Bourgain-Bulut 2D radial NLS on disc
Expanded details

Preliminaries and problem

Consider the defocusing 2-D NLS (radial only)

$$(*)\begin{cases} i\partial_t u+\Delta u = |u|^\alpha u & , x\in B_2 \\ u|_{t=0}=\phi \\ u|_{\partial B_2}=0 \end{cases}$$

where B_2 is the unit disc and $\alpha\in 2\mathbb{N}$, and ϕ is radial.
$(*)$ has Hamiltonian

$$H(u)=\frac{1}{2}\int_{B_2}|\nabla u|^2dx+\frac{1}{\alpha+2}\int_{B_2}|u|^{\alpha+2}dx,$$

which motivates to consider the "formal" Gibbs measure

$$\text{“}d\mu_G = Z^{-1}e^{-H(u)}du = Z^{-1}e^{-\frac{1}{\alpha+2}\int_{B_2}|u|^{\alpha+2}dx}d\mu_F\text{”}$$

where μ_F is the Gaussian ("free") measure

$$d\mu_F = \tilde{Z}^{-1}e^{-\frac{1}{2}\int_{B_2}|\nabla u|^2dx}du.$$

- μ_F is supported on $H^{1-\frac{1}{2}-}(B_2)=H^{\frac{1}{2}-}(B_2)$ where we have made use of the radial assumption, i.e. the "effective" spatial dimension of $(*)$ is 1.

Goal: Establish almost sure GWP for $(*)$ with random initial data lying in the support of the Gibbs measure.

With this goal in mind, introduce the truncated problem

$$(FNLS)\begin{cases} i\partial_t u_N+\Delta u_N = P_N(|u_N|^\alpha u_N) \\ u_N|_{t=0}=P_N\phi \\ u_N|_{\partial B_2}=0, \end{cases}$$

where $N\in\mathbb{Z}_+$ is a frequency cut-off defined by

$$P_N\Big(\sum_{n\in\mathbb{N}}a_ne_n(x)\Big)=\sum_{\{n:z_n\le N\}}a_ne_n(x)$$

where

$e_n(x)$ – Normalized in L^2, radial eigenfunctions of $-\Delta$ on B_2 with Dirichlet Boundary conditions.

(z_n^2) – Sequence of associated eigenvalues

$$e_n(x) = \frac{J_0(z_n r)}{\|J_0(z_n \cdot)\|_{L^2}},$$ where J_0 is the Bessel function of the first kind of order zero and (z_n) are the sequence of positive zeros of J_0 in increasing order.

Writing $u_N = P_N u_N + P_{>N} u_N$, we see by inserting into (FNLS) and projecting that $P_N u_N$ satisfies (FNLS) while $P_{>N} u_N$ satisfies the linear Schrödinger equation with zero initial data $\Rightarrow P_{>N} u_N \equiv 0$ for all t.

Hence

$$u_N(t,x) = P_N u_N = \sum_{\{n : z_n \le N\}}' u_n(t) e_n(x),$$

and $u_N(t)$ is global in-time as in frequency space, (FNLS) reduces to a finite system of ODEs for the coefficients

$$\{u_n(t)\}_{n\in\mathbb{N}}.$$

Local existence is then guaranteed by classical ODE theory. Global existence follows from L^2-conservation.

(FNLS) has associated Hamiltonian

$$H_N(u) = \frac{1}{2}\sum_{\{n : z_n \le N\}}' z_n^2 |\hat{u}(n)|^2 + \frac{1}{\alpha+2}\int_{B_2} |P_N u(x)|^{\alpha+2}\,dx.$$

By Liouville's Theorem and conservation of H_N under (FNLS) flow

$$\Phi_N : \phi_N \mapsto u_N(t),$$

the truncated Gibbs measure

$$d\mu_G^{(N)} = Z_N^{-1} e^{-H_N(u_N)}\,du_N = Z_N^{-1} e^{-\frac{1}{\alpha+2}\|P_N u\|_{L_x^{\alpha+2}}^{\alpha+2}}\,d\mu_F^{(N)}$$

is conserved as well. Here $\mu_F^{(N)}$ is the free Gaussian measure and is induced by the mapping

$$\omega \mapsto \frac{1}{\pi} \sum_{\{n\in\mathbb{N}: z_n \le N\}} \frac{g_n(\omega)}{z_n} e_n(x), \quad \omega \in \Omega$$

- $(\mathbb{P}, \Omega, \mathcal{F})$ some probability space.

Using estimates for the eigenfunctions $\{e_n(x)\}_{n\in\mathbb{N}}$ (to come) one can show that for every $\alpha \in 2\mathbb{N}$,

$$\|P_N \tilde{u}\|_{L_x^{\alpha+2}}^{\alpha+2} < \infty \quad a\text{-}s.$$

This implies that $\mu_G^{(N)}$ is a well-defined Probability measure.

Definition of "Solution":

We say $u, u_N : [0,T) \times B_2 \to \mathbb{C}$ are solutions of ⊛ and (FNLS) resp. if they belong to

$$C_t([0,T); H_x^\sigma(B_2))$$

for some $\sigma < 1/2$, and satisfy the integral equations

$S(t) := e^{it\Delta}$,

$$u(t) = S(t)\phi + i\int_0^t S(t-\tau)\left(|u(\tau)|^\alpha u(\tau)\right) d\tau, \quad t \in [0,T]$$

$$u_N(t) = S(t)P_N\phi + i\int_0^t S(t-\tau)\, P_N\left[|u_N(\tau)|^\alpha u_N(\tau)\right] d\tau, \quad t \in [0,T).$$

Define

$$\phi^{(\omega)}(x) := \frac{1}{\pi} \sum_{n\in\mathbb{N}} \frac{g_n(\omega)}{z_n} e_n(x).$$

The support of the truncated Gibbs measure $\mu_G^{(N)}$ corresponds to the set

$$\{P_N \phi^{(\omega)} : \omega \in \Omega\}.$$

Main Theorem: Fix $\alpha \in 2\mathbb{N}$. For $N \in \mathbb{N}$, $\omega \in \Omega$, denote u_N the sol$^{\underline{n}}$ to (FNLS) on the two dimensional unit disc with data $P_N\phi = P_N\phi^{(\omega)}$.

Then almost surely in Ω, for any $0 < T < \infty$, there exists

$$u_* \in C_t([0,T); H_x^s(B_2)) \quad (s < 1/2)$$

Such that $\{u_N\}_{N\in\mathbb{N}}$ converges to u_* in $C_t([0,T); H_x^s(B_2))$ and u_* is unique and solves ⊛ in the sense defined above.

Eigenfunction/value estimates

The following can be found in [Tzvetkov, Dyn-PDE, '06; Section 2].

$$z_n = \pi(n - 1/4) + O(1/n) \qquad \text{"}z_n \sim n \text{ for large } n\text{"}$$

$$\|e_n\|_{L^p_x(B_2)} \lesssim \begin{cases} 1, & p \in [2,4) \\ \log(2+n)^{1/4}, & p = 4 \\ n^{\frac{1}{2}-\frac{2}{p}}, & p \in [4,\infty]. \end{cases}$$

Also note that $\|\mathcal{J}(z_n \cdot)\|_{L^2(B_2)} \sim n^{-1/2}$.

//

The $X^{s,b}$-spaces

Fix $I_T := [0,T)$, $0 < T < 1/2$. Define $(s,b \in \mathbb{R})$

$$X^{s,b}(I_T) := \left\{ f : I_T \times B_2 \to \mathbb{C} : \|f\|_{X^{s,b}(I_T)} < \infty \right\},$$

where

$$\|f\|_{X^{s,b}(I_T)} := \inf \left\{ \left(\sum_{\substack{n\in\mathbb{N} \\ m\in\mathbb{Z}}} \langle z_n \rangle^{2s} \langle z_n^2 - m \rangle^{2b} |g_{n,m}|^2 \right)^{1/2} : \right.$$

g is a periodic extension of f over $\mathbb{T}_t := [-\frac{1}{4}, \frac{3}{4})$, i.e.

$$\left. f(x,t) = \sum_{\substack{n\in\mathbb{N} \\ m\in\mathbb{Z}}} g_{n,m} e_n(x) e(mt), \ (t,x) \in I_T \times B_2 \right\}.$$

For convenience, we sometimes write

$$\|g\|_{X^{s,b}(\mathbb{T}_t)} = \left(\sum_{\substack{n\in\mathbb{N} \\ m\in\mathbb{Z}}} \langle z_n \rangle^{2s} \langle z_n^2 - m \rangle^{2b} |g_{n,m}|^2 \right)^{1/2},$$

for periodic $g : \mathbb{T}_t \times B_2 \to \mathbb{C}$.

Embeddings for $X^{s,b}$

Lemma 2.3: Let $\frac{1}{4}<b<1$ and $2\le p<4$. Then, using

$$P_I f = \sum_{Z_n\in I} \hat{f}(n) e_n,$$

we have for any $\varepsilon>0$, $f\in \mathcal{S}_{x,t}(I_T\times B_2)$ and interval $I\subset\mathbb{R}$ (interval of spatial frequencies)

$$\|P_I f\|_{L^p_x L^4_t(I_T\times B_2)} \lesssim \begin{cases} |I|^{\varepsilon}\|P_I f\|_{X^{0,b}(I_T)}, & b>1/2 \\ |I|^{1-2b+\varepsilon}\|P_I f\|_{X^{0,b}(I_T)}, & b\le 1/2. \end{cases}$$

Remark: Notice the order of the space-time norm on the LHS.

Proof: Let $f\in X^{0,b}(I_T)$ and fix ~~any~~ representation/periodic extension g of f.
We have

$$P_I g = \sum_{m,Z_n\in I} g_{m,n}\, e_n(x) e(mt)$$

$$= \sum_{m,Z_n\in I} g_{m+\lfloor Z_n^2\rfloor,n}\, e_n(x) e(mt) e(\lfloor Z_n^2\rfloor t).$$

It suffices to prove $\|P_I g\|_{L^p_x L^4_t(\mathbb{T}_t\times B_2)} \lesssim |I|^{c(\varepsilon,b)}\|P_I g\|_{X^{0,b}(\mathbb{T}_t)}$ ⊛

for any extension g of f onto $\mathbb{T}_t$, and where

$$c(\varepsilon,b) := \begin{cases} \varepsilon, & b>1/2 \\ 1-2b+\varepsilon, & b\le 1/2. \end{cases}$$

The Lemma then follows since

$$\|P_I f\|_{L^p_x L^4_t(I_T\times B_2)} \le \|P_I g\|_{L^p_x L^4_t(\mathbb{T}_t\times B_2)} \qquad (\text{as } g|_{I_T}=f)$$

$$\lesssim |I|^{c(\varepsilon,b)}\|P_I g\|_{X^{0,b}(\mathbb{T}_t)}, \quad (\text{by } ⊛)$$

at which point we take an infimum over all periodic extensions of f.

Perform a dyadic decomposition into intervals of width $m \sim M$ ("$m \sim M$" $\Rightarrow$ $M \le m < 2M$), and so

$$\|P_I g\|_{L^p_x L^4_t} \lesssim \sum_M{}' \|g_M\|_{L^p_x L^4_t}$$

where

$$g_M := \sum_{m \sim M}{}' \Big(\sum_{z_n \in I} g_{m+[z_n^2],n}\, e_n(x)\, e([z_n^2]t) \Big) e(mt).$$

Now focus on estimating $\|g_M\|_{L^p_x L^4_t}$ for fixed $M \in 2^{\mathbb{N}}$.

We have

$$\|g_M\|_{L^p_x L^4_t} \lesssim \sum_{m \sim M}{}' \Big\| \Big(\sum_{z_n \in I} g_{m+[z_n^2],n}\, e_n(x)\, e([z_n^2]t) \Big) e(mt) \Big\|_{L^p_x L^4_t}$$

$$= \sum_{m \sim M} \Big\| \Big\| \sum_{z_n \in I} g_{m+[z_n^2],n}\, e_n(x)\, e([z_n^2]t) \Big\|_{L^4_t} \Big\|_{L^p_x}.$$

$\|F\|_{L^4_t} = \|F^2\|^{1/2}_{L^2_t}$

$$= \sum_{m \sim M} \Big\| \Big\| \Big| \sum_{z_n \in I} g_{m+[z_n^2],n}\, e_n(x)\, e([z_n^2]t) \Big|^2 \Big\|^{1/2}_{L^2_t} \Big\|_{L^p_x}$$

$$= \sum_{m \sim M}{}' \Big\| \Big\| \sum_{z_n, z_{n'} \in I}{}' g_{m+[z_n^2],n}\, g_{m+[z_{n'}^2],n'}\, e_n(x)\, e_{n'}(x)\, e\big(([z_n^2]+[z_{n'}^2])t\big) \Big\|^{1/2}_{L^2_t} \Big\|_{L^p_x}$$

$$= \sum_{m \sim M}{}' \Big\| \Big\| \sum_{\ell} \Big(\sum_{\substack{z_n, z_{n'} \in I \\ [z_n^2]+[z_{n'}^2]=\ell}} g_{m+[z_n^2],n}\, g_{m+[z_{n'}^2],n'}\, e_n(x)\, e_{n'}(x) \Big) e(\ell t) \Big\|^{1/2}_{L^2_t} \Big\|_{L^p_x}$$

$$= \sum_{m \sim M}{}' \Big\| \Big(\sum_{\ell} \Big| \sum_{\substack{z_n, z_{n'} \in I \\ [z_n^2]+[z_{n'}^2]=\ell}}{}' g_{m+[z_n^2],n}\, g_{m+[z_{n'}^2],n'}\, e_n(x)\, e_{n'}(x) \Big|^2 \Big)^{1/2} \Big\|^{1/2}_{L^{p/2}_x}$$

Plancherel in t.

We now perform a "Cauchy-Schwarz argument" which we detail.

Define

$$\circledast_{n,n'} := \{ z_n, z_{n'} \in I \;:\; [z_n^2] + [z_{n'}^2] = \ell \},$$

$$A_{nn'} := g_{m+[z_n^2],n}\, g_{M+[z_{n'}^2],n'}\, e_n(x)\, e_{n'}(x).$$

Then using Cauchy-Schwarz $\xrightarrow{\text{in }(n,n')}$ we have

$$\Big(\sum_\ell \Big|\sum_{\circledast_{n,n'}} A_{nn'}\Big|^2\Big)^{1/2} \le \Big(\sum_\ell \Big|\Big(\sum_{\circledast_{n,n'}} 1\Big)^{1/2}\Big(\sum_{\circledast_{n,n'}} A_{nn'}^2\Big)^{1/2}\Big|^2\Big)^{1/2}$$

$$= \Big(\sum_\ell \Big(\sum_{\circledast_{n,n'}} 1\Big)\Big(\sum_{\circledast_{n,n'}} A_{nn'}^2\Big)\Big)^{1/2}$$

Take out 1-Sum out in ℓ

$$\le \Big(\sup_\ell \sum_{\circledast_{n,n'}} 1\Big)^{1/2}\Big(\sum_\ell\sum_{\circledast_{n,n'}} A_{nn'}^2\Big)^{1/2}.$$

So we have

$$\lesssim \sum_{m\sim M}\Big(\sup_\ell \sum_{\circledast_{n,n'}} 1\Big)^{1/4}\Big\|\Big(\sum_\ell\sum_{\circledast_{n,n'}} A_{nn'}^2\Big)^{1/2}\Big\|_{L_x^{p/2}}^{1/2}$$

$$= \sum_{m\sim M}\Big(\sup_\ell \sum_{\circledast_{n,n'}} 1\Big)^{1/4}\Big\|\sum_\ell\sum_{\circledast_{n,n'}} \underbrace{|g_{m+[z_n^2],n}|^2|e_n(x)|^2\,|g_{M+[z_{n'}^2],n'}|^2|e_{n'}(x)|^2}_{\text{Sum decouples in } n,n' \text{ } \ell \text{ summation vanishes}}\Big\|_{L_x^{p/2}}^{1/2}$$

$$= \sum_{m\sim M}\Big(\sup_\ell \sum_{\circledast_{n,n'}} 1\Big)^{1/4}\Big\|\sum_{z_n\in I} |g_{m+[z_n^2],n}|^2|e_n(x)|^2\Big\|_{L_x^{p/2}}^{1/2}. \quad (**)$$

Now

$$\sup_\ell\Big(\sum_{\circledast_{n,n'}} 1\Big) = \sup_\ell \Big|\{(n_1,n_2)\in\mathbb{Z}^2 : [z_n^2]+[z_{n'}^2]=\ell,\ (z_n,z_{n'})\in I\times I\}\Big|.$$

By a short/small modification of the proof of Lemma 2.2 on the paper, we have the following estimate:

$$\sup_\ell \Big|\{(n_1,n_2)\in\mathbb{N}^2 : [z_n^2]+[z_{n'}^2]=\ell,\ (z_n,z_{n'})\in I\times I\}\Big|$$

$$\lesssim |I|^\varepsilon, \text{ for any } \varepsilon>0.$$

Thus

$$(**) \lesssim \sum_{M\sim M} |I|^{\varepsilon} \Big\| \sum_{z_n\in I} |g_{m+[z_n^2],n}|^2 |e_n(x)|^2 \Big\|_{L_x^{p/2}}^{1/2}$$

(Finite summation and only e_n depends on x.)

$$\lesssim \sum_{M\sim M} |I|^{\varepsilon} \Big(\sum_{z_n\in I} |g_{m+[z_n^2],n}|^2 \|e_n\|_{L_x^p(B_2)}^2 \Big)^{1/2}$$

For $2\le p<4$, $\|e_n\|_{L_x^p(B_2)} \lesssim 1$ (eigenfunction estimates).

$$\lesssim \sum_{M\sim M} |I|^{\varepsilon} \Big(\sum_{z_n\in I} |g_{m+[z_n^2],n}|^2 \Big)^{1/2}$$

By Cauchy-Schwarz in m,

$$\lesssim |I|^{\varepsilon} \Big(\sum_{m\sim M} 1 \Big)^{1/2} \Big(\sum_{m\sim M} \sum_{z_n\in I} |g_{m+[z_n^2],n}|^2 \Big)^{1/2}$$

$$\sim |I|^{\varepsilon} M^{1/2} \Big(\sum_{\substack{z_n\in I \\ m-[z_n^2]\sim M}} M^{-2b} \langle m-[z_n^2]\rangle^{2b} |g_{m+[z_n^2],n}|^2 \Big)^{1/2}.$$

$$\lesssim |I|^{\varepsilon} M^{\frac{1}{2}-b} \Big(\sum_{\substack{z_n\in I \\ m-z_n^2\sim M}} \langle m-z_n^2\rangle^{2b} |g_{m+[z_n^2],n}|^2 \Big)^{1/2}$$

$$= |I|^{\varepsilon} M^{\frac{1}{2}-b} \|P_I g\|_{X^{0,b}(\mathbb{T}_t)},$$

since $|z_n^2-[z_n^2]| \lesssim 1 \Rightarrow \langle m-[z_n^2]\rangle^{2b} \lesssim \langle m-z_n^2\rangle^{2b}$.

For $b>1/2$, $\frac{1}{2}-b<0$ so summing this in M yields $(*)$, and hence the desired inequality when $b>1/2$.

For the case $\frac{1}{4}<b\le 1/2$, we perform the dyadic decomposition in m but estimate $\|g_M\|_{L_x^p L_t^4}$ differently.

We have

$$\|g_M\|_{L^p_x L^4_t} \le \sum_{z_n\in I} \left\| \left(\sum_{m\sim M} g_{m+[z_n^2],n}\, e_n(x)\, e(mt)\right) e([z_n^2]t)\right\|_{L^p_x L^4_t}$$

$$= \sum_{z_n\in I} \|e_n(x)\|_{L^p_x(B_2)} \left\|\sum_{m\sim M} g_{m+[z_n^2],n}\, e(mt)\right\|_{L^4_t}$$

$\|e_n\|_{L^p_x} \lesssim 1$ for $2\le p<4$

$$\lesssim \sum_{z_n\in I} \left\|\sum_{m\sim M} g_{m+[z_n^2],n}\, e(mt)\right\|_{L^4_t}$$

$$= \sum_{z_n\in I} \left\|\sum_{\ell} \left(\sum_{\substack{m\sim M\\ m'\sim M\\ m+m'=\ell}} g_{m+[z_n^2],n}\, g_{m'+[z_n^2],n}\right) e(\ell t)\right\|_{L^2_t}^{1/2}$$

Plancherel in t

$$= \sum_{z_n\in I} \left(\sum_{\ell} \left|\sum_{\substack{m,m'\sim M\\ m+m'=\ell}} g_{m+[z_n^2],n}\, g_{m'+[z_n^2],n}\right|^2\right)^{1/4}$$

Cauchy-Schwarz

$$\lesssim \sum_{z_n\in I} \left(\sup_{\ell} \sum_{\substack{m,m'\sim M\\ m+m'=\ell}} 1\right)^{1/4} \left(\sum_{\ell}\sum_{\substack{m,m'\sim M\\ m+m'=\ell}} |g_{m+[z_n^2],n}|^2 |g_{m'+[z_n^2],n}|^2\right)^{1/4}$$

$$\lesssim \sum_{z_n\in I} \left(\sum_{m\sim M} 1\right)^{1/4} \left(\sum_{m\sim M} |g_{m+[z_n^2],n}|^2\right)^{1/2}$$

$$\lesssim M^{1/4} \sum_{z_n\in I} \left(\sum_{m-[z_n^2]\sim M} |g_{m,n}|^2\right)^{1/2}$$

$$\lesssim M^{1/4} |I|^{1/2} \left(\sum_{z_n\in I}\sum_{m-[z_n^2]\sim M} |g_{m,n}|^2\right)^{1/2}$$

$$\lesssim M^{\frac{1}{4}-b} |I|^{1/2} \|P_I g\|_{X^{0,b}(\mathbb{T}_t)}.$$

Combining these two estimates we have

$$\|g_M\|_{L^4_x L^4_t} \lesssim \min\left(|I|^{\varepsilon} M^{\frac{1}{2}-b},\ |I|^{1/2} M^{\frac{1}{4}-b}\right) \|P_I g\|_{X^{0,b}(\mathbb{T}_t)}.$$

It remains to sum over dyadic M, when $\frac{1}{4}<b\leq 1/2$.
We illustrate two ways the sum can proceed with the second being quicker; the first more immediately straightforward.

Method 1: Recall that for M,N dyadic, we have from sums of geometric series, the formulae

$$\begin{cases} \sum_{M\leq N} M^{\alpha} \lesssim N^{\alpha} & (\alpha>0) \\ \sum_{M>N} M^{-\beta} \lesssim N^{-\beta} & (\beta>0). \end{cases}$$

Now $|I|^{\varepsilon} M^{\frac{1}{2}-b} \leq |I|^{1/2} M^{1/4-b} \Rightarrow M \leq |I|^{2-4\varepsilon}$.
Therefore

$$\sum_M \min\left(|I|^{\varepsilon} M^{\frac{1}{2}-b},\ |I|^{1/2} M^{\frac{1}{4}-b}\right)$$

$$= \sum_{M\leq |I|^{2-4\varepsilon}} |I|^{\varepsilon} M^{\frac{1}{2}-b} + \sum_{M>|I|^{2-4\varepsilon}} M^{\frac{1}{4}-b} |I|^{1/2}$$

$$\lesssim |I|^{\varepsilon} |I|^{(\frac{1}{2}-b)(2-4\varepsilon)} + |I|^{1/2} |I|^{(2-4\varepsilon)(\frac{1}{4}-b)}$$

$$\lesssim |I|^{1-2b+\varepsilon}.$$

Method 2:

$$\sum_M \min\left(|I|^{\varepsilon} M^{\frac{1}{2}-b},\ |I|^{1/2} M^{\frac{1}{4}-b}\right) = \sum_M M^{-\varepsilon} \underbrace{\min\left(|I|^{\varepsilon} M^{\frac{1}{2}-b+\varepsilon},\ |I|^{1/2} M^{\frac{1}{4}-b+\varepsilon}\right)}$$

Just need to show this quantity can be bounded by $|I|$ to some power.

At worst, $|I|^{\varepsilon} M^{\frac{1}{2}-b+\varepsilon} = |I|^{1/2} M^{1/4-b+\varepsilon}$

$$\Rightarrow M = |I|^{2-4\varepsilon}.$$

Therefore

$$\min\left(|I|^{\varepsilon} M^{\frac{1}{2}-b+\varepsilon},\ |I|^{1/2} M^{\frac{1}{4}-b+\varepsilon}\right) \leq |I|^{\varepsilon}\left(|I|^{2-4\varepsilon}\right)^{\frac{1}{2}-b+\varepsilon} \sim |I|^{1-2b+\mu},\ \mu \ll 1.$$

Thus
$$\sum_M M^{-\varepsilon} \min(-,-) \lesssim |I|^{1-2b+\mu} \left(\sum_M M^{-\varepsilon}\right) \lesssim |I|^{1-2b+\mu}.$$

$\square$

Remark: When $p=4$, we have $(b \geq 1/2)$,
$$\|P_I f\|_{L^4_x L^4_t(I_T)} \lesssim |I|^{\varepsilon} \|P_I f\|_{X^{\varepsilon,b}(I_T)}$$
because we can use
$$\|e_n\|_{L^4_x} \leq (\log(2+n))^{1/4} \lesssim n^{\varepsilon}.$$

Remark: From Lemma 2-3 and the previous remark, we have
$$\begin{cases} \|f\|_{L^p_x L^4_t} \lesssim \|f\|_{X^{\varepsilon,b}(I_T)}, & b \geq 1/2,\ p \leq 4 \\ \|f\|_{L^p_x L^4_t} \lesssim \|f\|_{X^{1-2b+\varepsilon,b}}, & \frac{1}{4} < b < 1/2. \end{cases}$$

To prove these follow from Lemma 2-3, perform a dyadic decomposition of f of the form
$$f = \sum_N P_{[N,2N)} f, \quad N \geq 1 \text{ dyadic}.$$

Then applying Lemma 2-3 we have

$b \geq 1/2$:
$$\begin{aligned} \|f\|_{L^p_x L^4_t} &\leq \sum_N \|P_{[N,2N)} f\|_{L^p_x L^4_t} \\ &\lesssim \sum_N N^{\varepsilon/2} \|P_{[N,2N)} f\|_{X^{0,b}} \quad {=}\varepsilon \text{ if } p=4 \\ &\lesssim \sum_N N^{\varepsilon/2} N^{-\varepsilon} \left(N^{\varepsilon} \|P_{[N,2N)} f\|_{X^{0,b}}\right) \\ &\lesssim \left(\sum_N N^{-\varepsilon/2}\right) \|f\|_{X^{\varepsilon,b}}. \end{aligned}$$

$1/4 < b < 1/2$:
$$\begin{aligned} \|f\|_{L^p_x L^4_t} &\lesssim \sum_N N^{1-2b+\varepsilon/2} \|P_{[N,2N)} f\|_{X^{0,b}} \\ &\lesssim \left(\sum_N N^{-\varepsilon/2}\right) \|f\|_{X^{1-2b+\varepsilon,b}}. \end{aligned}$$

Lemma 2.5 (Dual estimate to Strichartz estimate / Nonlinear estimate)

Let $I \subset \mathbb{R}$ be an interval. Then for $b = \frac{1}{2}+$ (sufficiently close to $1/2$), and for every $\varepsilon > 0$, there exists $C = C(b, \varepsilon) > 0$ s.t.

$$\left\| P_I\left(\int_0^t S(t-\tau) f(\tau) d\tau\right)\right\|_{X^{0,b}(\mathbb{T}_T)} \leq C|I|^{2b-1+\varepsilon} \|f\|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{\frac{4}{3}}}.$$

Moreover, the inequality

$$\left\| \int_0^t S(t-\tau) f(\tau) d\tau \right\|_{X^{0,b}(\mathbb{T}_T)} \leq C \left\| (\sqrt{-\Delta})^{2b-1+\varepsilon} f \right\|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}}.$$

also holds for all $f \in \mathcal{S}_{x,t}$.

Remark: In the process of obtaining the above estimates, we will derive the estimate

$$\left\| \int_0^t S(t-\tau) f(\tau) d\tau \right\|_{X^{0,b}(\mathbb{T}_T)} \lesssim \|f\|_{X^{0,b-1}(\mathbb{T}_T)},$$

(which ~~resembles the~~ is the version of the classical dual estimate for the torus case. Thus the content of Lemma 2.5 can be understood in a similar vein as for the case of the torus.

Proof: As in the previous remark, we first establish

$$\left\| P_I\left(\int_0^t S(t-\tau) f(\tau) d\tau\right)\right\|_{X^{0,b}(\mathbb{T}_T)} \lesssim \|P_I f\|_{X^{0,b-1}(\mathbb{T}_T)}.$$

It suffices to show that for any extension g of $P_I f$ onto $\mathbb{T}_t$, we have

$$(1) \ldots \quad \left\| P_I\left(\int_0^t S(t-\tau) f(\tau) d\tau\right)\right\|_{X^{0,b}(\mathbb{T}_T)} \lesssim \|g\|_{X^{0,b-1}(\mathbb{T}_t)},$$

with the constant independent of the explicit representative g.

Now notice that

$$\tilde{g}(x,t) := \sum_{\substack{|m-z_n^2|>1 \\ z_n \in I}} g_{m,n} e_n(x) \frac{e(mt)}{i(m-z_n^2)} - \sum_{\substack{|m-z_n^2|>1 \\ z_n \in I}} g_{m,n} e_n(x) \frac{e(z_n^2 t)}{i(m-z_n^2)} \varphi(t)$$
$$+ \sum_{\substack{|m-z_n^2| \leq 1 \\ z_n \in I}} g_{m,n} e_n(x) \frac{e(mt) - e(z_n^2 t)}{i(m-z_n^2)} \varphi(t)$$

is a periodic (on $\mathbb{T}_t$) extension of $P_I\left(\int_0^t S(t-\tau)f(\tau)d\tau\right)$,
where $\varphi \in C_0^\infty(\mathbb{R}_t)$ such that ~~supp φ~~

$$\varphi \equiv 1 \text{ on } [0,T) = I_T.$$

Then in order to obtain ① it suffices to show

$$\|\tilde{g}\|_{X^{0,b}(\mathbb{T}_t)} \lesssim_\varphi \|g\|_{X^{0,b-1}(\mathbb{T}_t)}, \quad \text{---}(2)$$

since by definition of the $X^{0,b}(I_T)$ norm,

$$\left\|P_I\left(\int_0^t S(t-\tau)f(\tau)d\tau\right)\right\|_{X^{0,b}(I_T)} \leq \|\tilde{g}\|_{X^{0,b}(\mathbb{T}_t)}$$

$$\text{by (2)}, \quad \lesssim_\varphi \|g\|_{X^{0,b-1}(\mathbb{T}_t)},$$

which is (1).

Let us establish (2). By the triangle inequality,

$$\|\tilde{g}\|_{X^{0,b}(\mathbb{T}_t)} \lesssim \|(\mathrm{I})\|_{X^{0,b}(\mathbb{T}_t)} + \|(\mathrm{II})\|_{X^{0,b}(\mathbb{T}_t)} + \|(\mathrm{III})\|_{X^{0,b}(\mathbb{T}_t)},$$

where

$$(\mathrm{I}) := \sum_{\substack{|m-z_n^2|>1 \\ z_n \in I}} g_{m,n}\, e_n(x) \frac{e(mt)}{i(m-z_n^2)},$$

$$(\mathrm{II}) := \sum_{\substack{|m-z_n^2|>1 \\ z_n \in I}} g_{m,n}\, e_n(x) \frac{e(z_n^2 t)}{i(m-z_n^2)} \varphi(t),$$

$$(\mathrm{III}) := \sum_{\substack{|m-z_n^2|\leq 1 \\ z_n \in I}} g_{m,n}\, e_n(x) \frac{e(mt)-e(z_n^2 t)}{i(m-z_n^2)} \varphi(t).$$

(I): By our definition of $\|\cdot\|_{X^{0,b}(\mathbb{T}_t)}$, we have

$$\|(\mathrm{I})\|_{X^{0,b}(\mathbb{T}_t)} = \left\|\sum_{m,n} \frac{g_{m,n}}{i(m-z_n^2)} \chi_{z_n\in I}\, \chi_{|m-z_n^2|>1}\, e_n(x) e(mt)\right\|_{X^{0,b}(\mathbb{T}_t)}$$

$$= \left(\sum_{\substack{z_n \in I \\ |m-z_n^2|>1}} \langle m-z_n^2\rangle^{2b} \frac{|g_{m,n}|^2}{|m-z_n^2|^2}\right)^{1/2}$$

$$\lesssim \Bigg(\sum_{\substack{z_n \in I \\ |m - z_n^2| > 1}} \langle m - z_n^2 \rangle^{2b} \frac{|g_{m,n}|^2}{\langle m - z_n^2 \rangle^2} \Bigg)^{1/2}$$

as $|m-z_n^2| \geq 1$
$\Rightarrow |m-z_n^2| \sim \langle m - z_n^2 \rangle$

$$\lesssim \|g\|_{X^{0,b-1}(\mathbb{T}_t)}.$$

(II): $\|(\mathrm{II})\|_{X^{0,b}(\mathbb{T}_t)} = \Bigg\| \sum_{\substack{m \in \mathbb{Z} \\ |m - z_n^2| > 1 \\ z_n \in I}} \frac{g_{m,n}}{i(m - z_n^2)} e_n(x) \varphi(t) e(z_n^2 t) \Bigg\|_{X^{0,b}(\mathbb{T}_t)}$

$$= \Bigg\| \sum_k \sum_{z_n \in I} \Bigg(\sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{g_{m,n}}{i(m - z_n^2)} \Bigg) \underset{\substack{\downarrow \\ \text{Fourier transform on } \mathbb{R}}}{\hat{\varphi}(k - z_n^2)} e_n(x) e(kt) \Bigg\|_{X^{0,b}(\mathbb{T}_t)}$$

$$= \Bigg(\sum_k \sum_{z_n \in I} \langle k - z_n^2 \rangle^{2b} |\hat{\varphi}(k - z_n^2)|^2 \Bigg| \sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{g_{m,n}}{(m - z_n^2)} \Bigg|^2 \Bigg)^{1/2}$$

$$= \Bigg(\sum_{z_n \in I} \Bigg| \sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{g_{m,n}}{|m - z_n^2|} \Bigg|^2 \underbrace{\sum_k \langle k - z_n^2 \rangle^{2b} |\hat{\varphi}(k - z_n^2)|^2}_{\text{Translate in } k, \ = \|\varphi\|^2_{H^b_t}} \Bigg)^{1/2}$$

$$\leq \|\varphi\|_{H^b_t} \Bigg(\sum_{z_n \in I} \Bigg| \sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{g_{m,n}}{|m - z_n^2|} \Bigg|^2 \Bigg)^{1/2}$$

$$\lesssim_{\varphi, b} \Bigg(\sum_{z_n \in I} \Bigg| \sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{g_{m,n}}{\langle m - z_n^2 \rangle^{1-b}} \langle m - z_n^2 \rangle^{-b} \Bigg|^2 \Bigg)^{1/2}$$

$$\lesssim_{\varphi, b} \Bigg(\sum_{z_n \in I} \sum_{\substack{m \\ |m - z_n^2| > 1}} \frac{|g_{m,n}|^2}{\langle m - z_n^2 \rangle^{2(1-b)}} \Bigg)^{1/2} \underbrace{\Bigg(\sum_{\substack{m, z_n \\ |m - z_n^2| > 1}} \frac{1}{\langle m - z_n^2 \rangle^{2b}} \Bigg)^{1/2}}_{< \infty \text{ as } b > 1/2}$$

$$\lesssim_{\varphi, b} \|g\|_{X^{0,b-1}(\mathbb{T}_t)}.$$

(III): $$\|(\mathrm{III})\|_{X^{0,b}(\Pi_t)} = \Big\| \sum_{\substack{z_n\in I\\ m\in\mathbb{Z}\\ |m-z_n^2|\le 1}} g_{m,n} e_n(x) \frac{\varphi(t)\left(e(mt)-e(z_n^2 t)\right)}{i(m-z_n^2)} \Big\|_{X^{0,b}(\Pi_t)}$$

$$= \Big\| \sum_k \sum_{z_n\in I} \Big(\sum_{\substack{m\in\mathbb{Z}\\ |m-z_n^2|\le 1}} \frac{g_{m,n}}{m-z_n^2}\left(\hat{\varphi}(k-m)-\hat{\varphi}(k-z_n^2)\right)\Big) e_n(x) e(kt) \Big\|_{X^{0,b}(\Pi_t)}$$

$$= \Big(\sum_k \sum_{z_n\in I} \langle k-z_n^2\rangle^{2b} \Big| \sum_{\substack{m\in\mathbb{Z}\\ |m-z_n^2|\le 1}} \frac{g_{m,n}}{(m-z_n^2)} \left(\hat{\varphi}(k-m)-\hat{\varphi}(k-z_n^2)\right)\Big|^2 \Big)^{1/2}.$$

By the mean value Theorem,

$$\frac{|\hat{\varphi}(k-m)-\hat{\varphi}(k-z_n^2)|}{|m-z_n^2|} \le |(\hat{\varphi})'(\tilde{m})|,$$

where $\tilde{m}\in(k-m, k-z_n^2)$. Since $|m-z_n^2|<1$, $\tilde{m}\sim k-z_n^2+o(1)$. Therefore,

$$\lesssim \Big(\sum_k \sum_{z_n\in I} \langle k-z_n^2\rangle^{2b} \Big| \sum_{\substack{m\in\mathbb{Z}\\ |m-z_n^2|\le 1}} g_{m,n} |(\hat{\varphi})'(k-z_n^2+o(1))| \Big|^2 \Big)^{1/2}$$

$$\lesssim \Big(\sum_{z_n\in I} \Big(\sum_k \langle k-z_n^2\rangle^{2b} |(\hat{\varphi})'(k-z_n^2+o(1))|^2 \Big) \cdot \Big| \sum_{|m-z_n^2|\le 1} \frac{g_{m,n}}{\langle m-z_n^2\rangle^{1-b}} \Big|^2 \Big)^{1/2}$$

$\lesssim$

$$\lesssim \|t\varphi\|_{H_t^b} \Big(\sum_{z_n\in I} \Big(\sum_{\substack{|m-z_n^2|\le 1\\ m}} 1^2 \Big) \Big(\sum_{\substack{|m-z_n^2|\le 1\\ m}} \frac{|g_{m,n}|^2}{\langle m-z_n^2\rangle^{2(1-b)}} \Big) \Big)^{1/2}$$

$$\lesssim_{\varphi,b} \Big(\sum_{\substack{z_n\in I\\ m\in\mathbb{Z}}} \frac{|g_{m,n}|^2}{(m-z_n^2)^{2(1-b)}} \Big)^{1/2} \lesssim_{\varphi,b} \|g\|_{X^{0,b-1}(\Pi_t)}.$$

This completes the verification of (2) and hence of (1).

To conclude, we argue by duality.

$$\|P_I f\|_{X^{0,b-1}(I_t)} = \sup\Big\{ \Big| \int (P_I f)(t,x)(P_I g)(t,x)\,dx\,dt \Big| : g\in L^2_{t,x}, \|P_I g\|_{X^{0,1-b}} \le 1 \Big\}$$

(Hölder) $\quad \leq \|f\|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}} \|P_I g\|_{L_x^{\frac{4+3\varepsilon}{1+3\varepsilon}} L_t^4}$

(Lemma 2.3) as $1-b<1/2$ $\quad \lesssim |I|^{1-2(1-b)+\varepsilon} \|f\|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}} \|P_I g\|_{X^{0,1-b}(I_T)}$

$$\lesssim |I|^{2b-1+\varepsilon} \|f\|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}}.$$

For the second inequality, we have

$$\left\| \int_0^t S(t-\tau) f(\tau)\, d\tau \right\|_{X^{-(2b-1+\varepsilon),b}(I_T)} = \left\| \int_0^t S(t-\tau) \langle\nabla\rangle^{-(2b-1+\varepsilon)} f(\tau)\, d\tau \right\|_{X^{0,b}(I_T)}$$

(Result just proved Dual estimate) $\quad \lesssim \|f\|_{X^{-(2b-1+\varepsilon),b-1}(I_T)}$

$$= \sup\left\{ \left| \int\!\!\int f g \, dt\, dx \right| : g \in X^{2b-1+\varepsilon,1-b}, \|g\|_{X^{2b-1+\varepsilon,1-b}} \leq 1 \right\}$$

$$\lesssim \sup\left(\|f\|_{L_x^{4/3+\varepsilon} L_t^{4/3}} \cdot \|g\|_{L_x^{\frac{4+3\varepsilon}{1+3\varepsilon}} L_t^4} \right).$$

$$\lesssim \|f\|_{L_x^{4/3+\varepsilon} L_t^{4/3}}.$$

Now change $f \longmapsto (\sqrt{-\Delta})^{2b-1+\varepsilon} f$. $\quad \boxtimes$

Probabilistic Estimates

Lemma 3.1: Fix $s<1/2$. Then we have the bound

$$\boxed{\mu_F^{(N)}\left(\left\{ \phi : N_0^{\frac{1}{2}-s} \|P_{\geq N_0} \phi\|_{H_x^s} > \lambda \right\}\right) \lesssim e^{-c\lambda^2},}$$

for all $N_0 \geq 1$ sufficiently large, where $\phi = \phi^{(\omega)} = \sum_{n\in\mathbb{N}} \frac{g_n(\omega)}{z_n} e_n$.

Remark: We can write $\mu_F = \mu_F^{(N)} \otimes (\mu_F^{(N)})^{\perp}$, and

$$\{\phi : N_0^{\frac{1}{2}-s} \|P_{\geq N_0} \phi\|_{H_x^s} > \lambda\} =: A_\lambda, \quad \mu_F(A_\lambda \times P_{>N} H^{\frac{1}{2}-}(B_2)) = \mu_F^{(N)}(A_\lambda) \underbrace{(\mu_F^{(N)})^{\perp}(P_{>N} H^{\frac{1}{2}-}(B_2))}_{=1}$$

Remark: Let us detail where the choice of $\mathcal{J}$; the extension of $P_I \int_0^t S(t-\tau) f(\tau) d\tau$ came from.

Suppose f were periodic on $\mathbb{T}_t$. Then we could write

$$f(x,t) = \sum_{m,n}' f_{m,n}\, e_n(x) e(mt).$$

Thus

$$P_I\left(\int_0^t S(t-\tau) f(\tau) d\tau\right) = \sum_{\substack{m\in\mathbb{Z}\\ z_n\in I}} \int_0^t e^{-i(t-\tau)z_n^2} f_{n,m}\, e_n(x) e(m\tau)\, d\tau$$

$$= \sum_{\substack{m\in\mathbb{Z}\\ z_n\in I}} f_{n,m}\, e_n(x) e(-t z_n^2) \int_0^t e((m-z_n^2)\tau)\, d\tau$$

$$= \sum_{m, z_n\in I} f_{n,m}\, e_n(x) e(t z_n^2) \left[\frac{e((m-z_n^2)t) - 1}{i(m-z_n^2)}\right]$$

$$= \sum_{m, z_n\in I} f_{n,m}\, e_n(x) \frac{e(mt) - e(z_n^2 t)}{i(m-z_n^2)}.$$

To be able to say $|m-z_n^2| \sim \langle m-z_n^2\rangle$, we split this into the regions $|m-z_n^2|>1$ & $|m-z_n^2|\le 1$. Finally, we insert the cutoffs on those terms which are not periodic over $\mathbb{T}_t$.

so we could replace $\mu_F^{(N)}$ by μ_F.

Proof: Fix $q_1 \geq 2$ to be determined. Then

$$\Big\| \|P_{\geq N_0}\phi\|_{H^s_x} \Big\|^{q_1}_{L^{q_1}_\omega(d\mu_F^{(N)})} \underset{\text{Minkowski}}{\leq} \Big\| \|P^{(\omega)}_{\geq N_0}\phi^{(\omega)}\|_{L^{q_1}(\Omega)} \Big\|^{q_1}_{L^2_x}$$

$$= \Big\| \Big\| \sum_{\{n \geq N_0\}} \frac{g_n(\omega)}{z_n^{1-s}} e_n(x) \Big\|_{L^{q_1}(\Omega)} \Big\|^{q_1}_{L^2_x}$$

(Wiener Chaos estimate)
$$\lesssim q_1^{q_1/2} \Big\| \Big(\sum_{n \geq N_0} \frac{|e_n(x)|^2}{z_n^{2(1-s)}} \Big)^{1/2} \Big\|^{q_1}_{L^2_x}$$

$$\sim (\sqrt{q_1})^{q_1} \Big(\sum_{n \geq N_0} \frac{\|e_n(x)\|^2_{L^2_x}}{z_n^{2(1-s)}} \Big)^{q_1/2}$$

$$\lesssim (\sqrt{q_1})^{q_1} \Big(\sum_{n \geq N_0}^{\infty} \frac{1}{z_n^{2(1-s)}} \Big)^{q_1/2}$$

Using $z_n \sim n$ for large n.
$$\lesssim (\sqrt{q_1})^{q_1} N_0^{(-1+2s)q_1/2}$$

$$\sim \left(\sqrt{q_1} N_0^{-\frac{1}{2}+s}\right)^{q_1}$$

By Chebyshev's inequality

$$\mu_F^{(N)}\left(\{\phi : N_0^{\frac{1}{2}-s}\|P_{\geq N_0}\phi\|_{H^s_x} > \lambda\}\right) \leq \frac{(N_0^{\frac{1}{2}-s})^{q_1} \Big\| \|P_{\geq N_0}\phi\|_{H^s_x}\Big\|^{q_1}_{L^{q_1}_\omega(d\mu_F^{(N)})}}{\lambda^{q_1}}$$

$$\lesssim \left(\frac{\sqrt{q_1}}{\lambda}\right)^{q_1}.$$

Now choose $q_1 = \lambda^2/e^2$. If $q_1 \geq 2$, then

$$\left(\frac{\sqrt{q_1}}{\lambda}\right)^{q_1} = e^{-c\lambda^2},$$

while if $q_1 < 2$, choose $C > 0$ so that $Ce^{-2} \geq 1$. Then

$$\mu_F^{(N)}\left(\{\phi : N_0^{\frac{1}{2}-s}\|P_{\geq N_0}\phi\|_{H^s_x} > \lambda\}\right) \leq 1 \leq Ce^{-2} \leq Ce^{-c\lambda^2}.$$

$\square$

The next proposition is one of the key new ideas introduced in this series of papers. In order to show convergence of the sequence of truncated solutions $\{u_N\}_N$, one needs uniform in N estimates of u_N.
This proposition supplies these estimates, in the sense that if one removes a "bad" set of measure zero, we obtain uniform in N estimates on the nonlinearly flowed u_N's.
The key is using the invariance of the truncated Gibbs measure under the truncated nonlinear flow.

Prop^n 3.2 (Probabilistic Uniform estimates)

Let $T>0$ be given. Then for every $0\leq\sigma<1/2$, $2\leq p<2/\sigma$ and $q<\infty$,

$$\mu_F^{(N)}\left(\left\{\phi_N : \|(\sqrt{-\Delta})^\sigma u_N^{(\phi_N)}\|_{L^p_x L^q_t(I_T\times B_2)}>\lambda\right\}\right)\leq e^{-c\lambda^c}, \quad \lambda>0,\ N\geq 1,$$

some $c>0$.

where $u_N^{(\phi_N)}=u_N$ is the solution of (TNLS) with initial data $P_N\phi$.

Remark: It will be apparent from the proof that the same statement with the same range of exponents holds if the L^p_x and L^q_t norms are switched. This will be important at the convergence step (Proof of Main Theorem).

The same statement also holds if we measure wrt μ_F.

Proof: For $\lambda,\lambda_1>0$, define

$$A_N^\lambda := \left\{\phi_N : \|(\sqrt{-\Delta})^\sigma u_N^{(\phi_N)}\|_{L^p_x L^q_t}>\lambda\right\}\subset P_N(H^{\frac12-}(B_2))$$

$$B_N^{\lambda_1} := \left\{\phi_N : \|\phi_N\|_{L_x^{\alpha+2}}>\lambda_1\right\}\subset P_N(H^{\frac12-}(B_2)).$$

Thus

$$\mu_F^{(N)}(A_N^\lambda) = \mu_F^{(N)}(A_N^\lambda\cap B_N^{\lambda_1}) + \mu_F^{(N)}(A_N^\lambda\setminus B_N^{\lambda_1})$$
$$=: (\mathrm{I}) + (\mathrm{II}).$$

(I): $\mu_F^{(N)}(A_N^\lambda \cap B_N^{\lambda_1}) \leq \mu_F^{(N)}(B_N^{\lambda_1})$

(Chebyshev) $q_2 \geq 2+\alpha$, $\leq \frac{1}{\lambda_1^{q_2}} \int \|\phi_N\|_{L_x^{\alpha+2}}^{q_2} \, d\mu_F^{(N)}(\phi_N)$

(Minkowski) $\leq \frac{1}{\lambda_1^{q_2}} \Big\| \|\phi_N\|_{L^{q_2}(d\mu_F^{(N)})} \Big\|_{L_x^{\alpha+2}}^{q_2}$

$$\sim \frac{1}{\lambda_1^{q_2}} \Big\| \Big\| \sum_{\{z_n \leq N\}} \frac{g_n(\omega)}{z_n} e_n(x) \Big\|_{L^{q_2}(\Omega)} \Big\|_{L_x^{\alpha+2}}^{q_2}$$

(Wiener Chaos estimate) $\lesssim \left(\frac{\sqrt{q_2}}{\lambda_1}\right)^{q_2} \Big\| \Big(\sum_{\{z_n \leq N\}} \frac{e_n^2(x)}{z_n^2} \Big)^{1/2} \Big\|_{L_x^{\alpha+2}}^{q_2}$

$$= C \left(\frac{\sqrt{q_2}}{\lambda_1}\right)^{q_2} \Big(\sum_{\{n : z_n \leq N\}} \frac{\|e_n(x)\|_{L_x^{\alpha+2}}^2}{z_n^2} \Big)^{q_2/2}$$

(L^p norm estimates of e_n) $\lesssim \left(\frac{\sqrt{q_2}}{\lambda_1}\right)^{q_2} \Big(\sum_{\{n : z_n \leq N\}} \frac{n^{1-4/2+\alpha}}{z_n^2} \Big)^{q_2/2}$

$$\lesssim \left(\frac{\sqrt{q_2}}{\lambda_1}\right)^{q_2} \Big(\sum_n \frac{n^{1-\frac{4}{2+\alpha}}}{n^2} \Big)^{q_2/2}$$

$$\lesssim \left(\frac{\sqrt{q_2}}{\lambda_1}\right)^{q_2}.$$

Choosing $q_2 = \lambda_1^2/e^2$, optimizes this inequality and we obtain

$$\mu_F^{(N)}(A_N^\lambda \cap B_N^{\lambda_1}) \lesssim e^{-c\lambda^2}.$$

(II): Let $R_N(\phi) := \exp\left(-\frac{1}{\alpha+2} \|\phi\|_{L_x^{\alpha+2}}^{\alpha+2}\right)$.

Then $\mu_F^{(N)}(A_N^\lambda \mid B_N^{\lambda_1}) = \int_{A_N^\lambda \mid B_N^{\lambda_1}} d\mu_F^{(N)}(u)$

$$= Z_N^{-1}\int_{A_N^\lambda \setminus B_N^{\lambda_1}} e^{\frac{1}{2+\alpha}\lambda_1^{2+\alpha}} Z_N R_N(u)\, d\mu_F^{(N)}$$

$$= Z_N e^{\frac{1}{2+\alpha}\lambda_1^{2+\alpha}}\left(\int_{A_N^\lambda\setminus B_N^{\lambda_1}} d\mu_G^{(N)}(u)\right).$$

(Chebyshev)

$e^{\frac{1}{2+\alpha}\lambda_1^{2+\alpha}} =: e_{\lambda_1}$

$$\lesssim \frac{e_{\lambda_1}}{\lambda^{q_1}}\int \|(\sqrt{-\Delta})^{\sigma} u_N^{(\phi_N)}\|_{L_x^p L_t^q}^{q_1}\, d\mu_G^{(N)}(\phi_N),\ q_1\geq \max(p,q)$$

$$\lesssim \frac{e_{\lambda_1}}{\lambda^{q_1}}\left\| \|(\sqrt{-\Delta})^{\sigma} u_N^{(\phi_N)}\|_{L^{q_1}(d\mu_G^{(N)})}\right\|_{L_x^p L_t^q}^{q_1}$$

$$= \frac{e_{\lambda_1}}{\lambda^{q_1}}\left\| \|(\sqrt{-\Delta})^{\sigma}\phi_N\|_{L^{q_1}(d\mu_G^{(N)})}\right\|_{L_x^p L_t^q}^{q_1},$$

where in the last step we made use of the invariance of the truncated Gibbs Measure under the nonlinear truncated flow.
We now use Cauchy-Schwarz and the Wiener Chaos estimate proceeding similar to before,

$$\lesssim \frac{e_{\lambda_1}}{\lambda^{q_1}}\left\|\left(\int |(\sqrt{-\Delta})^{\sigma}\phi_N|^{q_1} Z_N^{-1}R_N\, d\mu_F^{(N)}\right)^{1/q_1}\right\|_{L_x^p L_t^q}^{q_1}$$

$$\lesssim \frac{e_{\lambda_1}}{\lambda^{q_1}}\left\|\left(\int |(\sqrt{-\Delta})^{\sigma}\phi_N|^{2q_1} d\mu_F^{(N)}\right)^{\frac{1}{2q_1}}\left(\int Z_N^{-2}R_N^2\, d\mu_F^{(N)}\right)^{1/2q_1}\right\|_{L_x^p L_t^q}^{q_1}.$$

$$\lesssim \frac{e_{\lambda_1}}{\lambda^{q_1}}\left\| \left\|\sum_{\{n:\lambda_n\leq N\}} \frac{g_n(\omega)}{z_n^{2(1-\sigma)}} e_n(x)\right\|_{L^{2q_1}(\Omega)}\right\|_{L_x^p L_t^q}^{q_1}$$

$$\lesssim \left(\frac{\sqrt{q_1}\,T^{1/q}}{\lambda}\right)^{q_1} e_{\lambda_1}\left(\sum_{\{n:\lambda_n\leq N\}}\frac{\|e_n(x)\|_{L_x^p}^2}{z_n^{2(1-\sigma)}}\right)^{q_1/2}$$

$$\lesssim e_{\lambda_1}\left(\frac{\sqrt{q_1}\,T^{1/q}}{\lambda}\right)^{q_1}\left(\sum_{n\in\mathbb{N}}\frac{n^{1-4/p}}{n^{2(1-\sigma)}}\right)^{q_1/2}$$

converges as long as

$p<2/\sigma.$

Therefore

$$(\mathrm{II}) \lesssim e^{\frac{1}{\alpha+2}\lambda_1^{2+\alpha}} \left(\frac{\sqrt{q_1}\,T}{\lambda}\right)^{q_1}$$

$$\sim e^{\frac{1}{\alpha+2}\lambda_1^{2+\alpha}} e^{-c\lambda^2}, \quad \text{choosing } q_1 \sim \lambda^2/e.$$

Combining (I) & (II) gives

$$\mu_F^{(N)}(A_N^\lambda) \lesssim e^{-c\lambda^2} e^{\frac{1}{\alpha+2}\lambda_1^{2+\alpha}} + e^{-c\lambda_1^2}.$$

Optimize by setting $c_0\lambda_1^{2+\alpha} = \lambda^2$

$$\Rightarrow e^{-c\lambda^2 + \frac{c_0}{\alpha+2}\lambda^2} = e^{-(c - \frac{c_0}{\alpha+2})\lambda^2}$$

$\Rightarrow$ Choose c_0 small enough so that $\frac{c_0}{\alpha+2} < c$,

and we have the final bound

$$\mu_F^{(N)}(A_N^\lambda) \leq \exp(-c\lambda^{\frac{4}{2+\alpha}}).$$

$\boxtimes$

The following refinement is needed for the proof of the main theorem.

Proposition 3.3: Let T, p, q, σ and (u_N) be as given in Prop$^{\underline{n}}$ 3-2. Then for all $M, N \geq 1$ with $M < N$, we have

$$\mu_F^{(N)}\left(\{\phi_N : \|(\sqrt{-\Delta})^\sigma(u_N - P_M u_N)\|_{L_x^p L_t^q} > \lambda\}\right) < e^{-(\theta\lambda)^c},$$

where $\theta := T^{-1/q} M^{\frac{2}{p}-\sigma}$.

Proof: Notice that $u_N - P_M u_N = P_{\geq M} u_N$ so the only change over the previous proofs is in the sum for (II), where we now have

$$\left(\sum_{|n|\gtrsim M} n^{1-\frac{4}{p}-2+2\sigma}\right)^{1/2} \lesssim \left(M^{-1-\frac{4}{p}+2\sigma+1}\right)^{1/2}$$

$$\sim \left(M^{-\frac{4}{p}+2\sigma}\right)^{1/2} \sim M^{-\frac{2}{p}+\sigma}.$$

$\boxtimes$

Proposition 4-1 (High-low bilinear estimate)

Fix $0 \leq s \leq 1$, $b = \frac{1}{2}+$ (sufficiently close to $\frac{1}{2}$) and $N \in \mathbb{Z}_+$. Then for every $\mu > 0$, we have the inequality

$$\circledast \quad \left\| \int_0^t S(t-\tau)(fg)(\tau)\,d\tau \right\|_{X^{s,b}(I_T)} \lesssim \|f\|_{X^{s,b}(I_T)} \left\| (\sqrt{-\Delta})^{s(2b-1)+\mu} g \right\|_{L^2_x L^2_t},$$

for every f and g (radial) representable as

$$f(t,x) = \sum_{\substack{n > N \\ m \in \mathbb{Z}}} f_{n,m}\, e_n(x) e(mt) \qquad \text{(High)}$$

$$g(t,x) = \sum_{\substack{n \leq N \\ m \in \mathbb{Z}}} g_{n,m}\, e_n(x) e(mt) \qquad \text{(Low)}$$

Remark: In the following proof there does not appear to be any need for the upper restriction $s \leq 1$.

Proof: We first reduce to the case $s=0$. Suppose $\circledast$ holds for $s=0$. Then we can show that $\circledast$ also holds for $s=1$ and from which the range $0<s<1$ follows by interpolation.

$$\left\| \int_0^t S(t-\tau)(fg)(\tau)\,d\tau \right\|_{X^{1,b}} = \left\| \int_0^t S(t-\tau)\,(1-\Delta)^{1/2}(fg)(\tau)\,d\tau \right\|_{X^{0,b}}$$

$$\leq \left\| \int_0^t S(t-\tau)(fg)(\tau)\,d\tau \right\|_{X^{0,b}} + \left\| \int_0^t S(t-\tau)(-\Delta)^{1/2}(fg)(\tau)\,d\tau \right\|_{X^{0,b}}$$

$$\lesssim \left\| \int_0^t S(t-\tau)(fg)(\tau)\,d\tau \right\|_{X^{0,b}} + \left\| \int_0^t S(t-\tau)\left[(-\Delta)^{1/2} f\right] g(\tau)\,d\tau \right\|_{X^{0,b}}$$

(Using $s=0$ case of $\circledast$) $$\lesssim \left(\|f\|_{X^{0,b}} + \left\| (-\Delta)^{1/2} f \right\|_{X^{0,b}} \right) \left\| (-\Delta)^{\frac{1}{2}(s(2b-1)+\mu)} g \right\|_{L^2_{x,t}}$$

$$\lesssim \|f\|_{X^{1,b}} \left\| (-\Delta)^{\frac{1}{2}[s(2b-1)+\mu]} g \right\|_{L^2_{x,t}}.$$

For the third inequality, we have used the fact that f has higher frequencies than g which implies

$$(-\Delta)^{1/2}(fg) \lesssim \left[(-\Delta)^{1/2} f\right] g. \quad \text{(A Heuristic argument for this meaning.)}$$

From now fix $s=0$. Dyadically decompose g

$$g = {\sum_{K<N}}' g_K \, , \quad g_K(t,x) := \sum_{\substack{n \sim K \\ m \in \mathbb{Z}}} g_{m,n}\, e_n(x)\, e(mt),$$

and using the triangle inequality implies

$$\text{LHS of } (*) \leq {\sum_{K<N}}' \left\| \int_0^t S(t-\tau)(f g_K)(\tau)\, d\tau \right\|_{X^{0,b}}$$

We seek now to estimate each summand for a fixed dyadic K. Let $K_1 = K^5$, and $\mathcal{P}$ be a partition of $\mathbb{Z}$ into intervals I of size K_1 and write

$$f g_K = \left(\sum_{I \in \mathcal{P}} P_I f\right) g_K = \sum_{I' \in \mathcal{P}} P_{I'}\left(\sum_{I \in \mathcal{P}} P_I f \cdot g_K\right)$$

$$= {\sum_{\substack{I, I' \in \mathcal{P} \\ \operatorname{dist}(I,I') \leq K_1}}}' P_{I'}(P_I f \cdot g_K) + {\sum_{\substack{I, I' \in \mathcal{P} \\ \operatorname{dist}(I,I') > K_1}}}' P_{I'}(P_I f \cdot g_K)$$

$$=: (\mathrm{I}) + (\mathrm{II}).$$

Estimating (I):

$$\left\| \int_0^t S(t-\tau)\, (\mathrm{I})\, d\tau \right\|^2_{X^{0,b}} \leq \left\| {\sum_{I \in \mathcal{P}}}' P_{\tilde{I}}\left(\int_0^t S(t-\tau)(P_I f) g_K(\tau)\, d\tau \right) \right\|^2_{X^{0,b}},$$

where $\tilde{I}$ is either I or an adjacent neighbor. Since the intervals I are essentially disjoint,

$$\left\| \sum_{I \in \mathcal{P}} P_{\tilde{I}}\left(\int_0^t S(t-\tau)(P_I f) g_K(\tau)\, d\tau \right) \right\|^2_{X^{0,b}} \simeq {\sum_{I \in \mathcal{P}}}' \left\| P_{\tilde{I}}\left(\int - \right) \right\|^2_{X^{0,b}}.$$

Therefore

$$\left\| \sum_{I \in \mathcal{P}} P_{\tilde{I}}\left(\int_0^t S(t-\tau)(P_I f) g_K(\tau)\, d\tau \right) \right\|_{X^{0,b}} \lesssim \left[{\sum_{I \in \mathcal{P}}}' \left\| P_{\tilde{I}}\left(\int_0^t S(t-\tau) P_I f\, g_K(\tau)\, d\tau \right) \right\|^2_{X^{0,b}} \right]^{1/2} \quad (1)$$

Applying the dual estimate to the Strichartz estimate (Lemma 2.5) to each summand of (1) we obtain,

$$\text{RHS of (1)} \le K_1^{2b-1+\varepsilon}\Big(\sum_{I\in P}\|P_I f - g_K\|^2_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}}\Big)^{1/2}$$

Hölder in (x,t)
$x: \frac{1}{4/3+\varepsilon} = \frac{1}{4-\varepsilon''} + \frac{1}{2+\varepsilon'}$
$t: \frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

$$\le K_1^{2b-1+\varepsilon}\|g_K\|_{L_x^{2+\varepsilon'}L_t^2}\Big(\sum_{I\in P}\|P_I f\|^2_{L_x^{4-\varepsilon''}L_t^4}\Big)^{1/2}$$

$$\lesssim 3K_1^{2b-1+\varepsilon}\|g_K\|_{L_x^{2+\varepsilon'}L_t^2}\Big(\sum_{I\in P}\|P_I f\|^2_{L_x^{4-\varepsilon''}L_t^4}\Big)^{1/2}. \quad \text{...(2)}$$

Now Lemma 2.3, $\|P_I f\|_{L_x^{4-\varepsilon''}L_t^4} \le K_1^{\varepsilon''}\|P_I f\|_{X^{0,b}}$, ...(3)

while, Minkowski and Sobolev inequalities imply

$$\|g_K\|_{L_x^{2+\varepsilon'}L_t^2} \le \|g_K\|_{L_t^2 L_x^{2+\varepsilon'}}$$
$$\lesssim \|\langle\nabla\rangle^{\tilde\varepsilon} g_K\|_{L_t^2 L_x^2}$$
$$\lesssim K^{\tilde\varepsilon}\|g_K\|_{L^2_{x,t}}. \quad \text{..(4)}$$

Using (3) and (4) implies

$$(2) \le K_1^{5(2b-1)+\tilde\mu}\|g_K\|_{L^2_{x,t}}\Big(\sum_{I\in P}\|P_I f\|^2_{X^{0,b}}\Big)^{1/2}$$

$$\lesssim \big(K^{5(2b-1)+\tilde\mu-\varepsilon}K^{\varepsilon}\|g_K\|_{L^2_{x,t}}\big)\cdot\|f\|_{X^{0,b}}$$

(Bernstein)
$$\lesssim K^{-\varepsilon}\|(\partial_t-\Delta)^{5(2b-1)+\mu} g\|_{L^2_{x,t}}\|f\|_{X^{0,b}}.$$

We now sum over K to obtain ⊛ for the piece (I).

Estimating (II):
Recall the classical dual estimate (which we proved in the case of B_2 in Lemma 2.5) which implies

$$\Big\|P_I\Big(\int_0^t S(t-\tau)F(\tau)d\tau\Big)\Big\|_{X^{0,b}} \lesssim \|P_I F\|_{X^{0,b-1}} \le \|P_I F\|_{X^{0,0}=L^2_{x,t}}, \quad \text{...(5)}$$

for $b \overset{>1/2}{}$ sufficiently close to $1/2$.

Notice that essentially ½ a derivative was "thrown away" here, which implies the term (II) must be a "negligible error term".

Heuristic: f high, g low so expect

$$P_I((P_{I'}f)g_K) \approx (P_I P_{I'} f) g_K + \text{"Error"}$$
$$\approx (P_{I\cap I'} f) g_K + \text{"Error"},$$

but $\mathrm{dist}(I,I') > K$, so $I \cap I' = \phi$,

$$\Rightarrow P_I((P_{I'}f)g_K) \approx \text{"Error"}$$

By (5),

$$\left\| P_{\le}\left(\int_0^t S(t-\tau)(\mathrm{II})\,d\tau\right)\right\|_{X^{0,b}} \lesssim \|(\mathrm{II})\|_{L^2_{x,t}}$$

so want to estimate

$$\Big\| \sum_{\substack{I,I' \in \mathcal{P} \\ \mathrm{dist}(I,I') > K}} P_{I'}((P_I f) g_K) \Big\|_{L^2_{x,t}} =: \|F(x,t)\|_{L^2_{x,t}}.$$

By duality,

$$\|F\|_{L^2_x} = \sup_{\|a\|_{L^2_x} = \sum_{n\ge 1} |a_n|^2 \le 1} \langle a, F\rangle$$

$$= \sup_{\|a_n\|_{\ell^2_n} \le 1} \Big\langle \sum_{n'} a_{n'} e_{n'}, \sum_n \hat{F}(n) e_n \Big\rangle$$

$$= \sup_{\|a_n\|_{\ell^2_n} \le 1} \sum_{n'} a_{n'} \hat{F}(n'). \quad (\langle e_{n'}, e_n\rangle = \delta_{nn'}).$$

Now

$$\hat{F}(n') = \langle F, e_{n'}\rangle$$

$$= \sum_{\circledast_{I,I'}} \Big\langle \sum_{j\in I'} \langle (P_I f) g_K, e_j\rangle e_j, e_{n'} \Big\rangle$$

$$= \sum_{\circledast_{I,I'}} \sum_{j} \langle (P_I f) g_K, e_j\rangle \underbrace{\langle e_j, e_{n'}\rangle}_{\delta_{jn'}} [j \in I']$$

$$= \sum_{\circledast_{I,I'}} \langle (P_I f) g_K, e_{n'} \rangle \, [n' \in I'] .$$

$$= \sum_{\circledast_{I,I' \in P}} \sum_n f_n \langle e_n g_K, e_{n'} \rangle \, [n' \in I'] [n \in I] [\mathrm{dist}(I, I') > k_1] .$$

Therefore,

$$\|F\|_{L^2_x} \le \sup_{\|a_n\|_{\ell^2_n} \le 1} \sum_{n'} |a_{n'}| |\hat{F}(n')|$$

$$\le \sup_{\|a_n\|_{\ell^2_n} \le 1} \sum_{n'} \sum_n \sum_{I, I'} |a_{n'}| |f_n| |\langle e_n g_K, e_{n'} \rangle| \underbrace{[n \in I][n' \in I'][\mathrm{dist}(I, I') > k_1]}_{= [n \in I,\, n' \in I',\, \mathrm{dist}(I, I') > k_1]}$$

$$= \sup_{\|a_n\|_{\ell^2_n} \le 1} \sum_{\substack{n, n' \\ |n - n'| > k_1}}' |a_{n'}| |f_n| |\langle e_n g_K, e_{n'} \rangle| .$$

$$\Rightarrow \Big\| \sum_{\circledast_{I,I'}} P_{I'} (P_I f \cdot g_K) \Big\|_{L^2_x} \le \sup_{\|a_n\|_{\ell^2_n} \le 1} \sum_{n, n' \ge 1}' |f_n| |a_{n'}| M_{n,n'} , \quad \text{--- (6)}$$

$$M_{n,n'} := |\langle e_n g_K, e_{n'} \rangle| \, \chi_{|n - n'| > k_1}^{(n,n')} .$$

If we have an estimate on $\|M\|_{\ell^2 \to \ell^2}$ where $M = (M_{n,n'})_{n,n'}$, then we would have

$$\sum_{n, n' \ge 1} |f_n| |a_{n'}| M_{n,n'} \le \|M\|_{\ell^2 \to \ell^2} \Big(\sum_{n,n'} |f_n|^2 |a_{n'}|^2 \Big)^{1/2}$$

$$\le \|M\|_{\ell^2 \to \ell^2} \Big(\sum_n |f_n|^2 \Big)^{1/2}$$

$$= \|M\|_{\ell^2 \to \ell^2} \|f(t)\|_{L^2_x} , \quad \text{-- (7)}$$

where $f(t, x) = \sum_{n \ge 1} f_n e_n(x)$.

Schur's test (for matrices)

$$\boxed{\begin{aligned} &\text{Suppose} \quad \sup_j \sum_{k=1}^{\infty} |A_{jk}| \le \alpha < \infty, \\ &\qquad\qquad\quad \sup_k \sum_{j=1}^{\infty} |A_{jk}| \le \beta < \infty. \\ &\text{Then} \quad \|A = (A_{jk})\|_{\ell^2 \to \ell^2} \le \sqrt{\alpha\beta}. \end{aligned}}$$

Remark: If A is symmetric, only one of the above conditions needs to be checked.

Thus to bound the symmetric matrix M from $\ell^2 \to \ell^2$ it suffices to bound

$$\sup_n \sum_{n'} |M_{n,n'}| .$$

Using the eigenfunction equation $-\Delta e_n = z_n^2 e_n$, we have

$$|z_n^2 - z_{n'}^2| \, |\langle e_n g_k, e_{n'} \rangle| = |\langle (\Delta e_n) g_k, e_{n'} \rangle - \langle e_n g_k, \Delta e_{n'} \rangle| \quad \ldots (7)$$

By integration by parts and the Dirichlet boundary conditions, we compute

$$\begin{aligned} \langle e_n g_k, \Delta e_{n'} \rangle &= \langle e_n g_k, \partial^i \partial_i e_{n'} \rangle \\ &= \langle (\partial^i \partial_i e_n) g_k, e_{n'} \rangle + 2 \langle \partial^i e_n \partial_i g_k, e_{n'} \rangle \\ &\quad + \langle e_n \partial^i \partial_i g_k, e_{n'} \rangle \end{aligned}$$

$$\Rightarrow \langle e_n g_k, \Delta e_{n'} \rangle - \langle (\Delta e_n) g_k, e_{n'} \rangle = 2 \langle \nabla e_n \cdot \nabla g_k, e_{n'} \rangle + \langle e_n (\Delta g_k), e_{n'} \rangle.$$

Inserting this into (7) and using the asymptotics of the eigenvalues z_n^2, we have

$$|\langle e_n g_k, e_{n'} \rangle| \lesssim \frac{1}{|n-n'|(n+n')} \Big[|\langle \nabla e_n \cdot \nabla g_k, e_{n'} \rangle| + \underbrace{|\langle e_n \Delta g_k, e_{n'} \rangle|} \Big].$$

Fixing $n \in \mathbb{Z}_+$, we have

$$\sum_{n'} M_{n,n'} \le \sum_{n'} [|n-n'| > k] \frac{1}{|n-n'|(n+n')} \Big[\quad \Big],$$

$$\lesssim \sum_{\{\ell: 2^\ell > K_1\}} \sum_{\{n': |n-n'| \sim 2^\ell\}} \frac{1}{|n-n'|(n+n')} \left[\quad \right]$$

$$\sim \sum_{\{\ell: 2^\ell > K_1\}} 2^{-\ell} \sum_{\{n': |n-n'| \sim 2^\ell\}} \frac{1}{n+n'} \left(|\langle \nabla e_n \cdot \nabla g_K, e_{n'} \rangle| + |\langle e_n \Delta g_K, e_{n'} \rangle| \right)$$

C-S in n'

$$\sim \sum_{\{\ell: 2^\ell > K_1\}} \frac{2^{-\ell}}{n} \Big(\underbrace{\sum_{\{n': |n-n'| \sim 2^\ell\}} 1^2}_{\sim 2^{-\ell/2}} \Big)^{1/2} \Big(\sum_{n' \geq 1} |\langle \nabla e_n \cdot \nabla g_K, e_{n'} \rangle|^2 + |\langle e_n \Delta g_K, e_{n'} \rangle|^2 \Big)^{1/2}.$$

Summing in ℓ and using Plancherel $\left(\|\cdot\|_{L^2_x}^2 = \sum |\text{coeffs}|^2 \right)$

$$\lesssim \frac{K_1^{-1/2}}{n} \left(\|\nabla e_n \cdot \nabla g_K\|_{L^2_x} + \|e_n \Delta g_K\|_{L^2_x} \right)$$

$$\lesssim \frac{K_1^{-1/2}}{n} \left(\|\nabla e_n\|_{L^2_x} \|\nabla g_K\|_{L^\infty_x} + \|e_n\|_{L^\infty_x} \|\Delta g_K\|_{L^2_x} \right)$$

$$\lesssim \frac{K_1^{-1/2}}{n} \left(n K^2 \|g_K\|_{L^2_x} + n^{1/2} K^2 \|g_K\|_{L^2_x} \right) \quad \left(\text{Bernstein + eigenfunction estimates} \right)$$

$$\lesssim \frac{K^2}{K_1^{1/2}} \|g(t)\|_{L^2_x} \underset{K_1 := K^5}{\lesssim} K^{-1/2} \|g(t)\|_{L^2_x}.$$

Therefore

$$\|M\|_{\ell^2 \to \ell^2} \leq \sup_n \sum_{n'} |M_{n,n'}|$$

$$\leq \sup_n K^{-1/2} \|g(t)\|_{L^2_x} = K^{-1/2} \|g(t)\|_{L^2_x}.$$

$$\Rightarrow \|(\text{II})\|_{L^2_x} \lesssim \frac{1}{K^{1/2}} \|f(t)\|_{L^2_x} \|g(t)\|_{L^2_x}.$$

$$\Rightarrow \|(\mathrm{II})\|_{L^2_{x,t}} \lesssim K^{-1/2} \big\| \|f(t)\|_{L^2_x} \|g(t)\|_{L^2_x} \big\|_{L^2_t}$$

$$\lesssim K^{-1/2} \|f\|_{L^\infty_t L^2_x} \|g\|_{L^2_{x,t}}$$

$X^{0,b} \hookrightarrow L^\infty_t L^2_x$ iff $b > 1/2$.

$$\lesssim K^{-1/2} \|f\|_{X^{0,b}} \|g\|_{L^2_{x,t}};$$

Summing in K implies ⊛ for piece (II), which combined with (I) concludes the proof. ☑.

Proof of Theorem 1.1 (Main Theorem)

We obtain almost sure convergence in $X^{s,b}([0,T))$ which will imply a.s. convergence in $C_t([0,T); H^s_x(B_2))$.

Let $0<s<1/2$ and $T>0$ be given. We can assume $T<1/2$, otherwise we essentially just need to iterate for longer in the following proof.
We split the proof into two parts:

1) Convergence of a subsequence

We show the subsequence $(u_{N_K})_K$, where $N_K = 2^K$, converges.

Put $\sigma \in (0, 1/2)$, $r \in (2, 2/\sigma)$, $p, q \in (2, \infty)$ which will be determined later.
Fix $N_0 < N_1$ ($N_j < N_{j+1}$) and for each $\omega \in \Omega$, let u_{N_0}, u_{N_1} be the solutions of (FNLS) with corresponding data $P_{N_0}\phi^{(\omega)}$, $P_{N_1}\phi^{(\omega)}$ resp.

Let $B_{N_0} > 0$ be a parameter to be determined s.t. $B_{N_0} \lesssim N_0^{\delta}$ for some $\delta > 0$
(We will choose $B_{N_0} \sim (\log N_0)^{\delta}$ so this will be satisfied).

Define the "BAD" set

$$\Omega(N_0, N_1) := \Big\{ \omega \in \Omega : \| P_{N_1}\phi^{(\omega)} - P_{N_0}\phi^{(\omega)} \|_{H^s_x} \gtrsim N_0^{s-1/2} B_{N_0},$$
$$\max\big(\|u_{N_0}\|_{L^p_x L^q_t}, \|u_{N_1}\|_{L^p_x L^q_t}, \|u_{N_0}\|_{L^q_t L^p_x}, \|u_{N_1}\|_{L^q_t L^p_x},$$
$$\|(\sqrt{-\Delta})^{\sigma} u_{N_0}\|_{L^r_x L^q_t \cap L^q_t L^r_x}, \|(\sqrt{-\Delta})^{\sigma} u_{N_1}\|_{L^r_x L^q_t \cap L^q_t L^r_x} \big) > B_{N_0} \Big\}.$$

By the probabilistic tail estimates (Prop^n 3.2 and Lemma 3.1),

$$\mu_F(\Omega(N_0,N_1)) = \mu_F^{(N)}(\Omega(N_0,N_1)) \lesssim \exp(-B_{N_0}^{c}).$$
$\ast R$

Note for the application of Lemma 3.1,

$$\|P_{N_1}\phi^{(\omega)} - P_{N_0}\phi^{(\omega)}\|_{H^s_x} = \|P_{N_0 \leq \cdot \leq N_1}\phi^{(\omega)}\|_{H^s_x} \leq \|P_{\geq N_0}\phi^{(\omega)}\|_{H^s_x}$$

$$\boxed{\text{Fix } \omega \in \Omega \setminus \Omega(N_0, N_1)}$$

We want to estimate the difference $u_{N_1} - u_{N_0}$ over $[0,T)$ in $X^{s,b}([0,T])$. To this end, partition $[0,T)$ into $[T/\eta]$ intervals $[t_i, t_{i+1})$ of length $t_{i+1} - t_i = \eta$.

Since u_{N_1}, u_{N_0} solve (FNLS), then for $t \in [t_i, t_{i+1})$,

$$u_{N_1}(t) - u_{N_0}(t) = S(t-t_i)\left(u_{N_1}(t_i) - u_{N_0}(t_i)\right)$$
$$- i\int_{t_i}^{t} S(t-\tau)\left[P_{N_1}(|u_{N_1}|^\alpha u_{N_1})(\tau) - P_{N_0}(|u_{N_1}|^\alpha u_{N_1})(\tau)\right] d\tau \quad \text{introduced.}$$

$(*)$

$$- i\int_{t_i}^{t} S(t-\tau) P_{N_0}\left[|u_{N_1}|^\alpha u_{N_1} - |u_{N_0}|^\alpha u_{N_0}\right](\tau)\, d\tau$$

Linear estimate

Define $v(t_i) := S(-t_i)\left[u_{N_1}(t_i) - u_{N_0}(t_i)\right]$, and let $\varphi \in C_0^\infty(\mathbb{R})$ be a cutoff function supported over ~~$[t_i, t_{i+1}]$~~ $\mathbb{T}_t \supset [t_i, t_{i+1})$ and such that $\varphi(t) \equiv 1$ on $[t_i, t_{i+1})$.

Then $\varphi(t) S(t) v(t_i)$ is a periodic extension of $S(t) v(t_i)$ and hence

$$\|S(t-t_i)(u_{N_1}(t_i) - u_{N_0}(t_i))\|_{X^{s,b}([t_i,t_{i+1}))} \leq \|\varphi(t) S(t) v(t_i)\|_{X^{s,b}(\mathbb{T}_t)}$$

$$= \Big\|\sum_{n,m} \hat{\varphi}(m - z_n^2)\, \hat{v}_n(t_i) e_n(x) e(mt)\Big\|_{X^{s,b}(\mathbb{T}_t)}$$

$$= \Big(\sum_{n} \langle z_n\rangle^{2s} |\hat{v}_n(t_i)|^2 \sum_m \langle m - z_n^2\rangle^{2b} |\hat{\varphi}(m - z_n^2)|^2\Big)^{1/2}$$

$$= \|\varphi\|_{H^b_t} \|v(t_i)\|_{H^s_x}$$

$$\leq C(\varphi)\|u_{N_1}(t_i) - u_{N_0}(t_i)\|_{H^s_x(B_2)}. \quad \text{---}(1)$$

Nonlinear estimates

Fix $s' \in (s, 1/2)$.
All $X^{s,b}$, L_t^p norms will be taken on the interval $[t_i, t_{i+1})$ unless otherwise stated.
We begin with the first nonlinear term in ⊛. We have

$$\Big\| \int_{t_i}^{t} S(t-\tau) \underbrace{[P_{N_1}(|u_{N_1}|^\alpha u_{N_1})(\tau) - P_{N_0}(|u_{N_1}|^\alpha u_{N_1})(\tau)]}_{P_{N_0 < \cdot \le N_1}} d\tau$$

$$\lesssim N_0^{-(s'-s)} \Big\| \int_{t_i}^{t} S(t-\tau) P_{N_0 < \cdot \le N_1} (|u_{N_1}|^\alpha u_{N_1})(\tau) d\tau \Big\|_{X^{s',b}}$$

$$\lesssim N_0^{-(s'-s)} \| (\sqrt{-\Delta})^{2b-1+s'+\varepsilon} (|u_{N_1}|^\alpha u_{N_1}) \|_{L_x^{\frac{4}{3}+\varepsilon} L_t^{4/3}}$$ (Dual estimate to Strichartz)

$$\lesssim N_0^{-(s'-s)} \| (\sqrt{-\Delta})^{2b-1+s'+\varepsilon} (|u_{N_1}|^\alpha u_{N_1}) \|_{L_t^{4/3} L_x^{4/3+\varepsilon}}$$ (Minkowski)

$$\lesssim N_0^{-(s'-s)} \|u_{N_1}\|^\alpha_{L_t^{\frac{4(\alpha+1)}{3}} L_x^{r_1}} \| (\sqrt{-\Delta})^{2b-1+s'+\varepsilon} u_{N_1} \|_{L_t^{\frac{4(\alpha+1)}{3}} L_x^{r_2}}$$

Fractional Leibniz rule + Hölder in x, when $b \sim \frac{1}{2}+$, σ chosen so that

$$2b-1+s'+\varepsilon < \sigma.$$

(Note: $2b-1 \sim \varepsilon$, $s' \in (s, 1/2) \Rightarrow \sigma > \frac{1}{2}- \Rightarrow 2/\sigma < 4$.
So $r_1, r_2 \geq 2$ satisfy $r_1 \le p$, $r_2 \le r_1$, $\frac{4(\alpha+1)}{3} < q$ and

$$\frac{3}{4+3\varepsilon} = \frac{\alpha}{r_1} + \frac{1}{r_2}.$$

Prop^n 3.2 $\Rightarrow r_1 < 2/\sigma$ but $\sigma = 0$ for this term (no derivatives) so free to choose r_1 sufficiently large so that r_2 can satisfy $2 \le r_2 < 2/\sigma_2$.)

Since $\omega \in \Omega \setminus \Omega(N_0, N_1)$ we get

$$\Big\| \int_{t_i}^{t} S(t-\tau)[P_{N_1}(|u_{N_1}|^\alpha u_{N_1}) - P_{N_0}(|u_{N_1}|^\alpha u_{N_1})] d\tau \Big\|_{X^{s,b}([t_i,t_{i+1}))} \lesssim N_0^{-(s'-s)} B_{N_0}^{\alpha+1}. \quad \ldots (2$$

Now we estimate the second non-linear term in ⊛,

$$\left\| \int_{t_i}^{t} S(t-\tau) P_{N_0}\left[|u_{N_1}|^\alpha u_{N_1} - |u_{N_0}|^\alpha u_{N_0} \right] d\tau \right\|_{X^{s,b}([t_i,t_{i+1}))} \quad \cdots (3)$$

We can write

$$|u_{N_1}|^\alpha u_{N_1} - |u_{N_0}|^\alpha u_{N_0} = (u_{N_1} - u_{N_0}) F_+(u_{N_0}, u_{N_1}, \overline{u_{N_0}}, \overline{u_{N_1}})$$
$$+ (\overline{u_{N_1}} - \overline{u_{N_0}}) F_-(u_{N_0}, u_{N_1}, \overline{u_{N_0}}, \overline{u_{N_1}}),$$

where $F_\mp$ are homogeneous polynomials of degree α.
We consider only F_+ (the first term).
Dyadically decompose F_+ into $P_{K<\cdot\leq 2K} F_+$ and then write
$$u_{N_1} - u_{N_0} = P_{>2K}(u_{N_1} - u_{N_0}) + P_{\leq 2K}(u_{N_1} - u_{N_0}),$$
and insert into (3) to get

$$(3) \lesssim \sum_K \left\| \int_{t_i}^{t} S(t-\tau) P_{N_0}\left[(P_{>2K}(u_{N_1} - u_{N_0}))\, P_{K\leq\cdot\leq 2K} F_+ \right] d\tau \right\|_{X^{s,b}}$$
$$+ \sum_K \left\| \int_{t_i}^{t} S(t-\tau) P_{N_0}\left[P_{\leq 2K}(u_{N_1} - u_{N_0}) \cdot P_{K<\cdot\leq 2K} F_+ \right] d\tau \right\|_{X^{s,b}}$$
$$=: \sum_K (\mathrm{I})_K + \sum_K (\mathrm{II})_K .$$

$(\mathrm{I})_K$: $\quad (\mathrm{I})_K \leq \left\| \int_{t_i}^{t} S(t-\tau) P_{>2K}(u_{N_1} - u_{N_0})\, P_{\sim K} F_+(\tau)\, d\tau \right\|_{X^{s,b}}$

(Bilinear estimate) $\lesssim \|u_{N_1} - u_{N_0}\|_{X^{s,b}} \left\| (\sqrt{-\Delta})^{s(2b-1)+2\varepsilon} (\sqrt{-\Delta})^{-\varepsilon} P_{\sim K} F_+ \right\|_{L^2_{t,x}}$

$$\lesssim K^{-\varepsilon} \|u_{N_1} - u_{N_0}\|_{X^{s,b}} \left\| (\sqrt{-\Delta})^{s(2b-1)+2\varepsilon} P_{\sim K} F_+ \right\|_{L^2_{t,x}}$$

(Hölder in space-time) $\lesssim K^{-\varepsilon} \eta^{1/4} \|u_{N_1} - u_{N_0}\|_{X^{s,b}} \left\| (\sqrt{-\Delta})^{s(2b-1)+2\varepsilon} P_{\sim K} F_+ \right\|_{L^4_{t,x}}$

(Frac Leibniz in space + Hölder in time; $\frac{1}{4} = \frac{1}{8} + \frac{\alpha-1}{8(\alpha-1)}$) $\lesssim K^{-\varepsilon} \eta^{1/4} \|u_{N_1} - u_{N_0}\|_{X^{s,b}} \|u_{N_0}\|^{\alpha-1}_{L^{8(\alpha-1)}_{t,x}} \left\| (\sqrt{-\Delta})^{s(2b-1)+2\varepsilon} u_{N_0} \right\|_{L^8_{t,x}}$

$$\lesssim K^{-\varepsilon} \eta^{1/4} B^\alpha_{N_0} \|u_{N_1} - u_{N_0}\|_{X^{s,b}}$$

Irrelevant if u_{N_1} or u_{N_0}

for $\varepsilon>0$ sufficiently small and $b>1/2$ sufficiently close to $1/2$ to ensure

$$8 < \frac{2}{5(2b-1)+2\varepsilon} = \frac{2}{\sigma}.$$

$(\mathrm{II})_K$:

$$(\mathrm{II})_K \le \left\| \int_{t_i}^{t} S(t-\tau)\left[P_{\le 2K}(u_{N_1}-u_{N_0}) \cdot P_{\sim K} F_+\right](\tau)\,d\tau \right\|_{X^{s,b}}$$

$$= \left\| \int_{t_i}^{t} S(t-\tau)(I-\Delta)^{s/2}\left[P_{\le 2K}(u_{N_1}-u_{N_0}) P_{\sim K} F_+\right] d\tau \right\|_{X^{0,b}}$$

$$\lesssim \left\| \int_{t_i}^{t} S(t-\tau)\left(P_{\lesssim K}(u_{N_1}-u_{N_0}) \cdot (I-\Delta)^{s/2} P_{\sim K} F_+\right) d\tau \right\|_{X^{0,b}}$$

Dual to Strichartz

$$\lesssim \left\| (\sqrt{-\Delta})^{2b-1+\varepsilon}\left(P_{\le K}(u_{N_1}-u_{N_0}) \cdot (I-\Delta)^{s/2} P_{\sim K} F_+\right) \right\|_{L_x^{\frac43+\varepsilon} L_t^{4/3}}.$$

$$\lesssim \left\| P_{\le K}(u_{N_1}-u_{N_0}) \cdot (I-\Delta)^{\frac{2b-1+s+\varepsilon}{2}} P_{\sim K} F_+ \right\|_{L_t^{\frac43+\varepsilon} L_x^{4/3}}$$

Hölder:
$\frac{3}{4+\varepsilon} = \frac{1}{4+\varepsilon'} + \frac12$
$\frac34 = \frac14 + \frac12$

$$\lesssim \left\| P_{\le K}(u_{N_1}-u_{N_0}) \right\|_{L^2_{t,x}} \left\| \langle\nabla\rangle^{\varepsilon} P_{\sim K}\left(\langle\nabla\rangle^{2b-1+s} F_+\right) \right\|_{L_x^{4+\varepsilon'} L_t^4}$$

$$\lesssim \| u_{N_1}-u_{N_0} \|_{L^2_{t,x}} \left\| \langle\nabla\rangle^{\varepsilon+\varepsilon''} P_{\sim K}\left(\langle\nabla\rangle^{2b-1+s} F_+\right) \right\|_{L_x^4 L_t^4}$$

Minkowski + Sobolev in x
$\varepsilon'' = \frac14 - \frac{1}{4+\varepsilon'}$

Hölder int

$$\lesssim \eta^{1/2} \| u_{N_1}-u_{N_0} \|_{L_t^\infty L_x^2} \left\| \langle\nabla\rangle^{-(\varepsilon+\varepsilon'')} \langle\nabla\rangle^{2(\varepsilon+\varepsilon'')} P_{\sim K}\left(\langle\nabla\rangle^{2b-1+s} F_+\right) \right\|_{L_t^4 L_x^4}$$

$$\lesssim K^{-(\varepsilon+\varepsilon'')} \eta^{1/2} \| u_{N_1}-u_{N_0} \|_{X^{0,b}} \left\| \langle\nabla\rangle^{2b-1+s+2(\varepsilon+\varepsilon'')} F_+ \right\|_{L_t^4 L_x^4}$$

$$\lesssim K^{-(\varepsilon+\varepsilon'')} \eta^{1/2} \| u_{N_1}-u_{N_0} \|_{X^{0,b}} \| u_{N_\square} \|^{\alpha-1}_{L_t^{8(\alpha-1)} L^{\frac{4(4+\varepsilon)(\alpha-1)}{\varepsilon}}}$$

$$\times \left\| |\nabla|^{2b-1+s+2(\varepsilon+\varepsilon'')} u_{N_\square} \right\|_{L_t^8 L_x^{4+\varepsilon'}}$$

Provided that $\quad 4+\tilde{\varepsilon} < \dfrac{2}{2b-1+s+2(\varepsilon+\varepsilon'')}$, $\tilde{\varepsilon}$ suff. small.

$$= \frac{2}{\sigma}.$$

$$\Rightarrow (\mathrm{II})_K \lesssim K^{-(\varepsilon+\varepsilon'')} \eta^{1/2} B_{N_0}^{\alpha} \| u_{N_1} - u_{N_0} \|_{X^{s,b}}.$$

Summing over K for $(\mathrm{I})_K$ & $(\mathrm{II})_K$, we obtain

$$\left\| \int_{t_i}^{t} S(t-\tau) P_{N_0}\left[|u_{N_1}|^{\alpha} u_{N_1} - |u_{N_0}|^{\alpha} u_{N_0} \right] d\tau \right\|_{X^{s,b}([t_i,t_{i+1}])} \lesssim \eta^{1/4} B_{N_0}^{\alpha} \| u_{N_1} - u_{N_0} \|_{X^{s,b}}.$$

Combining everything gives

$$\left. \begin{aligned} \| u_{N_1} - u_{N_0} \|_{X^{s,b}([t_i,t_{i+1}])} &\le C_1 \| u_{N_1}^{(t_i)} - u_{N_0}^{(t_i)} \|_{H_x^s} + C_2 N_0^{-(s'-s)} B_{N_0}^{\alpha+1} \\ &\quad + C_3 \eta^{1/4} B_{N_0}^{\alpha} \| u_{N_1} - u_{N_0} \|_{X^{s,b}([t_i,t_{i+1}])}. \end{aligned} \right\} \cdots (4)$$

Choose

$$B_{N_0} = (c \log N_0)^{\frac{1}{8\alpha}}, \quad \eta = c B_{N_0}^{-4\alpha} \sim (\log N_0)^{-1/2},$$

and N_0 sufficiently large so that

$$C_3 \eta^{1/4} B_{N_0}^{\alpha} < \frac{1}{2}. \quad \cdots (5)$$

Then (4) becomes

$$\begin{aligned} \| u_{N_1} - u_{N_0} \|_{X^{s,b}([t_i,t_{i+1}])} &\le C_1 \| u_{N_1}(t_i) - u_{N_0}(t_i) \|_{H_x^s} \\ &\quad + C_2 N_0^{-\left(\frac{s'-s}{2}\right)} \quad \cdots (6). \end{aligned}$$

Our end goal is to get an estimate of the form

$$\| u_{N_1} - u_{N_0} \|_{X^{s,b}([0,T])} \lesssim_T N_0^{-\theta}, \quad \theta > 0.$$

We thus need to iterate (6).

Iteration procedure:

(1) On $[0,t_1)$ $\to$ Use difference in H^s_x at $t=t_0=0$.
$\Downarrow$ Get
$X^{s,b}([0,t_1))$ estimate
$\Downarrow$ Embedding into $C_tH^s_x$
Bound of diff. in H^s_x at $t=t_1$

$\Downarrow$ (Proceed)

(2) On $[t_1,t_2)$ $\to$ Diff at $t=t_1$ in H^s_x
$\Downarrow$ Get
$X^{s,b}([t_1,t_2))$ estimate
$\Downarrow$ Embedding $\qquad \omega\in\Omega\setminus\Omega(N_0,N_1)$
Bound in H^s_x at $t=t_2$

$\Downarrow$ (Proceed)

$\vdots$

$\Downarrow$

$(i+1)$ On $[t_i,t_{i+1})$ $\to$ H^s_x diff at $t=t_i$
$\Downarrow$ Get
$X^{s,b}([t_i,t_{i+1}))$
$\Downarrow$
H^s_x diff at $t=t_{i+1}$

$\Downarrow$

$\vdots$

Proceed $\lceil T/\eta\rceil$ times.

On $[t_0\overset{=0}{},t_1)$, $\|u_{N_1}(0)-u_{N_0}(0)\|_{H^s_x}=\|\phi^{(\omega)}_{N_1}-\phi^{(\omega)}_{N_0}\|_{H^s_x}\le N_0^{s-\frac{1}{2}}B_{N_0}$.

$$\overset{(6)}{\Rightarrow}\ \|u_{N_1}-u_{N_0}\|_{X^{s,b}([0,t_1))}\le C_1\cdot N_0^{s-\frac{1}{2}}B_{N_0}+C_2N_0^{-\left(\frac{s'-s}{2}\right)}.$$

$$\underset{(\text{Choice of } B_{N_0})}{\le} C_2 N_0^{-\frac{1}{2}(s'-s)}$$

Embedding $X^{s,b}\hookrightarrow C_tH^s_x$,

$$\Rightarrow \|u_{N_1}(t_1)-u_{N_0}(t_1)\|_{H^s_x}\le C_{\text{Embed}}C_2N_0^{-\frac{1}{2}(s'-s)}.$$

In $[t_1, t_2)$, we have (using the previous inequality) by (6),

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([t_1,t_2))} \le C_{Embed} C_2^2 N_0^{-\frac{1}{2}(s'-s)} + C_2 N_0^{-\frac{1}{2}(s'-s)}$$

$$\le C_{Emb} C_2^2 N_0^{-\frac{1}{2}(s'-s)}$$

$$\Rightarrow \|u_{N_1}(t_2) - u_{N_0}(t_2)\|_{H^s_x} \le C_{Embed}^2 C_2^2 N_0^{-\frac{1}{2}(s'-s)}$$

Iterate $\lceil T/\eta \rceil$ times

$$\Rightarrow \|u_{N_1} - u_{N_0}\|_{L^\infty_t([0,T];H^s_x)} \le C^{\lceil T/\eta \rceil} N_0^{-\frac{1}{2}(s'-s)}.$$

Then by our choice of B_{N_0} and hence η,

$$C^{\lceil T/\eta \rceil} N_0^{-\frac{1}{2}(s'-s)} = C^{T\sqrt{\log N_0}} N_0^{-\frac{1}{2}(s'-s)}$$

$$\sim e^{T\sqrt{\log N_0}} N_0^{-\frac{1}{2}(s'-s)}$$

$$\ll N_0^{\varepsilon T} N_0^{-\frac{1}{2}(s'-s)} \lesssim N_0^{-\frac{1}{4}(s'-s)}$$

provided we choose N_0 large enough so that

$$T\sqrt{\log N_0} \le \varepsilon T \log N_0 .$$

Thus, on any of the subintervals $[t_i, t_{i+1}) \subset [0,T)$ we have

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([t_i,t_{i+1}))} \lesssim N_0^{-\frac{1}{4}(s'-s)}.$$

Hence

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([0,T])} \le \left(\sum_{i=0}^{\lceil T/\eta \rceil} \|u_{N_1} - u_{N_0}\|^2_{X^{s,b}([t_i,t_{i+1}))} \right)^{1/2}$$

$$\lesssim \left\lceil \frac{T}{\eta} \right\rceil N_0^{-\frac{1}{4}(s'-s)}$$

$$\lesssim_T (\log N_0)^{1/2} N_0^{-\frac{1}{4}(s'-s)}$$

$$\lesssim_T N_0^{-\frac{1}{8}(s'-s)}.$$

which holds for all initial data $f^{(\omega)}$ with $\omega \in \Omega \setminus \Omega(N_0, N_1)$.

Now we show $\{u_{N_k}\}_k$ is a Cauchy sequence in $X^{s,b}([0,T])$.

Set

$$\Omega_0 := \limsup_{j \to \infty} \Omega(N_j, N_{j+1})$$

$$= \bigcap_{J \geq 1} \bigcup_{j \geq J} \Omega(N_j, N_{j+1}),$$

Then

$$\mu_F(\Omega_0^c) = \mu_F(\liminf_{j \to \infty} \Omega(N_j, N_{j+1}))$$

$$= \mu_F(\Omega \setminus \Omega(N_j, N_{j+1}) \quad \text{e.v.}),$$

and

$$\mu_F(\Omega_0) \leq \sum_{j \geq J} \mu_F(\Omega(N_j, N_{j+1}))$$

$$\leq \sum_{j \geq J} \exp(-B_{N_j}^c)$$

$$\leq \sum_{j \geq J} \exp(-c (\log N_j)^{c/8\alpha}).$$

Since $N_j = 2^j$ we can sum the above and take $J \to \infty$ to get

$$\mu_F(\Omega_0) = 0.$$

Then for every $\omega \in \Omega \setminus \Omega_0$, there exists J_0 s.t. for all $j \geq J_0$ we have the bounds in the definition of $\Omega(N_j, N_{j+1})$ holding true and, by the previous arguments,

$$\| u_{N_{j+1}} - u_{N_j} \|_{X^{s,b}([0,T])} \lesssim_T N_j^{-\frac{1}{8}(s'-s)}.$$

Then for M, N dyadic, $M \geq N \geq 2^{J_0}$,

$$\| u_M - u_N \|_{X^{s,b}([0,T])} \leq \sum_{j=n}^{m} \| u_{N_{j+1}} - u_{N_j} \|_{X^{s,b}([0,T])}$$

$(N = 2^n, M = 2^m)$

$$\lesssim_T \sum_{j=n}^{m} N_j^{-\frac{1}{8}(s'-s)}$$

$$\longrightarrow 0 \quad \text{as } n, m \to \infty \quad (N, M \to \infty).$$

Thus for any $\omega \in \Omega \setminus \Omega_0$, $\{u_{N_k}\}_k$ is a Cauchy sequence in $X^{s,b}([0,T])$. By completeness, we have convergence to a limit $u \in X^{s,b}([0,T])$ μ_F-a.s.

2) Convergence of full sequence

As we can no longer assume our indexing is lacunary, we cannot obtain convergence of the summation bounding $\mu_F(\Omega_0)$.

The modification here entails carefully bounding elements of the sequence which indexed closely.

For each $N_0 \gg 1$, consider the set

$$\Omega'(N_0) = \Big\{ \omega \in \Omega : \|P_{>N_0}\phi^{(\omega)}\|_{H^s_x} \gtrsim N_0^{s-1/2}, \ \|u_{N_0}\|_{L^p_x L^q_t} > B_{N_0},$$
$$\|(\sqrt{-\Delta})^{\sigma} u_{N_0}\|_{L^r_x L^p_t} > B_{N_0}, \ \max_{N_0 \le N < 2N_0} \|u_N - P_M u_N\|_{L^p_x L^q_t} > 1,$$
$$\max_{N_0 \le N < 2N_0} \|u_N - P_M u_N\|_{L^p_x L^q_t} > 1 \Big\},$$

By the tail estimates Prop^n 3.2 & Prop^n 3.3,

$$\mu_F(\Omega'(N_0)) < \exp(-B_{N_0}^c) + 2N_0 \exp(-(M^{\frac{2}{p}-\sigma})^c).$$

Note that $\Omega'(N_0)$ will replace the "Bad" set $\Omega(N_0, N_1)$ from the previous part.

Consider

$$M = (\log N_0)^{C(p)},$$

where $C(p) \gg 1$ is chosen sufficiently large so that

$$N_0 \exp(-(\log N_0)^{C(p)c(\frac{2}{p}-\sigma)}) \sim \exp(-B_{N_0}^c).$$

Then

$$\mu_F(\Omega'(N_0)) < 3\exp(-B_{N_0}^c).$$

Fix $N_0 \le N_1 < 2N_0$. Then, for any $\omega \in \Omega \setminus \Omega'(N_0)$,

$$\|u_{N_1}\|_{L^p_x L^q_t} \le \|u_{N_1} - P_M u_{N_1}\|_{L^p_x L^q_t} + \|P_M(u_{N_1} - u_{N_0})\|_{L^p_x L^q_t}$$
$$+ \|u_{N_0} - P_M u_{N_0}\|_{L^p_x L^q_t} + \|u_{N_0}\|_{L^p_x L^q_t}$$

Hölder) $n(x,t)$.)
$$\le 2 + T^{1/q} \|P_M(u_{N_1} - u_{N_0})\|_{L^\infty_{t,x}} + B_{N_0}$$
$$\le 2 + T^{1/q} M^{1-s} \|u_{N_1} - u_{N_0}\|_{X^{s,b}} + B_{N_0}$$
$$\le 2B_{N_0} + (\log N_0)^c \|u_{N_1} - u_{N_0}\|_{X^{s,b}},$$

(Bernstein $\|P_M f\|_{L^\infty_x} \le M^{2(\frac{1}{2}-\frac{1}{\infty})} \|P_M f\|_{L^2_x}$ $\le M^{1-s} \|P_M f\|_{H^s_x}$ + $(L^\infty_t H^s_x \supset X^{s,b})$).

and

$$\|(\sqrt{-\Delta})^\sigma u_{N_1}\|_{L^r_x L^q_t} \le \|(\sqrt{-\Delta})^\sigma (u_{N_1} - P_M u_{N_1})\|_{L^r_x L^q_t} + \|(\sqrt{-\Delta})^\sigma P_M(u_{N_1} - u_{N_0})\|_{L^r_x L^q_t}$$
$$+ \|(\sqrt{-\Delta})^\sigma (u_{N_0} - P_M u_{N_0})\|_{L^r_x L^q_t} + \|(\sqrt{-\Delta})^\sigma u_{N_0}\|_{L^r_x L^q_t}$$
$$\le 2 + T^{1/q} M^\sigma \|P_M(u_{N_1} - u_{N_0})\|_{L^\infty_{t,x}} + B_{N_0}$$
$$\le 2 + T^{1/q} M^{1+\sigma-s} \|P_M(u_{N_1} - u_{N_0})\|_{X^{s,b}} + B_{N_0}$$
$$\le 2B_{N_0} + (\log N_0)^c \|u_{N_1} - u_{N_0}\|_{X^{s,b}}.$$

Following the same estimates and methods as in the first part of the proof we obtain on each subinterval $[t_i, t_i+\eta)$, $(N_0 \le N_1 < 2N_0)$

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([t_i,t_i+\eta))} \lesssim \|u_{N_1}(t_i) - u_{N_0}(t_i)\|_{H^s_x} + \eta^{1/4} B_{N_0}^\alpha \|u_{N_1} - u_{N_0}\|_{X^{s,b}([t_i,t_i+\eta))}$$
$$+ (\log N_0)^\alpha \|u_{N_1} - u_{N_0}\|^{\alpha+1}_{X^{s,b}([t_i,t_i+\eta))}$$
$$+ N_0^{-(s^L - s)} B_{N_0}^{\alpha+1}.$$

As last time, with N_0 suff. large, $C\eta^{1/4} B_{N_0}^\alpha < 1/2$, and hence

$$\|u_{N_1}-u_{N_0}\|_{X^{s,b}([t_i,t_i+\eta])} \le C\|u_{N_1}(t_i)-u_{N_0}(t_i)\|_{H^s_x}$$

$$(**)\quad \left\{ \begin{aligned} &+C(\log N_0)^{C\alpha}\,\|u_{N_1}-u_{N_0}\|^{\alpha+1}_{X^{s,b}([t_i,t_i+\eta])} \\ &+C N_0^{-\frac{1}{2}(s_*-s)}. \end{aligned} \right.$$

To obtain an estimate on the difference $u_{N_1}-u_{N_0}$ in $X^{s,b}([t_i,t_i+\eta])$ in terms of a negative power of N_0, we need to modify the iteration argument by performing a bootstrap argument at each step.

Iteration Procedure:

(1) On $[0,t_1)$ $\to$ Difference in H^s_x at $t=t_0=0$.

$\Downarrow$ Bootstrap argument for (**)

$X^{s,b}([0,t_1))$ estimate

$\Downarrow$ Embedding into $C_t H^s_x$

Bound of diff. in H^s_x at $t=t_1$

$\Downarrow$

2) On $[t_1,t_2)$ $\to$ Diff in H^s_x at $t=t_1$

$\Downarrow$ Bootstrap for (**)

$X^{s,b}([t_1,t_2))$ estimate

$\Downarrow$ Embedding

Diff in H^s_x at $t=t_2$

$\Downarrow$

$\vdots$

Proceed $\lceil T/\eta \rceil$ times.

New steps

We will detail the bootstrap argument at the first step.

Denote $X(t) := \|u_{N_1} - u_{N_0}\|_{X^{s,b}([0,t])}$. Note $t \longmapsto X(t)$ is continuous. We have

$$X(0) \leq C N_0^{-\frac{1}{4}(s'-s)}$$

$$\|u_{N_1}(0) - u_{N_0}(0)\|_{H^s_x} \leq N_0^{s-1/2} \leq N_0^{-\frac{1}{2}(s'-s)}.$$

~~From these~~

By continuity, there exists $\delta > 0$ such that

$$X(\delta) \leq 4C N_0^{-\frac{1}{4}(s'-s)}. \quad \ldots (a)$$

Then (**) at $t = \delta$ implies

$$X(\delta) \leq C_0 N_0^{-\frac{1}{2}(s'-s)} + C_0 (\log N_0)^{c\alpha} X(\delta)^{\alpha+1} + C_0 N_0^{-\frac{1}{2}(s'-s)}$$

$$\leq C N_0^{-\frac{1}{2}(s'-s)} + C (\log N_0)^{c\alpha} X(\delta)^{\alpha+1}.$$

Now (a) implies

$$X(\delta) \leq C N_0^{-\frac{1}{2}(s'-s)} + C (\log N_0)^{c\alpha} 4^{\alpha+1} C^{\alpha+1} N_0^{-\frac{\alpha+1}{4}(s'-s)}$$

$$= 2C N_0^{-1/4(s'-s)} \underbrace{\left[\frac{1}{2} N_0^{-1/4(s'-s)} + 2C^{\alpha+1} (\log N_0)^{c\alpha} 4^{\alpha} N_0^{-\frac{\alpha}{4}(s'-s)} \right]}_{<1 \text{ for } N_0 \text{ large enough}}$$

$$\leq 2C N_0^{-1/4(s'-s)}.$$

By a process of continuation, we conclude that

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([0,\eta])} \leq 2C N_0^{-1/4(s'-s)}.$$

Hence

$$\|u_{N_1}(\eta) - u_{N_0}(\eta)\|_{H^s_x} \leq 2C C_{Embed} N_0^{-\frac{1}{4}(s'-s)}.$$

Proceeding in this way, we have

$$\|u_{N_1}(T) - u_{N_0}(T)\|_{H^s_x} \leq 2(C C_{Embed})^{\lceil T/\eta \rceil} N_0^{-1/4(s'-s)}$$

$$\leq N_0^{-\frac{1}{8}(s'-s)}.$$

by our choice of $\gamma = \gamma(N_0)$. Since $C^{[T/\gamma]}$ is the largest possible constant here, ~~by choos~~ this step guarantees that N_0 can be chosen independent of the partition of $[0,T]$, i.e. independent of any given interval $[t_i, t_{i+1}]$.

Thus,

$$\|u_{N_1} - u_{N_0}\|_{X^{s,b}([0,T])} \lesssim N_0^{-\frac{1}{16}(s'-s)} =: N_0^{-\gamma}.$$

Set

$$\Omega_1 = \limsup_{k\to\infty} \Omega'(2^k N_0). \qquad (N_0 \gg 1).$$

Then for any $N_1 \geq N_0$, and $\omega \in \Omega \setminus \Omega_1$,

$$\|u_{N_0} - u_{N_1}\|_{X^{s,b}([0,T])} \leq \|u_{N_0} - u_{2N_0}\|_{X^{s,b}} + \|u_{2N_0} - u_{4N_0}\|_{X^{s,b}}$$

(K smallest integer s.t. $2^k N_0 \leq N_1$)

$$+ \dots + \|u_{2^k N_0} - u_{N_1}\|_{X^{s,b}}$$

$$\leq N_0^{-\gamma} + (2N_0)^{-\gamma} + \dots + (2^k N_0)^{-\gamma}$$

$$= N_0^{-\gamma} \sum_k 2^{-k\gamma}$$

$$\lesssim N_0^{-\gamma} \longrightarrow 0 \quad \text{as } N_0 \to \infty.$$

$\Rightarrow \{u_N\}_N$ is a Cauchy sequence in $X^{s,b}([0,T])$,

and $\mu_F(\Omega_1) = 0$ by the same argument as in the first part.

$\Rightarrow \{u_N\}_N$ converges, in $X^{s,b}([0,T])$, to a limit $u \in X^{s,b}([0,T])$ μ_F - a.s.