Global dynamics above the ground state energy for the 3D cubic nonlinear Klein-Gordon equation

Part 2

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Recap from William

The setup

We consider the cubic nonlinear Klein-Gordon equation (NLKG):

$$\partial_t^2 u - \Delta u + u = u^3, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}$$
$$\vec{u}(0) = (u(0), \partial_t u(0)) \in \mathcal{H} := H^1_{\mathrm{rad}}(\mathbb{R}^3) \times L^2_{\mathrm{rad}}(\mathbb{R}^3)$$

Energy:

$$E(\vec{u}(t)) := \int_{\mathbb{R}^3} \left(\frac{1}{2}u^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{4}u^4 \right) dx$$

Dynamics of NLKG understood below the ground state, Q, the unique, positive, radial solution in H^1 of

$$-\Delta Q + Q = Q^3.$$

Properties:

- $Q \in C^\infty(\mathbb{R}^3)$ and exponentially decaying
- Minimizer of the stationary energy

$$J(\varphi) := \int_{\mathbb{R}^3} \left(\frac{1}{2}\varphi^2 + \frac{1}{2}|\nabla \varphi|^2 - \frac{1}{4}\varphi^4 \right) dx$$

Dynamics below the ground state

Payne-Sattinger, 1975: Behaviour dictated by the sign of a functional K_0:

$$K_0(\varphi) := \int \left(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\right)\, dx,$$

$$\{\vec{u} \in \mathcal{H} \,:\, E(\vec{u}) < J(Q),\ K_0(u) \geq 0\} \implies \text{Global Solutions}$$

$$\{\vec{u} \in \mathcal{H} \,:\, E(\vec{u}) < J(Q),\ K_0(u) < 0\} \implies \text{Finite time blow-up}$$

Above the ground state

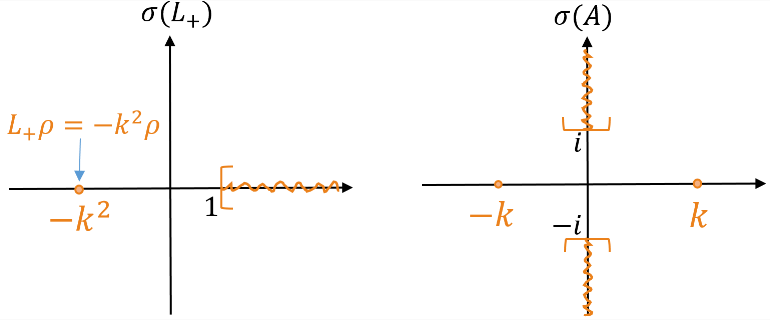
$E(\vec{u}) < J(Q) + \epsilon^2$

Perturb about the ground state $u = Q + v$; obtain the system

$$\partial_t \begin{pmatrix} v \\ \dot{v} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -L_+ & 0 \end{bmatrix}}_{=:A} \begin{pmatrix} v \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 \\ N(v) \end{pmatrix},$$

where

$$L_+ := -\Delta + 1 - 3Q^2.$$



$E(\vec{u}) < J(Q) + \epsilon^2$

$$u = Q + v$$

$$\Downarrow$$

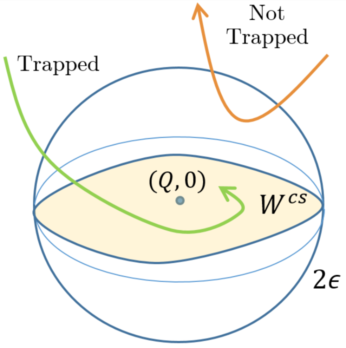
$$u(t,x) = Q(x) + \lambda(t)\rho(x) + (P_\rho^\perp v)(t,x)$$

Projecting away unstable modes of λ: Center-stable manifold W^{cs}
(Interim presentations)

Dynamics off center-stable manifold

Control sign $K_0(u)$

Mechanics of the game



The pieces:

1. sign K_0 can *only change* if you re-enter the 2ϵ-ball
2. Solutions *not trapped* by 2ϵ-ball are *ejected* (Ejection Lemma)
3. Upon exit from 2ϵ-ball, solution *cannot* re-enter (One-Pass)

$\implies$ "Either Trapped or Ejected."

A notion of distance

Main technical tool: The *non-linear distance function*

$$d_Q(\vec{u}(t)) \simeq \|\vec{u}(t) - (Q, 0)\|_{\mathcal{H}}.$$

"Distance measure in $\mathcal{H}$ taking into account the non-linearity in NLKG."

$$d_Q(\vec{u}) \leq \delta_E \ll 1$$

$$\Downarrow$$

$$d_Q^2(\vec{u}(t)) = E(\vec{u}) - J(Q) + k^2\lambda(t)^2.$$

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$\Longrightarrow \lambda$ dominance: $d_Q(\vec{u}) \simeq |\lambda|$

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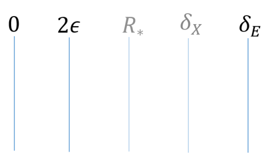

(2) Ejection Lemma

Ejection Lemma (Nakanishi-Schlag, 2011)

There exists an abs. constant $0 < \delta_X \leq \delta_E$ with the following property. Let $u(t)$ be an NLKG solution satisfying

$$0 < d_Q(\vec{u}(0)) \leq \delta_X, \quad E(\vec{u}) < J(Q) + \epsilon^2,$$

and

$$\left.\frac{d}{dt}\right|_{t=0} d_Q(\vec{u}(t)) \geq 0.$$

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Then $d_Q(\vec{u}(t))$ monotonically increases until hitting δ_X while

$$d_Q(\vec{u}(t)) \simeq d_Q(\vec{u}(0)) e^{kt}, \quad \mathfrak{s} K_0(u(t)) \gtrsim d_Q(\vec{u}(t)) - C_* d_Q(\vec{u}(0)),$$

where $\mathfrak{s} \in \{\pm 1\}$ is a fixed sign and C_* an abs. constant.

(2) Ejection Lemma cont.

Proof (sketch):

Differentiating $d_Q^2(\vec{u}(t)) = E(\vec{u}) - J(Q) + k^2\lambda(t)^2$, using

$$\ddot{\lambda} - k^2\lambda = P_\rho N(v)$$

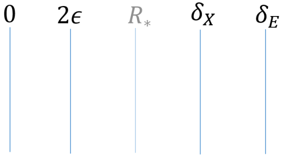
and λ dominance implies

$$\partial_t^2 d_Q^2(\vec{u}(t)) \simeq k^2 d_Q^2(\vec{u}(t)).$$

If we had equality, solve the ODE to get e^{kt} and e^{-kt} modes. $\square$

(2) Not trapped implies ejection

Corollary 1: There is no circulating solution ($2\epsilon < d_Q(\vec{u}(t)) < \delta_X$ for all $t \geq 0$).



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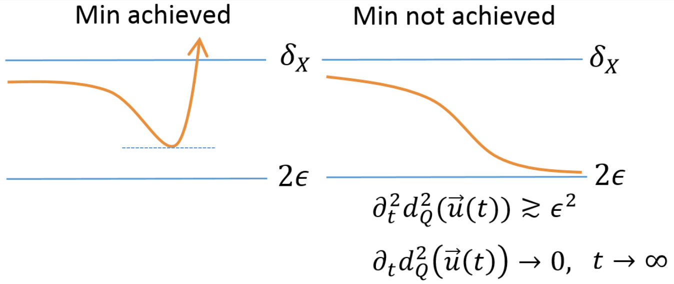

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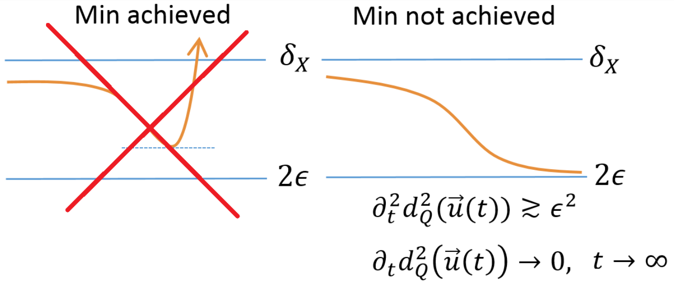

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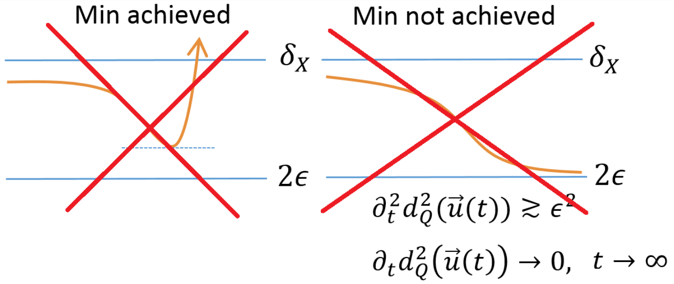

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Corollary 2: Suppose $d_Q(\vec{u}(0)) \ll \delta_X$ and $\vec{u}(t)$ is not trapped by the 2ϵ-ball about $(Q, 0)$. Then $\vec{u}$ is ejected to δ_X.

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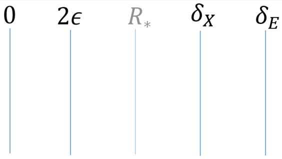
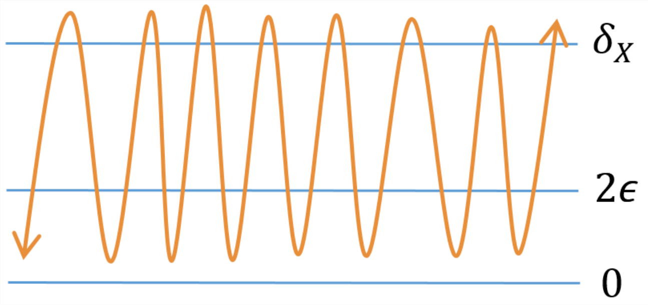
Proof.
Never enters 2ϵ-ball: Cannot circulate $\Rightarrow$ Ejection Lemma

Enters 2ϵ-ball: Not trapped $\Rightarrow$ exits at some time $t = T$. Apply Ejection Lemma at $t = T$. □



(1)+(2)=Insufficient

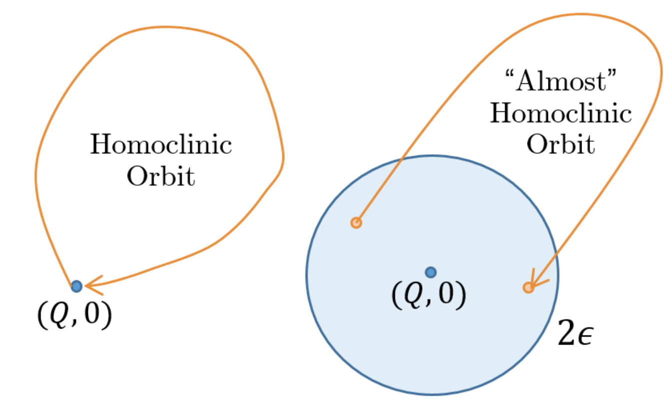
1. sign K_0 can *only change* if you re-enter the 2ϵ-ball.



No chance for sign K_0 to stabilize!

(3) One-Pass

Idea: Limit number of times solution can return to 2ϵ-ball.



Can only make 'one-pass.'

(3) One-Pass

One-Pass (Nakanishi-Schlag, 2011)

There exists an abs. constant $2\epsilon \ll R_* \ll \delta_X$ such that if an NLKG solution u satisfies for some $R \in (2\epsilon, R_*]$ and $t_1 < t_2$,

$$E(\vec{u}) < J(Q) + \epsilon^2, \quad d_Q(\vec{u}(t_1)) < R = d_Q(\vec{u}(t_2)),$$

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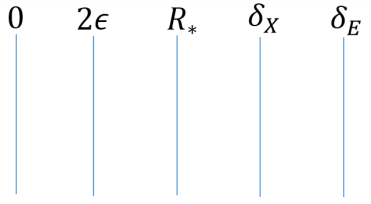

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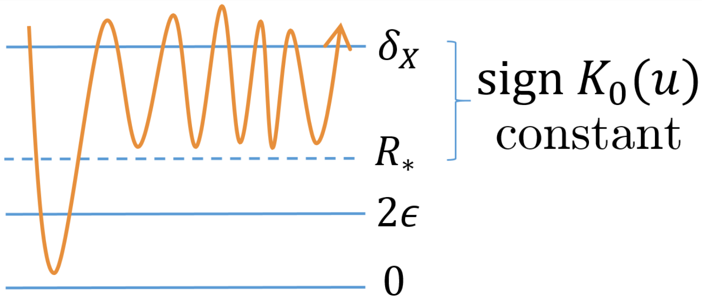
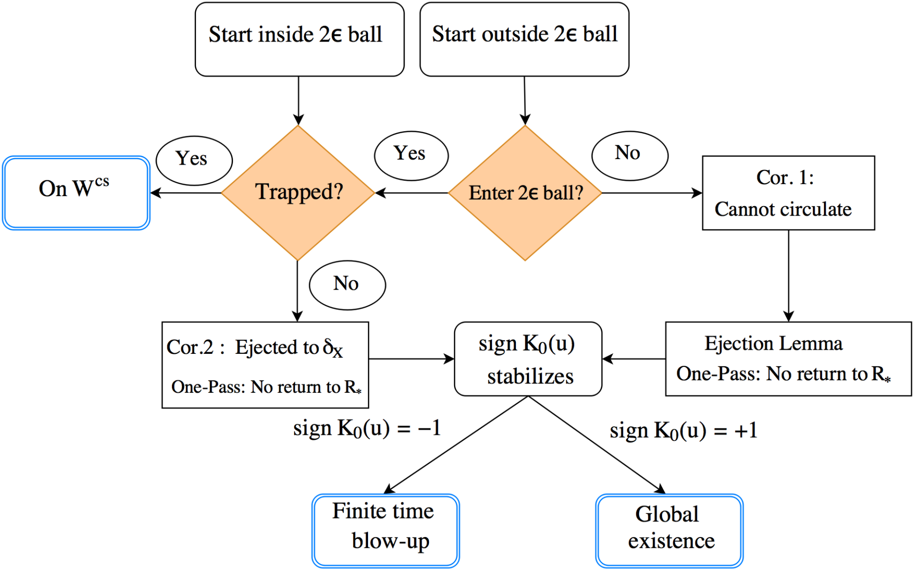


The full picture

Classification of global behaviour



Thank you.

References

- Payne, L. E., Sattinger, D. H., *Saddle points and instability of non-linear hyperbolic equations*. Israel J. Math. 22 (1975), no. 3-4, 273-303.
- Nakanishi, K., Schlag, W., *Invariant manifolds and dispersive hamiltonian equations* European Mathematical Society Publishing House, (2011) Zurich, Switzerland.

The $3 \times 3 = 9$ scoops