Invariant manifolds and dispersive PDE

Part 2

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Recap from William

The setup

We consider the cubic nonlinear Klein-Gordon equation (NLKG):

$$\partial_t^2 u - \Delta u + u = u^3, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}$$
$$\vec{u}(0) = (u(0), \partial_t u(0)) \in \mathcal{H} := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$$

Energy:

$$E(\vec{u})(t) := \int_{\mathbb{R}^3} \left(\frac{1}{2} u^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 - \frac{1}{4} u^4 \right) dx$$

Dynamics of NLKG understood about the ground state, Q, the unique, positive, radial solution in H^1 of

$$-\Delta Q + Q = Q^3.$$

Properties:

- $Q \in C^\infty(\mathbb{R}^3)$ and exponentially decaying
- Minimizer of the stationary energy

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2} \varphi^2 + \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{4} \varphi^4 \right) dx$$

Dynamics below the ground state

Payne-Sattinger, 1975: Behaviour dictated by the sign of a functional K_0:

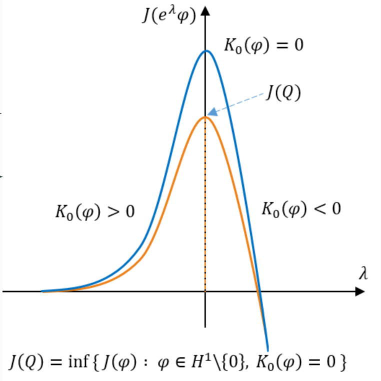
$$K_0(\varphi) := \partial_\lambda|_{\lambda=0} J(e^\lambda \varphi) = \int \left(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\right) dx,$$

$$\mathcal{PS}_+ = \{\vec{u} \in \mathcal{H} \ : \ E(\vec{u}) < J(Q), \ K_0(u) \geq 0\}$$

$$\mathcal{PS}_- = \{\vec{u} \in \mathcal{H} \ : \ E(\vec{u}) < J(Q), \ K_0(u) < 0\}$$

$\mathcal{PS}_+$: Global solutions

$\mathcal{PS}_-$: Finite time blow-up



Scattering in $\mathcal{PS}_+$

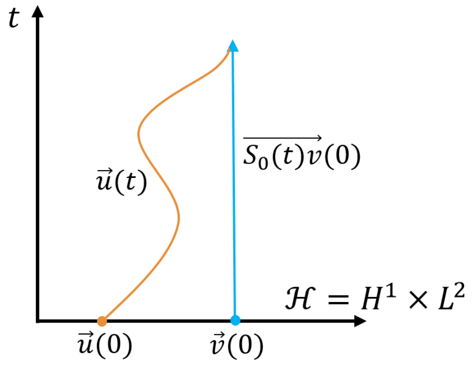
Scattering

A *global* solution u scatters if there exists initial data $(v_0, v_1) \in \mathcal{H}$ such that

$$\|\vec{u}(t) - S_0(t)(v_0, v_1)\|_{\mathcal{H}} \to 0, \quad \text{as } t \to \infty,$$

where $S_0(t)$ is the free Klein-Gordon evolution.

A global solution u scatters $\iff \|u\|_{L_t^3([0,\infty);L_x^6(\mathbb{R}^3))} < \infty.$



Improvement to Payne-Sattinger theory

Theorem (Ibrahim-Masmoudi-Nakanishi, 2011)

All solutions $\vec{u}(t) \in \mathcal{PS}_+$ scatter as $t \to \pm\infty$ and $\|u\|_{L^3_t L^6_x} < \infty$.

Ingredients of proof

Radial case:

- Small data, scattering theory
- Profile decomposition/ Concentrated compactness
- Perturbation lemma
- Virial-type argument

Non-radial case:

- More involved profile decomposition
- Refined virial argument using momentum and Lorentz transformations

Idea of proof

Proof.

□

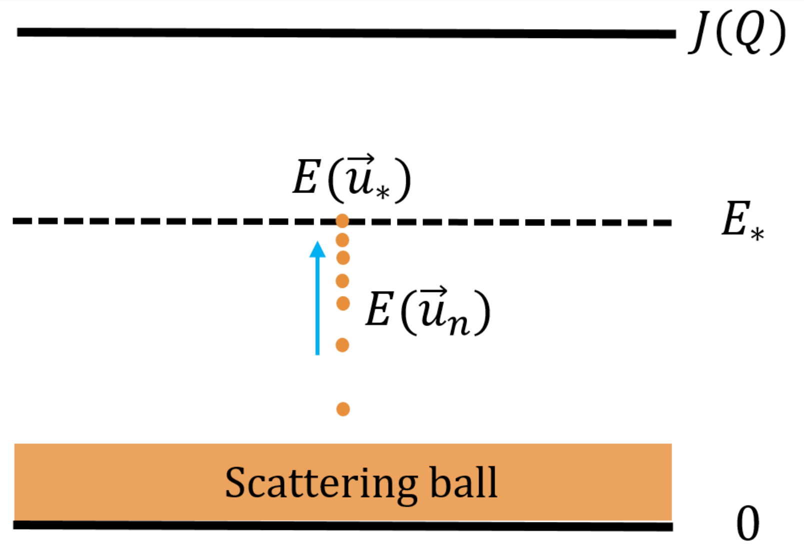
Idea of proof

Proof.

By contradiction ($\times 9$)

□

Idea of proof



Invariant manifolds

Above the ground state

$$J(Q) \leq E(\vec{u}) < J(Q) + \epsilon^2$$

We perturb about the ground state by writing $u = Q + v$; obtaining the system

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}}_{=: A} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(v) \end{pmatrix},$$

where

$$L_+ := -\Delta + 1 - 3Q^2.$$

Spectral properties of L_+ and A on L^2_{rad} :

- $\sigma(L_+) = \{-k^2\} \cup [1, \infty)$
- $\Rightarrow \sigma(A) = \{\pm k\} \cup i\,[1, \infty) \cup i\,(-\infty, -1]$
- Gap property: L_+ has no eigenvalues in $(0, 1]$ and no resonance at 1.

The center-stable manifold

Theorem (Nakanishi-Schlag, 2011)

Assume that the gap property for L_+ holds.
Then there exists a smooth graph $\mathcal{M}$ contained in a small ball $B(Q, 0) \subset \mathcal{H}_{\text{rad}}$ with tangent plane

$$T_Q\mathcal{M} = \{(v_0, v_1) \in \mathcal{H} \mid \langle kv_0 + v_1 \mid \rho\} = 0\}.$$

Any data $(u_0, u_1) \in \mathcal{M}$ lead to global evolutions of the form $u = Q + v$ where v scatters to a free KG solution in $\mathcal{H}$. Furthermore, $\mathcal{M}$ is invariant under the flow for all $t \geq 0$.

- The stable and unstable manifolds are obtained as corollaries.
- Lyapunov-Perron method: Requires full description of the spectrum of L_+ but gives scattering and stability information. Generalizes to other powers of the non-linearity.

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"*Needs more but gives more*"

Proof: The Lyapunov-Perron method

Further decompose $v = \lambda(t)\rho + \gamma(t)$, $\gamma \perp \rho$, where $L_+\rho = -k^2\rho$. Then for $(\lambda, \gamma) \in \mathbb{R} \times P_\rho^\perp(H^1)$

$$\begin{cases} \ddot{\lambda} - k^2\lambda = \langle N(v) \,|\, \rho \rangle =: N_\rho(v), \\ \ddot{\gamma} + \omega^2\gamma = P_\rho^\perp N(v), \qquad \omega := \sqrt{P_\rho^\perp L_+} \end{cases}$$

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Under the stability condition

$$\dot{\lambda}(0) + k\lambda(0) = -\int_0^\infty e^{-ks} N_\rho(v)(s)\, ds,$$

we look for solutions to

$$\Gamma_\lambda(t) = e^{-kt}\left[\lambda(0) + \frac{1}{2k}\int_0^\infty e^{-ks} N_\rho(v)(s)\, ds\right] + \frac{1}{2k}\int_0^\infty e^{-k|t-s|} N_\rho(v)(s)\, ds,$$

$$\Gamma_\gamma(t) = \cos(\omega t)\gamma(0) + \frac{\sin(\omega t)}{\omega}\dot{\gamma}(0) + \int_0^t \frac{\sin(\omega(t-s))}{\omega} P_\rho^\perp N(v)(s)\, ds.$$

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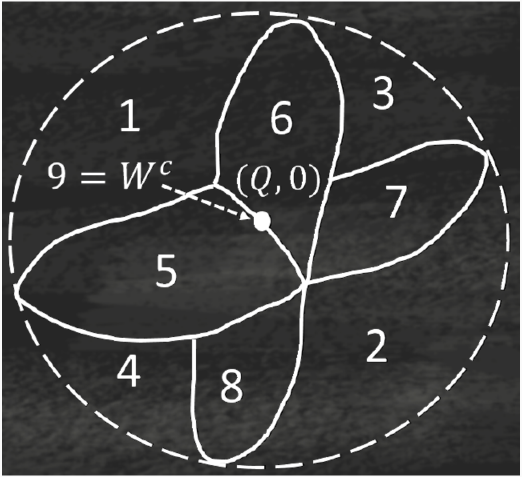
Need Strichartz estimates for $e^{it\omega}$: Heavy machinery!

What's up next?

Cool manifolds, so what?



Thank you.

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