ON THE QUOTIENT RING BY DIAGONAL INVARIANTS

IAIN GORDON

Abstract. For a finite Coxeter group, W, and its reflection representation $\mathfrak{h}$, we find the character and Hilbert series for a quotient ring of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ by an ideal containing the W–invariant polynomials without constant term. This confirms conjectures of Haiman.

1. Introduction

1.1. Let W be a finite Coxeter group, $\mathfrak{h}$ its reflection representation, and $\mathbb{C}[\mathfrak{h}]$ the ring of polynomial functions on $\mathfrak{h}$. The action of W on $\mathfrak{h}$ extends to an action by algebra automorphisms on $\mathbb{C}[\mathfrak{h}]$ by $w \cdot f(x) = f(w^{-1} \cdot x)$. It is a classical fact that the ring of invariants

$$\mathbb{C}[\mathfrak{h}]^W = \{f \in \mathbb{C}[\mathfrak{h}] : w \cdot f = f \text{ for all } w \in W\}$$

is a polynomial ring in $\dim(\mathfrak{h})$ variables, [16, Chapter 3]. Furthermore, the coinvariant ring

$$\mathbb{C}[\mathfrak{h}]^{coW} = \frac{\mathbb{C}[\mathfrak{h}]}{\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle}$$

is a finite dimensional vector space, isomorphic as a W–module to the regular representation $\mathbb{C}W$ of W, [16, Chapter 3]. (Here, $\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle$ denotes the ideal of $\mathbb{C}[\mathfrak{h}]$ generated by invariant polynomials without constant term.) These important results are fundamental in algebraic geometry, the representation theory of finite simple groups of Lie type and the theory of Lie algebras.

1.2. Recently, attention has focused on a "double" analogue of the above results. The space $\mathfrak{h}$ is replaced by $\mathfrak{h} \oplus \mathfrak{h}^*$, and its corresponding diagonal W–action. The orbit space $\mathfrak{h} \oplus \mathfrak{h}^*/W$ is a particularly interesting example of a symplectic singularity, of current interest in algebraic geometry, [22]. Moreover, the process of "doubling up" is crucial to the study of Nakajima quiver varieties and preprojective algebras, the Hilbert scheme of points in the plane, [19], and in Lie theoretic work on principal nilpotent pairs, [12].

1.3. The ring of invariants $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$ is never smooth and for general W, its generators and relations are poorly understood, [23]. It is, however, expected that the ring of coinvariants

$$\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{coW} = \frac{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]}{\langle \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W_+\rangle}$$

should display interesting combinatorial properties, relating to its W–action and its natural grading arising from setting $\deg(x) = 1$ and $\deg(y) = -1$ for $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$, [14].

In the case $W = \mathfrak{S}_n$, the work of Haiman on the $n!$–conjecture shows that the Hilbert series of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{coW}$ is $t^{-n(n-1)/2}(1+t+\cdots+t^n)^{n-1}$, and determines its decomposition as an $\mathfrak{S}_n$–module, [15]. This confirms several conjectures in [14].

1.4. In this paper we extend the above results to all finite Coxeter groups. Our main theorem is the following, confirming a conjecture of Haiman [14, Section 7].

Theorem. *Let W be a finite Coxeter group. Let n be the rank of W and h the Coxeter number. Let $D_W = \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{coW}$. Then there exists a W–stable quotient ring R_W of D_W satisfying the following properties:*

1. $\dim R_W = (h+1)^n$;
2. *R_W is $\mathbb{Z}$–graded with Hilbert series $t^{-hn/2}(1+t+\cdots+t^h)^n$;*
3. *The image of $\mathbb{C}[\mathfrak{h}]$ in R_W is the classical coinvariant algebra, $\mathbb{C}[\mathfrak{h}]^{coW}$;*
4. *If W is a Weyl group then as a W–module $R_W\otimes\epsilon$ is isomorphic to the permutation representation of W on the reduction of the root lattice modulo $h+1$, written $Q/(h+1)Q$.*

It is known that for W not of type A or I, R_W is usually a *proper* quotient of D_W. For dihedral groups the theorem is known by calculations of Alfano and Reiner.

1.5. Let us make some comments on the method of proof, which is rather indirect. Our strategy is to find a non–commutative deformation of R_W which has all the properties discussed in the theorem. This non–commutative deformation is a simple module of a rational Cherednik algebra associated to W. The rational Cherednik algebras, introduced by Etingof and Ginzburg [11], are members of a certain flat family of deformations of the skew group algebra $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*] * W$. The parameter set for this family is the W–invariant functions $c : \mathcal{S} \longrightarrow \mathbb{C}$, where $\mathcal{S}$ denotes the set of reflections attached to W. We denote the corresponding algebra by H_c. It is known that $\mathbb{C}W \subset H_c$ for all c.

Of crucial importance is the existence of a faithful "Dunkl representation" for these algebras, allowing us to relate the representation theory of a rational Cherednik algebra

H_c through monodromy to the representation theory of the Hecke algebra associated to W at parameter $q = exp(2\pi ic)$.

Using known results on the representation theory of Hecke algebra at roots of unity, we can describe quite well a simple module, L, for H_c at a particular value of c. In particular, it is possible to understand the decomposition of L as a W–module, and describe a certain $\mathbb{Z}$–grading on it.

Let $e_\epsilon \in \mathbb{C}W$ be the sign idempotent. Define the *spherical algebra* to be $e_\epsilon H_c e_\epsilon$ for any c and note that $H_c e_\epsilon$ is a $(H_c, e_\epsilon H_c e_\epsilon)$–bimodule. The spherical algebra has a one–dimensional module $\mathbb{C}$. It turns out that L has an alternative description as the induced module $H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} \mathbb{C}$. Passing to the associated graded module we find a surjective homomorphism from $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]e_\epsilon \otimes_{\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W e_\epsilon} \mathbb{C}$ to grL. For Theorem 1.4 we set $R_W = grL$.

1.6. It seems likely that the proof can be adapted to deal with a larger class of complex reflection groups. In particular for the wreath products $\mathbb{Z}_m \wr \mathfrak{S}_n$, that is complex reflection groups of type $G(m,1,n)$, results of Weyl on the generation of $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$ by generalised polarisations should allow one to prove an analogue of the important Proposition 4.8. Using the work of Dunkl and Opdam, [10, Section 3.5] on shifts of Dunkl operators, and the work of Ariki, [1], and Broué, Malle and Rouquier, [5], on cyclotomic Hecke algebras and their representations, it should be possible to extend Theorem 1.4 to this larger class of groups.

1.7. The paper is organised as follows. In Section 2 we recall basic definitions from rational Cherednik algebras. Section 3 discusses Hecke algebras and their relation to rational Cherednik algebras. In Section 4 we find the finite dimensional simple H_c–module described above. Finally, in Section 5, we relate the earlier sections to the coinvariants of W on $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$, proving Theorem 1.4.

1.8. In recent work, [3], Berest, Etingof and Ginzburg have found a character formula for *all* finite dimensional simple H_c–modules in type A, and for many in other types. Their techniques, which were discovered independently, are similar to those used here and point towards links with Hilbert schemes on the plane and other "doubled" geometric objects. We thank Ginzburg for bringing this work to our attention. Moreover, we are indebted to Etingof for showing how to extend our earlier version of Proposition 4.8 and of Theorem 1.4(4) from classical Weyl groups to the exceptional types. Without this, our results would be less complete.

2. Rational Cherednik algebras

2.1. The following is due to Etingof and Ginzburg, [11]. Let W be a finite Coxeter group, and $\mathfrak{h}$ its reflection representation over $\mathbb{C}$, which is the complexification of a real Euclidean vector space. Let $\mathcal{S} \subset W$ be the set of reflections in W. To each W–invariant function $c : \mathcal{S} \longrightarrow \mathbb{C}$ one can attach an associative algebra H_c, called the *rational Cherednik algebra*. Given $s \in \mathcal{S}$ let $\alpha_s \in \mathfrak{h}^*$ be a linear functional vanishing on the reflecting hyperplane of s, and write $\alpha_s^\vee = 2(\alpha_s, -)/(\alpha_s, \alpha_s) \in \mathfrak{h}$ for the corresponding coroot. The rational Cherednik algebra H_c is generated by the vector spaces $\mathfrak{h}, \mathfrak{h}^*$, and the group W, with defining relations given by

$$w \cdot x \cdot w^{-1} = w(x), \quad w \cdot y \cdot w^{-1} = w(y), \qquad \text{for all } y \in \mathfrak{h}, x \in \mathfrak{h}^*, w \in W$$

$$x_1 \cdot x_2 = x_2 \cdot x_1, \quad y_1 \cdot y_2 = y_2 \cdot y_1, \qquad \text{for all } y_1, y_2 \in \mathfrak{h}, x_1, x_2 \in \mathfrak{h}^*$$

$$y \cdot x - x \cdot y = \langle y, x \rangle - \sum_{s \in \mathcal{S}} c_s \cdot \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle \cdot s, \qquad \text{for all } y \in \mathfrak{h}, x \in \mathfrak{h}^*.$$

2.2. The elements $x \in \mathfrak{h}^*$ generate a subalgebra $\mathbb{C}[\mathfrak{h}] \subset H_c$ of polynomial functions on $\mathfrak{h}$, the elements $y \in \mathfrak{h}$ generate a subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset H_c$, and the elements $w \in W$ span a copy of the group algebra $\mathbb{C}W$ sitting naturally inside H_c. By [11, Theorem 1.3] there is a Poincaré–Birkhoff–Witt isomorphism

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\sim} H_c. \tag{1}$$

2.3. Let $\{x_i\}$ and $\{y_i\}$ be a pair of dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Viewing $\mathfrak{h}$ and $\mathfrak{h}^*$ as subspaces of H_c, we let $\mathbf{h} = \frac{1}{2}\sum_i (x_i y_i + y_i x_i) \in H_c$, which is independent of the choice of bases. Let $(\, , \,)$ be the complex bilinear form on $\mathfrak{h}$ extending the Euclidean inner product. Define $\mathbf{x}^2 \in \mathbb{C}[\mathfrak{h}]$ to be the squared norm function $x \mapsto (x, x)$ and let $\mathbf{y}^2 \in \mathbb{C}[\mathfrak{h}^*]$ be defined similarly. By [2, (2.6)], $\{\mathbf{x}^2, \mathbf{h}, \mathbf{y}^2\}$ forms an $\mathfrak{sl}_2$–triple in the algebra H_c.

2.4. **A filtration.** There exists a filtration on H_c with $\deg(x) = \deg(y) = 1$ and $\deg(w) = 0$ for $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$. By the Poincaré-Birkhoff-Witt isomorphism, the associated graded ring of H_c with respect to this filtration, $gr H_c$, equals $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$. Using the non–commutativity of H_c, a standard procedure, [11, Section 2], produces a Poisson bracket on $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. This agrees with the Poisson bracket induced by the canonical W–invariant symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$, [11, Lemma 2.23].

2.5. **Category $\mathcal{O}_c$.** We recall the definition of a special subcategory of H_c–mod from [2, Definition 2.4]. Let $\mathcal{O}_c$ be the abelian category of finitely–generated H_c–modules

M, such that the action on M of the subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset H_c$ is locally finite, that is $\dim \mathbb{C}[\mathfrak{h}^*] \cdot m < \infty$ for any $m \in M$, and such that for any $P \in \mathbb{C}[\mathfrak{h}^*]^W$ the action of $P - P(0)$ is locally nilpotent. Let $\mathsf{Irrep}(W)$ denote the set of simple W–modules, up to isomorphism. Given $\tau \in \mathsf{Irrep}(W)$, we define an object of $\mathcal{O}_c$, called the *standard* module $M_c(\tau)$, to be the induced module

$$M_c(\tau) = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*] * W} \tau,$$

where $\mathbb{C}[\mathfrak{h}^*] * W$ acts on τ by sending $pw \cdot m = p(0)(w \cdot m)$, for $p \in \mathbb{C}[\mathfrak{h}^*]$, $w \in W$ and $m \in \tau$. It is shown in [2, Section 2] that each $M_c(\tau)$ has a unique simple quotient $L_c(\tau)$, that the set $\{L_c(\tau) : \tau \in \mathsf{Irrep}(W)\}$ provides a complete list of non–isomorphic simple objects in $\mathcal{O}_c$, and that every object in $\mathcal{O}_c$ has finite length. Note that it follows from the Poincaré–Birkhoff–Witt theorem that the standard module $M_c(\tau)$ is a free $\mathbb{C}[\mathfrak{h}]$-module of rank $\dim(\tau)$.

3. Hecke Algebras

3.1. Let B_W be the braid group of W, in other words the fundamental group of the variety $\mathfrak{h}^{\text{reg}}/W$. Fix a point $* \in \mathfrak{h}^{\text{reg}}$ and for each simple reflection $s \in W$, let T_s be the class in B_W corresponding to a straight path from the point $*$ to the point $s(*)$ with an inserted anti–clockwise semi–circle around the hyperplane defining $\alpha_s = 0$. Given $\mathbf{q} : \mathcal{S} \longrightarrow \mathbb{C}^*$ $(s \mapsto q_s)$, a W–invariant function, we define $\mathcal{H}_W(\mathbf{q})$ to be the quotient of the group algebra $\mathbb{C}[B_W]$ by the ideal generated by $(T_s + 1)(T_s - q_s)$. The algebra $\mathcal{H}_W(\mathbf{q})$ is the *Hecke algebra* of W.

3.2. It is well–known that $\dim \mathcal{H}_W(\mathbf{q}) = |W|$. When $\mathbf{q} = \mathbf{1}$, $\mathcal{H}_W(\mathbf{q}) = \mathbb{C}W$. For a generic choice of $\mathbf{q}$ the algebra $\mathcal{H}_W(\mathbf{q})$ is semisimple and isomorphic to $\mathbb{C}W$. For any choice of $\mathbf{q}$, there exist a set of $\mathcal{H}_W(\mathbf{q})$-modules labelled by the simple W–modules, [17]. These are called *Specht modules* and written $S_{\mathbf{q}}(\lambda)$ for $\lambda \in \mathsf{Irrep}(W)$. These are simple–headed and give a complete set of non–isomorphic irreducible $\mathcal{H}_W(\mathbf{q})$–modules in the semisimple case. If $\mathcal{H}_W(\mathbf{q})$ is not semisimple the composition factors of $S_{\mathbf{q}}(\lambda)$ are the entries of the decomposition matrix for $\mathcal{H}_W(\mathbf{q})$.

3.3. We need to recall a few results about Dunkl operators, which link rational Cherednik algebras to Hecke algebras, see [2, Section 2]. According to [11, Proposition 4.5], the algebra H_c has a faithful "Dunkl representation", an injective algebra homomorphism

$H_c \longrightarrow D(\mathfrak{h}^{\mathrm{reg}}) * W$. This allows us to consider $M|_{\mathfrak{h}^{\mathrm{reg}}}$, the localisation of an H_c–module M, as a W–equivariant D–module on $\mathfrak{h}^{\mathrm{reg}}$.

3.4. In particular, the standard module $M_c(\tau)|_{\mathfrak{h}^{\mathrm{reg}}}$, viewed as a D-module on $\mathfrak{h}^{\mathrm{reg}}$, is the trivial vector bundle $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \otimes \tau$ equipped with a flat connection. It is well–known that the connection is the Knizhnik-Zamolodchikov connection with values in τ. Moreover, the monodromy representation of the fundamental group $B_W = \pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$ corresponding to this connection factors through the Hecke algebra $\mathcal{H}_W(exp(2\pi i c))$.

The germs of the horizontal holomorphic sections of $M_c(\tau)|_{\mathfrak{h}^{\mathrm{reg}}}$ form a locally constant sheaf on $\mathfrak{h}^{\mathrm{reg}}/W$. Let $Mon_c(\tau)$ be the corresponding monodromy representation of the fundamental group $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$ in the fibre over $*$, where $*$ is some fixed point in $\mathfrak{h}^{\mathrm{reg}}/W$. The following Lemma was proved by Opdam–Rouquier and presented in [2, Lemma 2.10].

Lemma. *Let $\tau, \mu \in \mathsf{Irrep}(W)$. The natural map*

$$Hom_{H_c}(M_c(\tau), M_c(\mu)) \longrightarrow Hom_{B_W}(Mon_c(\tau), Mon_c(\mu))$$

is injective.

4. A simple module for rational Cherednik algebras

4.1. Let ϵ be the sign representation of W and $e_\epsilon = |W|^{-1}\sum_w \epsilon(w)w$ the sign idempotent. Let $e = |W|^{-1}\sum_w w$, the trivial idempotent. Let h be the Coxeter number of W. Given $0 \leq k \leq n$, set $\mathfrak{h}_k = \wedge^k \mathfrak{h}$, the k–th exterior power of $\mathfrak{h}$. In particular $\mathfrak{h}_1 = \mathfrak{h}$ and $\mathfrak{h}_0 = \mathsf{triv}$. Thanks to [6, Theorem 9.13] $\mathfrak{h}_k \in \mathsf{Irrep}(W)$. Let $n = \dim \mathfrak{h}$. Recall $\mathcal{S}$ is the set of reflections in W. Set $N = |\mathcal{S}|$.

4.2. Each module M in category $\mathcal{O}_c$ splits into a direct sum of generalised $\mathbf{h}$–eigenspaces, [13, Section 2]. Moreover, the action of $\mathbf{h}$ on the standard and simple modules in $\mathcal{O}_c$ is diagonalisable. To describe the $\mathbf{h}$–eigenspaces of $M_c(\tau)$ we introduce some notation. Let

$$\kappa_c = \sum_{s \in \mathcal{S}} c_s(1-s)$$

be the central element of $\mathbb{C}W$ canonically attached to $c \in \mathbb{C}[R]^W$. This element acts by a scalar, say $\kappa_c(\tau)$, on each irreducible representation $\tau \in \mathsf{Irrep}(W)$. Thanks to [13, Section 2], for $p \in \mathbb{C}[\mathfrak{h}]_m$ and $v \in \tau$ we have in $M_c(\tau)$

$$\mathbf{h}(p \otimes v) = (m + \frac{n}{2} + \kappa_c(\tau) - \sum_{s \in \mathcal{S}} c_s) p \otimes v. \tag{2}$$

Lemma. *Keep the above notation. If c is constant, then $\kappa_c(\mathfrak{h}_k) = hck$.*

Proof. Recall $N = |R_+|$ and $\mathfrak{h}_k = \wedge^k \mathfrak{h}$. Since κ_c is central, for all $\tau \in \mathsf{Irrep}(W)$ we have

$$\kappa_c(\tau) = \frac{\chi_\tau(\kappa_c)}{\dim \tau} = Nc - \frac{c \sum \chi_\tau(s)}{\dim \tau}, \tag{3}$$

where χ_τ is the character of τ.

For any $s \in \mathcal{S}$, the reflection s fixes a hyperplane pointwise and acts as -1 on its orthogonal complement, so that $\mathfrak{h} = F \oplus S$, where $\dim F = n-1$ and $\dim S = 1$. Thus, the action of s on $\mathfrak{h}_k$ is described by $\mathfrak{h}_k = \wedge^k F \oplus \wedge^{k-1} F \otimes S$. We deduce that

$$\chi_{\mathfrak{h}_k}(s) = \binom{n-1}{k} - \binom{n-1}{k-1}.$$

Combining this with (3) yields

$$\kappa_c(\mathfrak{h}_k) = Nc - \frac{Nc\left(\binom{n-1}{k} - \binom{n-1}{k-1}\right)}{\binom{n}{k}} = Nc - \frac{Nc(n-2k)}{n} = \frac{2Nck}{n}.$$

Since the Coxeter number of W equals $2N/n$, [16, Chapter 3], the lemma follows. ☐

4.3. In the following lemma, we make use of the relationship between rational Cherednik algebras and Hecke algebras.

Lemma. *Suppose $c = (1+mh)/h$ for m a positive integer. Let $\tau, \mu \in \mathsf{Irrep}(W)$ be non–isomorphic and suppose that there is a non–zero H_c–homomorphism $M_c(\tau) \longrightarrow M_c(\mu)$. Then $\tau \cong \mathfrak{h}_i$ and $\mu \cong \mathfrak{h}_j$ for some non–negative integers i and j.*

Proof. By [9, Section 6] the composition multiplicities of the $\mathcal{H}_W(q)$–modules $Mon_c(\mu)$ are described by the decomposition matrix for $\mathcal{H}_W(q)$. The decomposition matrix for $\mathbf{q} = \exp(2\pi i c)$ is described in [4, Section 6.4] for Weyl groups and in [18] for the non-crystallographic groups. In particular, [4, Lemma 6.5] and [18] shows that the Specht modules associated to λ, for $\lambda \neq \mathfrak{h}_i$ $(0 \le i \le n)$, are irreducible and projective. Hence, to each such λ, there corresponds μ such that $Mon_c(\mu)$ is irreducible and projective. By [8, Corollary 3.5] we must have that $\mu = \lambda$. Thus we have shown that $Mon_c(\lambda)$ is an irreducible and projective $\mathcal{H}_W(q)$–module for $\lambda \neq \mathfrak{h}_i$, $0 \le i \le n$. Furthermore, by [4, Theorem 6.6] and [18] there is a unique non–trivial block for $\mathcal{H}_W(q)$ containing the Specht modules associated to $\mathfrak{h}_i$, so we deduce that $Mon_c(\mathfrak{h}_i)$ is not isomorphic to $Mon_c(\lambda)$.

We have $\dim \mathrm{Hom}_{H_c}(M_c(\tau), M_c(\mu)) \le \mathrm{Hom}_{\mathcal{H}_W(q)}(Mon_c(\tau), Mon_c(\mu))$ by Lemma 3.4. The result follows from the above paragraph. ☐

4.4. Let $e_1, \ldots, e_n$ be the exponents of W and $d_i = e_i + 1$, the degrees of the fundamental invariants for W, [16, Chapter 3]. Let

$$p = \prod_{i=1}^{n} (1 - t^{d_i})^{-1},$$

the Hilbert series of the invariant ring $\mathbb{C}[\mathfrak{h}]^W$.

Lemma. *Suppose $c = (1 + mh)/h$ for m a positive integer. The Hilbert series of $e_\epsilon M_c(\mathfrak{h}_{n-i})$ with respect to the $\mathbf{h}$–eigenspaces is*

$$p(e_\epsilon M_c(\mathfrak{h}_{n-i}), t) = t^{-mN+(n-i)(mh+1)} p \sum_{1 \le j_1 < \cdots < j_i \le n} t^{e_{j_1} + \cdots + e_{j_i}}.$$

Proof. As a $\mathbb{C}[\mathfrak{h}] * W$–module we have $M_c(\mathfrak{h}_{n-i}) \cong \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{h}_{n-i}$. Hence, the sign module appears in $M_c(\mathfrak{h}_{n-i})$ with multiplicity one for each appearance of $(\mathfrak{h}_{n-i})^* \otimes \epsilon$ in $\mathbb{C}[\mathfrak{h}]$. Since $\mathfrak{h}_n = \epsilon$, there is a homomorphism $\mathfrak{h}_{n-i} \wedge \mathfrak{h}_i \longrightarrow \epsilon$, and hence an isomorphism $\mathfrak{h}_i \cong (\mathfrak{h}_{n-i})^* \otimes \epsilon$.

To determine the appearances of $\mathfrak{h}_i$ in $\mathbb{C}[\mathfrak{h}]$, we study

$$\mathbb{C}[\mathfrak{h}]^{coW} = \frac{\mathbb{C}[\mathfrak{h}]}{\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle}.$$

By [20] there is a copy of $\mathfrak{h}$ in degree e_i for $1 \le i \le n$. Since $\mathbb{C}[\mathfrak{h}]^{coW}$ is isomorphic to the regular representation of W, we know that $\mathfrak{h}_i$ appears $\binom{n}{i}$ times. Taking i distinct copies of $\mathfrak{h}$ in degrees $e_{j_1}, \ldots, e_{j_i}$ we obtain a copy of $\mathfrak{h}_i$ in degree $e_{j_1} + \cdots + e_{j_i}$ by taking their wedge product. Running through all such products yields $\binom{n}{i}$ copies of $\mathfrak{h}_i$. Since the wedge product of the n distinct copies of $\mathfrak{h}$ above yields the unique copy of ϵ in $\mathbb{C}[\mathfrak{h}]^{coW}$, all the copies of $\mathfrak{h}_i$ described above are distinct. Hence the Hilbert series of $\mathfrak{h}_i$ in $\mathbb{C}[\mathfrak{h}]^{coW}$ is given by $\sum_{1 \le j_1 < \cdots < j_i \le n} t^{e_{j_1} + \cdots + e_{j_i}}$.

It follows from the W–invariant graded decomposition

$$\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}[\mathfrak{h}]^W$$

that the Hilbert series for $\mathfrak{h}_i$ in $\mathbb{C}[\mathfrak{h}]$ is given by $p \sum_{1 \le j_1 < \cdots < j_i \le n} t^{e_{j_1} + \cdots + e_{j_i}}$.

The lowest $\mathbf{h}$–eigenspace in $M_c(\mathfrak{h}_{n-i})$ can be calculated from formula (2) on $1 \otimes v$. Since $c = (1+mh)/h$ and $nh/2 = |R_+|$, Lemma 4.2 shows that this equals $-mN + (n-i)(1+mh)$. The lemma follows. □

4.5. The following lemma is crucial.

Lemma. *Suppose $c = (1 + mh)/h$ for m a positive integer. Then $e_\epsilon L_c(\mathfrak{h}_i) \neq 0$.*

Proof. If $L_c(\mathfrak{h}_i) = M_c(\mathfrak{h}_i)$ the claim is clear, so suppose $M_c(\mathfrak{h}_i)$ is not simple. Let $R_c(\mathfrak{h}_i)$ be its radical (unique maximal submodule). Then $M_c(\mathfrak{h}_i)/R_c(\mathfrak{h}_i) = L_c(\mathfrak{h}_i)$. Moreover, the lowest $\mathbf{h}$–eigenspace of $R_c(\mathfrak{h}_i)$ yields a homomorphism $M_c(\tau) \longrightarrow R_c(\mathfrak{h}_i)$ for some $\tau \in \mathsf{Irrep}(W)$. By Lemma 4.3 we know that τ must have the form $\mathfrak{h}_j$ for some j. By [13, Section 3] we have $\kappa_c(\mathfrak{h}_i) < \kappa_c(\mathfrak{h}_j)$ which by Lemma 4.2 shows that $j > i$. By Lemma 4.4, the difference in the lowest $\mathbf{h}$–eigenspaces of $M_c(\mathfrak{h}_i)$ and $M_c(\mathfrak{h}_j)$ which contain a copy of ϵ is at least $mh+1-e_i$ for some i. As all exponents e_i are less than the Coxeter number, $M_c(\mathfrak{h}_j)$ cannot map onto the lowest $\mathbf{h}$–eigenspace of $M_c(\mathfrak{h}_i)$ containing a copy of ϵ. Thus means that $R_c(\mathfrak{h}_i)$ cannot contain all copies of ϵ. □

4.6. In this section we consider only the value $c = 1/h$.

Lemma. *Let $c = 1/h$. Then H_c has a one dimensional module, whose restriction to W is the trivial module.*

Proof. For type A this follows from [2, Remark following Proposition 5.7]. We will prove this for type B, the other cases being similar.

Let $\{x_1, \dots, x_n\}$ be the standard basis for $\mathfrak{h}^*$ and $\{y_1, \dots, y_n\}$ for $\mathfrak{h}$. Recall, [16, Chapter 2], that $W = \mathbb{Z}_2 \wr \mathfrak{S}_n$ and that the positive roots of W can be chosen to be the elements of $\mathfrak{h}$ of the form $x_i \pm x_j$ for $1 \le i < j \le n$ (short roots) and $2x_i$ for $1 \le i \le n$ (long roots).

Let c_s and c_l be the values of $c : \mathcal{S} \longrightarrow \mathbb{C}$ on the reflections corresponding to short and long roots respectively. Calculation gives the following formula:

$$[y_i, x_j] = \delta_{ij} + \begin{cases} -c_s \sum_{t \neq i} (s_{x_{i+t}} + s_{x_{i-t}}) - 2c_l s_{x_i} & \text{if } i = j, \\ -c_s (s_{x_{i+j}} - s_{x_{i-j}}) & \text{if } i \neq j. \end{cases}$$

The trivial representation sends the group elements to 1, so setting $c_s = c_l = \frac{1}{2n} = 1/h$ proves that $[y_i, x_j] \mapsto 0$. The lemma follows. □

Remark. *For $c = 1/h$ it can be shown using a Koszul resolution that we have*

$$[M_c(\mathfrak{h}_i) : L_c(\mathfrak{h}_j)] = \begin{cases} 1 & \text{if } j = i, i+1 \\ 0 & \text{otherwise,} \end{cases}$$

and $M_c(\tau) = L_c(\tau)$ if $\tau \neq \mathfrak{h}_i$ for all i. This is greatly generalised in [3].

4.7. Consider the functor $F : H_c - mod \longrightarrow e_\epsilon H_c e_\epsilon - mod$ which sends M to $e_\epsilon M$. Since e_ϵ is an idempotent, this functor is exact. All objects in category $\mathcal{O}_c$ have finite length, hence Lemma 4.5 shows that if $c = (1+mh)/h$ then $M_c(\tau)$ is simple for $\tau \neq \mathfrak{h}_i$ and $[M_c(\mathfrak{h}_i) : L_c(\mathfrak{h}_j)] = [e_\epsilon M_c(\mathfrak{h}_i) : e_\epsilon L_c(\mathfrak{h}_j)]$. Let D be the $n+1 \times n+1$ matrix whose i,j–th entry is given by $[M_c(\mathfrak{h}_i) : L_c(\mathfrak{h}_j)]$. By Lemma 4.2 and [13, Section 3, Proof of Theorem 8] this is an upper triangular matrix. This means that there is a unique way to write $e_\epsilon L_c(\mathfrak{h}_j)$ in terms of $e_\epsilon M_c(\mathfrak{h}_i)$.

Lemma. *Suppose that $c = (1+h)/h$. We have $\sum_{i=0}^n (-1)^i p(e_\epsilon(M_c(\mathfrak{h}_i), t) = 1$.*

Proof. By Lemma 4.4 we have

$$\begin{aligned}\sum_{i=0}^n (-1)^i p(e_\epsilon(M_c(\mathfrak{h}_i)) &= p\sum_{i=0}^n (-1)^i t^{-N+i(h+1)} \sum_{1\leq j_1<\cdots<j_{n-i}\leq n} t^{e_{j_1}+\cdots+e_{j_{n-i}}} \\ &= pt^{-N}\prod_{k=1}^n (t^{e_k} - t^{h+1}) \\ &= pt^{-N}\prod_{k=1}^n t^{e_k}(1-t^{h+1-e_k}).\end{aligned}$$

By [16, Chapter 3] the sum $\sum_{k=1}^n e_k = N$, whilst the multiset $\{h+1-e_k : 1\leq k\leq n\}$ equals $\{d_k : 1\leq k\leq n\}$. We deduce that

$$\sum_{i=0}^n (-1)^n p(e_\epsilon(M_c(\mathfrak{h}_i)) = p\prod_{k=1}^n (1-t^{d_k}) = 1,$$

as required. □

4.8. **Shifting.** We now need an analogue of a theorem of Berest, Etingof and Ginzburg for non–regular values of c. Let $\mathbf{1} : \mathcal{S} \longrightarrow \mathbb{C}$ be the map taking the value 1 everywhere. To indicate provenance, we give two versions of the following Proposition. The second is due to Berest, Etingof and Ginzburg and will appear in [3]. We thank the authors for allowing us to reproduce the result here.

Proposition (Version 1)**.** *Let W be a finite Coxeter group of type A, B, D or I. Then for any function c there is an algebra isomorphism $e_\epsilon H_c e_\epsilon \cong eH_{c-\mathbf{1}}e$.*

Proof. The proof follows the same lines as [2, Proposition 4.11] with one exception. We need to know for *all* values of c that eH_ce is generated by its positive part $\mathbb{C}[\mathfrak{h}]^We$ and the element $\mathbf{y}^2$. This follows from [23, Appendix 2] which shows that diagonal invariants of W on $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ are generated by $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W$ and their Poisson brackets. Since $gr(eH_ce) = eS(\mathfrak{h}\oplus\mathfrak{h}^*)e \cong S(\mathfrak{h}\oplus\mathfrak{h}^*)^W$ and the Poisson bracket is induced by the commutator in eH_ce, Section 2.4, it follows that eH_ce is generated by $\mathbb{C}[\mathfrak{h}]^We$ and $\mathbb{C}[\mathfrak{h}^*]^We$. Finally, [2, Corollary 4.9] shows that eH_ce is generated by $\mathbb{C}[\mathfrak{h}^*]^We$ and $\mathbf{y}^2e$. □

Proposition (Version 2, [3])**.** *For any function c there is an algebra isomorphism $e_\epsilon H_c e_\epsilon \cong eH_{c-\mathbf{1}}e$.*

Proof. Let $\theta_c : eH_{c-\mathbf{1}}e \longrightarrow \mathrm{End}(\mathbb{C}[\mathfrak{h}])$ be the spherical Harish–Chandra homomorphism, and $\theta_c^- : e_\epsilon H_c e_\epsilon \longrightarrow \mathrm{End}(\mathbb{C}[\mathfrak{h}])$ the antispherical Harish–Chandra homomorphism, [11, Section 4]. The image of these maps lies in differential operators on $\mathfrak{h}^{\mathrm{reg}}$, preserving $\mathbb{C}[\mathfrak{h}]$. Give H_c the filtration induced by setting $\deg(x) = \deg(w) = 0$ and $\deg(y) = 1$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $w \in W$, and put the filtration on differential operators induced by order. By [11, Proposition 4.5] both θ_c and θ_c^- are flat families of injective filtration preserving homomorphisms, such that the associated graded maps are injective. Since, by [2, Propositions 4.10 and 4.11], the images of θ_c and θ_c^- are equal for generic values of c, the images are equal for all values of c. The isomorphism from $e_\epsilon H_c e_\epsilon$ to $eH_{c-\mathbf{1}}e$ is thus given by $\theta_c^{-1}\theta_c^-$. ☐

4.9. For $c = (1+h)/h$ we can now describe the Hilbert series of $L_c(\mathsf{triv})$.

Theorem. *Set $c = (1+h)/h$. The simple H_c–module $L_c(\mathsf{triv})$ is finite dimensional. Its Hilbert series is given by*

$$p(L_c(\mathit{triv}), c) = t^{-N}(1 + t + t^2 + \cdots + t^h)^n.$$

Proof. Under the shift isomorphism of Proposition 4.8(Version 2) $\mathbf{x}^2 e$ is sent to $\mathbf{x}^2 e_\epsilon$ and $\mathbf{y}^2 e$ to $\mathbf{y}^2 e_\epsilon$, [2, Proof of Proposition 4.11]. Since $\{\mathbf{x}^2 e, \mathbf{h}e, \mathbf{y}^2 e\}$ and $\{\mathbf{x}^2 e_\epsilon, \mathbf{h}e_\epsilon, \mathbf{y}^2 e_\epsilon\}$ form $\mathfrak{sl}_2$–triples in $eH_{c-\mathbf{1}}e$ and $e_\epsilon H_c e_\epsilon$ respectively, it follows that the shift isomorphism sends $\mathbf{h}e = [\mathbf{x}^2 e, \mathbf{y}^2 e]$ to $\mathbf{h}e_\epsilon = [\mathbf{x}^2 e_\epsilon, \mathbf{y}^2 e_\epsilon]$.

Thanks to Lemma 4.6 and Proposition 4.8(Version 2) there is a one–dimensional $e_\epsilon H_c e_\epsilon$–module. Call this one–dimensional module $\mathbb{C}$. By the above paragraph $\mathbf{h}e_\epsilon$ acts on $\mathbb{C}$ with weight zero since $\mathbf{h}e$ does so on the trivial module. Thus Lemma 4.7 shows how to write $\mathbb{C}$ in terms of the standard modules.

Define a functor $G : e_\epsilon H_c e_\epsilon - \mathit{mod} \longrightarrow H_c - \mathit{mod}$ sending M to $G(M) = H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} M$. Since $H_c e_\epsilon$ is a finite module over $e_\epsilon H_c e_\epsilon$, G preserves finite dimensionality. Suppose M is a finite dimensional simple $e_\epsilon H_c e_\epsilon$–module. Consider a composition series for $G(M)$

$$G(M) = X_0 \supset X_1 \supset \cdots \supset X_n \supset 0.$$

By [7, Théorème 4.1] each finite dimensional simple H_c–module belongs to the category $\mathcal{O}_c$. By Lemma 4.3 if $L_c(\tau)$ is finite dimensional then $\tau \cong \mathfrak{h}_i$ for some i. Thus Lemma 4.5 shows that F is exact and faithful on the finite dimensional simple modules of $\mathcal{O}_c$. We deduce a composition series

$$F(G(M)) = F(X_0) \supset F(X_1) \supset \cdots \supset F(X_n) \supset 0.$$

But $F(G(M)) = e_\epsilon(H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} M) = M$. Hence $F(X_1) = 0$, implying that $X_1 = 0$. Hence G preserves the simplicity of finite dimensional representations.

Let $L = G(\mathbb{C})$. By construction L contains a copy of ϵ in degree 0. By Lemma 4.4 the unique standard module $M_c(\mathfrak{h}_i)$ with ϵ appearing in degree 0 is $M_c(\mathfrak{h}_0)$. Thus L must be a factor of $M_c(\mathfrak{h}_0)$. In other words $L \cong L_c(\mathfrak{h}_0)$. By Lemma 4.7 we deduce that

$$L_c(\mathfrak{h}_0) = \sum_{i=0}^{n} (-1)^i M_c(\mathfrak{h}_i).$$

Thus we find that

$$\begin{aligned} p(L_c(\mathfrak{h}_0), t) &= \sum_{i=0}^{n} (-1)^i p(M_c(\mathfrak{h}_i), t) \\ &= \sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i} t^{-N+(h+1)i}}{(1-t)^n} \\ &= t^{-N} \frac{(1-t^{h+1})^n}{(1-t)^n} \\ &= t^{-N}(1+t+t^2+\cdots+t^h)^n. \end{aligned}$$

□

The result of this theorem is consistent with the conjecture in [2, Section 5] on the character formula for representations of H_c in type A. This conjecture has now been confirmed in [3].

5. Diagonal harmonics

The following theorem was conjectured by Haiman for all finite reflection groups, [14, Section 7]. For type A a stronger version of the theorem was obtained by Haiman in [15]. Results for dihedral groups were obtained by Alfano and Reiner.

Recall n is the rank of W, h its Coxeter number of W and ϵ the sign representation of W. If W is a Weyl group let Q be the root lattice associated with W.

Theorem. *Let W be a finite reflection group. Let the quotient ring by diagonal invariants be*

$$D_W = \frac{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]}{\langle \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]_+^W \rangle}.$$

Then there exists a W–stable quotient ring R_W of D_W satisfying the following properties:

(1) $\dim R_W = (h+1)^n$;
(2) *R_W is $\mathbb{Z}$–graded with Hilbert series* $t^{-hn/2}(1+t+\cdots+t^h)^n$;
(3) *The image of $\mathbb{C}[\mathfrak{h}]$ in R_W is the classical coinvariant algebra,* $\mathbb{C}[\mathfrak{h}]^{coW}$;

(4) *If W ia a Weyl group then as a W–module $R_W \otimes \epsilon$ is isomorphic to the permutation representation of W on $Q/(h+1)Q$.*

Proof. Let $L = H_c e_\epsilon \otimes_{e_\epsilon H_c e_\epsilon} \mathbb{C}$ for $c = (1+h)/h$. Consider the associated graded module grL. Since $gr(H_c e_\epsilon) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon$ and $gr(e_\epsilon H_c e_\epsilon) = e_\epsilon \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon$ we have a natural surjection

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon \otimes_{e_\epsilon \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon} \mathbb{C} \longrightarrow grL.$$

Left multiplication by e_ϵ provides a graded $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$–isomorphism between $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W e_\epsilon$ and $e_\epsilon \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon$. Right multiplication by e_ϵ provides a graded $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$–isomorphism between $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \otimes \epsilon$ and $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W e_\epsilon$. We deduce a graded $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$–surjection

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \otimes_{\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W} \mathbb{C} \longrightarrow grL \otimes \epsilon.$$

We set $R_W = grL \otimes \epsilon$.

By the second paragraph of the proof of Theorem 4.9 the element $e_\epsilon \otimes 1$ has degree 0 in L. Since the grading induced by $\mathbf{h}$ gives $x \in \mathfrak{h}^*$ degree 1 and $y \in \mathfrak{h}$ degree -1, we see that R_W has the same Hilbert series as L, so (2) follows from Theorem 4.9. Part (1) is obtained by setting $t = 1$.

The image of $\mathbb{C}[\mathfrak{h}]$ in R_W corresponds to the subspace $\mathbb{C}[\mathfrak{h}]e_\epsilon \otimes 1$ of L. If $p \in \mathbb{C}[\mathfrak{h}]^W_+ e_\epsilon$ then

$$p \otimes 1 = e_\epsilon p e_\epsilon \otimes 1 = e_\epsilon \otimes p.1 = 0.$$

Thus the ideal generated by $\mathbb{C}[\mathfrak{h}]^W_+$ annihilates $e_\epsilon \otimes 1$. On the other hand, the quotient $\mathbb{C}[\mathfrak{h}]^{coW}$ contains a unique (up to scalar) element of maximal degree N, say q, [16, Chapter 3]. The space $\mathbb{C}q$ is the socle of $\mathbb{C}[\mathfrak{h}]^{coW}$ since the ring of coinvariants is Frobenius (using Poincaré duality, for instance). We claim $qe_\epsilon \otimes 1 \neq 0$. By (1) any element of H_c can be written as a sum of terms of the form $p_- w p_+$ where $p_- \in \mathbb{C}[\mathfrak{h}^*]$, $p_+ \in \mathbb{C}[\mathfrak{h}]$ and $w \in W$. Since p_- and w do not increase degree, it would follow if $qe_\epsilon \otimes 1$ were zero, then L could have no subspace in degree N. But the Hilbert series of L has highest order term $t^{-N+hn} = t^N$. Thus $qe_\epsilon \otimes 1$ is non–zero and $\mathbb{C}[\mathfrak{h}]e_\epsilon \otimes 1$ is isomorphic to $\mathbb{C}[\mathfrak{h}]^{coW} \otimes e_\epsilon$. This proves (3).

It remains to check (4). Notice it is enough to calculate the W–decomposition of L, since passing to associated graded module is W–equivariant. Recall from the proof of Theorem 4.9 we can calculate the W–decomposition of L from the formula

$$L = \sum_{i=0}^{n} (-1)^i M_c(\mathfrak{h}_i).$$

Therefore, with the obvious notation, the graded character of w on L is given by

$$\mathrm{ch}_L(w,t) = \sum_{i=0}^{n} (-1)^i \mathrm{ch}_{M_c(\mathfrak{h}_i)}(w,t). \tag{4}$$

Recall, as a graded W–module we have $M_c(\mathfrak{h}_i) = \mathbb{C}[\mathfrak{h}] \otimes \mathfrak{h}_i$. It is well known that the graded trace of w on $\mathbb{C}[\mathfrak{h}]$ is given by $1/\det(1-tw)$. Thus

$$\mathrm{ch}_{M_c(\mathfrak{h}_i)}(w,t) = \frac{t^{\kappa_c(\mathfrak{h}_i)}\mathrm{ch}_{\mathfrak{h}_i}(w)}{\det(1-tw)}. \tag{5}$$

Combining (4), (5) and the equality $\kappa_c(\mathfrak{h}_i) = i(1+h)$ from Lemma 4.2 yields

$$\mathrm{ch}_L(w,t) = t^{-N}\frac{\det(1-t^{h+1}w)}{\det(1-tw)}.$$

Evaluating this expression at $t=1$ gives the character of L

$$\mathrm{ch}_L(w) = h^{\dim\ker(1-w)}. \tag{6}$$

On the other hand, since h and $h=1$ are coprime it follows from [21] that the character of w on the permutation representation $Q/(h+1)Q$ equals $h^{\dim\ker(1-w)}$. This proves (4) Weyl groups. □

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Department of Mathematics, University of Glasgow, Glasgow, G12 8QW, U.K.

E-mail address: `ig@maths.gla.ac.uk`