]} E_{\text{ext}}(T_j) \lesssim \max_{j=1,2} E_{\text{ext}}(T_j),$$

which implies that $E_{\text{ext}}(t) \lesssim R^2$ for all $t \in [T_1, T_2]$ and hence

$$\dot{V}_w(t) = -K_2(u(t)) + \mathcal{O}(R^2), \quad t \in [T_1, T_2]. \tag{4.4.4}$$

Writing $u = Q + v$ and $Au = (3/2)u + 2x \cdot \nabla u$, we find

$$\begin{aligned} |V_w(t)|_{T_1}^{T_2} &\lesssim |\langle \chi(x/S)\partial_t u \,|\, AQ \rangle| + |\langle \chi(x/S)\partial_t u \,|\, Au \rangle| \\ &\lesssim \|\partial_t v\|_{L^2} + \|v\|_{L^2}^2 + \|\partial_t v\|_{L^2}^2 + \|\chi(x/S)|x||\nabla v|\|_{L^2}^2 \\ &\lesssim \sum_{t = T_1, T_2} \|\partial_t v(t)\|_{L^2} + S\|\vec{v}(t)\|_{\mathcal{H}}^2 \\ &\lesssim R + SR^2. \end{aligned}$$

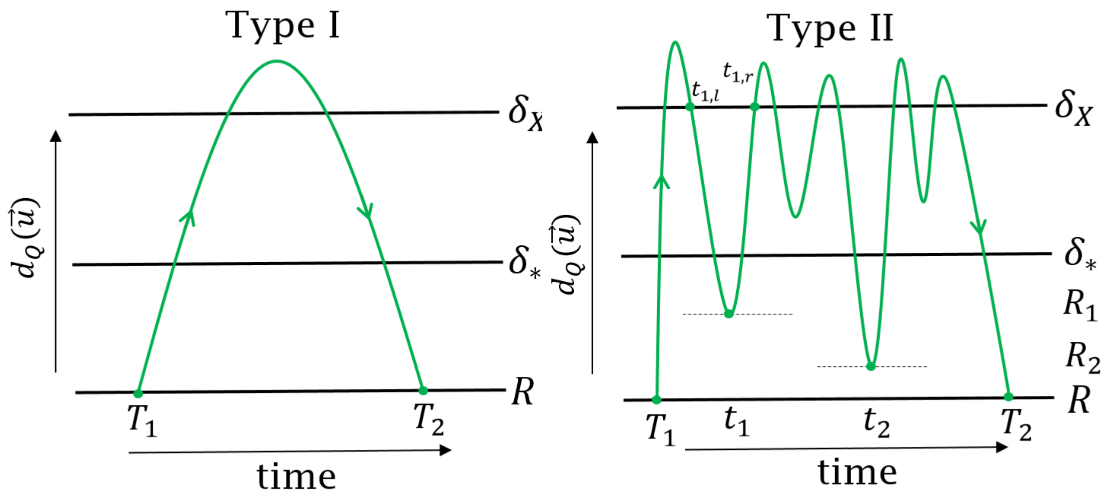

Figure 4.1: Type I trajectories are initially ejected to the δ_X ball and do not re-enter the δ_* ball until it makes its passage back into the R ball. On the other hand, a type II trajectory can make multiple passages into the δ_* ball before the time of re-entry into the R ball.

Choosing S such that $|\log R| \ll S \ll 1/R$, we obtain the upper bound

$$|V_w(t)|_{T_1}^{T_2} \lesssim R. \tag{4.4.5}$$

Step 2: Bounding K_2 Lemma 4.10 gives us a fixed sign $\{\pm 1\} \ni \mathfrak{s} := \mathfrak{S}(u(t))$ on $[T_1, T_2]$. There are two types of orbits that can occur as described in Figure 4.1. We will only consider type II trajectories as they are slightly more involved. Let $t_m \in [T_1, T_2]$ be a time for which $d_Q(\vec{u}(t))$ attains a local minima, $R_m := d_Q(\vec{u}(t_m))$, and $R_m \in [R, \delta_*)$ and set $M := \#\{t_m\}$. Clearly both $T_1 =: t_0$ and $T_2 =: t_{M+1}$ satisfy these conditions. We define the left and right ejection times

$$t_{m,l} := \sup\{t < t_m \,|\, d_Q(\vec{u}(t)) = \delta_X\}, \quad m = 1, 2, \ldots, M+1$$
$$t_{m,r} := \sup\{t > t_m \,|\, d_Q(\vec{u}(t)) = \delta_X\}, \quad m = 0, 1, \ldots, M$$

and the intervals about t_m by $I_m := [t_{m,l}, t_{m,r}]$ for $m = 1, \cdots, M$ while $I_0 := [T_1, t_{1,r}]$ and $I_{M+1} := [t_{M+1,l}, T_2]$.
We apply the Ejection Lemma to $u(t_m - t)$ and $u(t - t_m)$ and that we have a fixed sign, we obtain

$$d_Q(\vec{u}(t)) \simeq R_m e^{k|t-t_m|}, \quad \mathfrak{s}K_2(u(t)) \gg d_Q(\vec{u}(t)) - C_* R_m, \tag{4.4.6}$$

for all $t \in I_m$. Multiplying (4.4.4) by $-\mathfrak{s}$, integrating over a single I_m and using (4.4.6), we obtain the lower bound

$$[-\mathfrak{s}V_w(t)]_{I_m} \gtrsim \int_{I_m} \left[d_Q(\vec{u}(t)) - C_* R_m - \mathcal{O}(R^2)\right] \, dt \simeq \delta_X, \tag{4.4.7}$$

where we have used the exponential growth from (4.4.6) and that $R \le R_m \le \delta_* \ll \delta_X$.
As for the remainder portion

$$I' := [T_1, T_2] \setminus \cup_m I_m,$$

we always have $d_Q(\vec{u}(t)) > \delta_*$ and $\epsilon \le \epsilon_0(\delta_*)$. Therefore we may apply Proposition 4.9 to deduce

$$\mathfrak{s}K_2(u) \ge \kappa_1(\delta_*) > 0, \quad (\mathfrak{s} = -1)$$
$$\mathfrak{s}K_2(u) \ge \min(\kappa_1(\delta_*), \kappa_0\|\nabla u\|_{L^2}^2). \quad (\mathfrak{s} = +1)$$

Now as $d_Q(\vec{u}(t)) > \delta_* \gg R$, there exists $\tau > 0$ such that for any $t \in I'$, $[t - \tau, t + \tau] \subset [T_1, T_2]$ for any $t \in I'$. To show this, notice that from Figure 4.1, the 'worst case' scenario is when

$t \in (t_{0,r}, t_{1,l}) \cup (t_{M,r}, t_{M+1,l})$ and hence we should take, say, $\tau < \min\{|t_{0,r} - T_1|, |T_2 - t_{M+1,l}|\}$. Using the exponential growth from ejection (4.4.6), we take

$$\tau < \frac{1}{k} \min_{j=1,2} \left(\log \left(\frac{\delta_X}{C_j R} \right) \right),$$

for some constants C_1, C_2 coming from (4.4.6).
In the case $\mathfrak{s} = -1$, we have

$$\int_{t-\tau}^{t+\tau} \mathfrak{s} K_2(u) dt \gtrsim \kappa_1(\delta_*) \gg R_*^2. \tag{4.4.8}$$

When $\mathfrak{s} = +1$, we deal with the potential for $\nabla u(t)$ to vanish at some time by noticing that Lemma 4.4.2 implies

$$\int_{t-\tau}^{t+\tau} \|\nabla u(s)\|_{L^2}^2 ds > \mu^2.$$

If this did not hold, then $\|u\|_{L_t^3 L_x^6} \ll \mu^{1/6} \ll J(Q)^{1/2}$ which contradicts $d_Q(\vec{u}(T_1)) = R \ll J(Q)^{1/2}$. Therefore we have

$$\int_{t-\tau}^{t+\tau} \mathfrak{s} K_2(u) dt \gtrsim \min\left(\kappa_1(\delta_*), \kappa_0 \mu^2\right) \gg R_*^2. \tag{4.4.9}$$

So (4.4.8) and (4.4.9) imply that, for either sign $\mathfrak{s}$, the portion of the integral of (4.4.4) over I' is negligible in comparison to that portion over $\cup_m I_m$. Upon summing (4.4.7) over m, we find

$$[-\mathfrak{s} V_w(t)]_{T_1}^{T_2} \gtrsim \delta_X \times \#\{t_m\} \gtrsim \delta_X. \tag{4.4.10}$$

However this contradicts the upper bound (4.4.5) as $R_* \ll \delta_X$. □

Chapter 5

The full charactersiation

We can now complete the characterisation for solutions with energies

$$E(\vec{u}) < J(Q) + \epsilon^2.$$

Suppose we have a solution that begins outside the small 2ϵ-ball about one of the ground states. It either enters the ball or never does. In the latter case, Lemma 4.6 forbids circulation which implies the solution hits δ_X. In doing so the Ejection Lemma would have applied along its path which means it gets ejected in the sense of Definition 4.2.10, fixing $\operatorname{sign} K_0$ at the time of ejection. The One-Pass Theorem says we cannot even return to a distance R_* and thus $\operatorname{sign} K_0$ stabilizes. In the former case, we may either be trapped or not-trapped by the 2ϵ-ball. If we are trapped, we hit the center-stable manifold by Theorem 3.4 and are expressible in the long time limit as $Q +$ radiation. Else, not trapping implies ejection by Corollary 4.8 and then the same arguments as above imply that $\operatorname{sign} K_0$ stabilizes. This discussion is summarised in Figure 5.1.

Once $\operatorname{sign} K_0$ stabilizes, we can mimic the Payne-Sattinger theory to conclude global existence or finite time blow-up forward in time. The sign may also stabilize backwards in time leading to global existence or blow-up as $t \to -\infty$. It is precisely only solutions with $J(Q) \leq E(\vec{u}) \leq J(Q) + \epsilon^2$ that can exhibit a different behaviour forwards in time as it does backwards in time. The reason is that such solutions may freely make one pass through the 2ϵ-ball, keeping their energy fixed while allowing $\operatorname{sign} K_0$ to change. This is not possible for solutions with $E(\vec{u}) < J(Q)$.

We are thus led to the 9-set Theorem 1.2. We have not proved here though that solutions with $\operatorname{sign} K_0 = +1$ either for all $t > 0$ or $t < 0$, scatter to zero and that each of the nine sets are non-empty. The scattering statement largely follows that described in Chapter 2 with the crucial difference being the extraction of the critical element. One needs to ensure that not only does it have energy slightly below the ground state but also that it remains a distance R_{ast} away from a ground state. Showing the sets are non-empty amounts to choosing initial data that launch solutions with a fixed sign for λ and that satisfy the Ejection lemma. The details can be found in Chapter 5 of [5].

We stress here that the results we have described are but the tip of the iceberg for the full dynamical picture at arbitrary energies. The time-independent NLKG admits a countable sequence of solutions with increasing energies; the excited states. The analysis here is solely for energies very close to that of the ground state. At this stage the dynamics, even at the first excited state, are still conjecture. If these excited states are unstable, then we may expect a similar blow-up/scattering/trapping trichotomy although it is unclear how the results of Chapter 4 may be generalised to consider dynamics far from the ground state. Such results would be an important step on the road to resolving The Soliton Resolution conjecture.

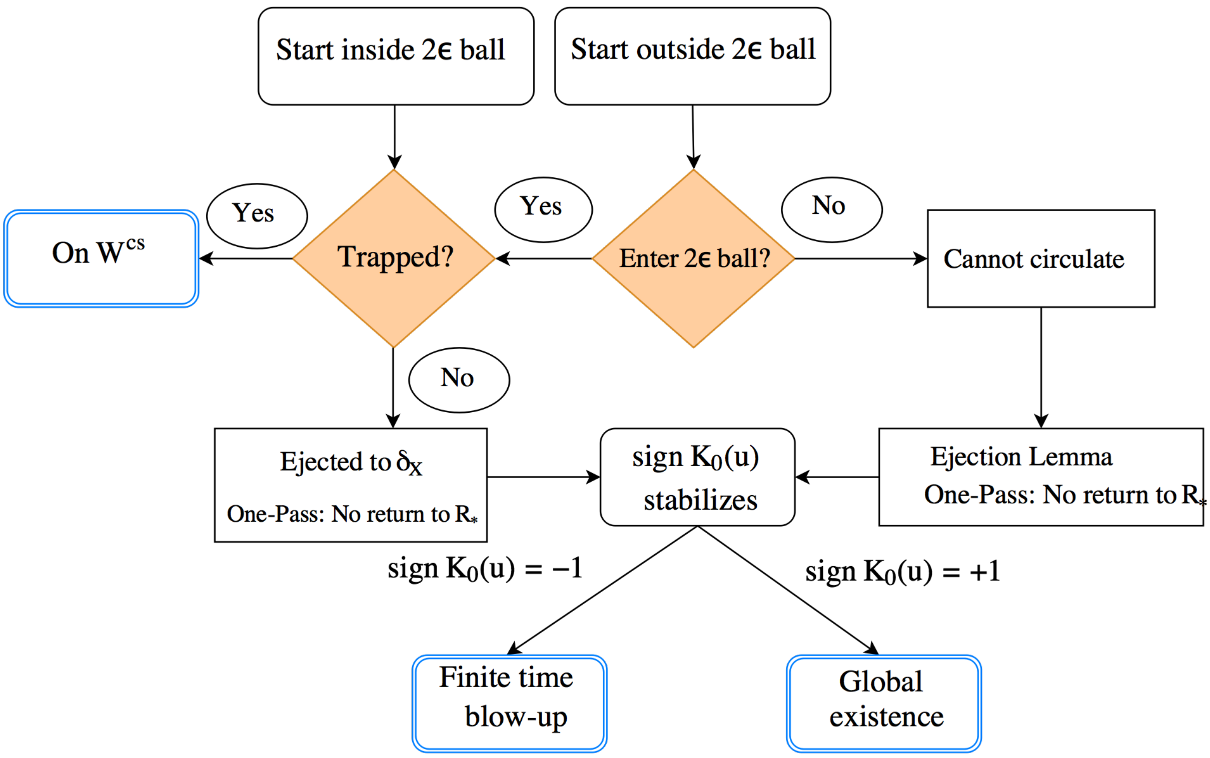

Figure 5.1: Flow chart of behaviour

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Appendix A

Appendix

A.1 Results used in radial scattering proof

Lemma A.1 (Log-convexity of $L_t^q L_x^p$ norms)**.** *Let* $0 < p_0 < p_1 \leq \infty$, $0 < q_0 < q_1 \leq \infty$ *and* $f \in L_t^{q_0} L_x^{p_0} \cap L_t^{q_1} L_x^{p_1}$. *Then* $f \in L_t^{q_\theta} L_x^{p_\theta}$ *where for any* $0 \leq \theta \leq 1$,

$$\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{A.1.1}$$

and we have

$$\|f\|_{L_t^{q_\theta} L_x^{p_\theta}} \leq \|f\|_{L_t^{q_0} L_x^{p_0}}^{1-\theta} \|f\|_{L_t^{q_1} L_x^{p_1}}^{\theta}. \tag{A.1.2}$$

Proof. By the log-convexity for L^p norms, we have

$$\|f\|_{L_x^{p_\theta}} \leq \|f\|_{L_x^{p_0}}^{1-\theta} \|f\|_{L_x^{p_1}}^{\theta}.$$

Multiplying both sides by q_θ and integrating over time we obtain

$$\begin{aligned}\|f\|_{L_t^{q_\theta} L_x^{p_\theta}}^{q_\theta} &\leq \| \|f\|_{L_x^{p_0}}^{1-\theta} \|f\|_{L_x^{p_1}}^{\theta} \|_{L_t^{q_\theta}}^{q_\theta} \\ &= \| \|f\|_{L_x^{p_0}}^{q_\theta(1-\theta)} \|f\|_{L_x^{p_1}}^{q_\theta\theta} \|_{L_t^1}.\end{aligned}$$

Applying Hölder's inequality with

$$1 = \frac{1}{\frac{q_0}{q_\theta(1-\theta)}} + \frac{1}{\frac{q_1}{q_\theta\theta}}$$

yields (A.1.2). □

Lemma A.2. *Let* $B_1 \hookrightarrow B_2$ *be Banach spaces such that the sequence* $\{f_n\} \subset B_1$ *weakly converges to* f_1 *in* B_1 *and weakly converges to* f_2 *in* B_2. *Then* $f_1 = f_2$.

Proof. Notice that any continuous linear functional on B_2 is also a continuous linear functional on B_1. Therefore for any $l \in B_2^*$, $l(f_n) \to l(f_1)$. By uniqueness of weak limits we have the result. □

Lemma A.3. *Let* $\{f_n\}$ *be a radial sequence in* $L^p(\mathbb{R}^d)$ *and* $f_n \to f$ *in* $L^p(\mathbb{R}^d)$. *Then* f *is radial.*

Proof. For $\mathcal{R} \in SO(\mathbb{R}^d)$, we have by the radial assumption on the sequence f_n and a change of variables,

$$\begin{aligned}\|f(\mathcal{R}\cdot) - f(\cdot)\|_{L^p} &\leq \|f(\mathcal{R}\cdot) - f_n(\mathcal{R}\cdot)\|_{L^p} + \|f_n(\mathcal{R}\cdot) - f(\cdot)\|_{L^p} \\ &\leq \|f(\cdot) - f_n(\cdot)\|_{L^p} + \|f_n(\cdot) - f(\cdot)\|_{L^p} \\ &= 2\|f_n(\cdot) - f(\cdot)\|_{L^p} \to 0,\end{aligned}$$

as $n \to 0$. Therefore $f(R\cdot) = f(\cdot)$ almost everywhere. □

Lemma A.4. *Suppose that* $f_n \rightharpoonup f$ *and* $g_n \to g$*. Then* $\langle f_n \mid g_n \rangle \to \langle f \mid g \rangle$.