Infinite-dimensional Lie algebras

Lecture course by Iain Gordon

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Contents

1 Introduction

Lie algebras may arise in the following ways in the wild:

- Derivations of an associative algebra.
- Vector fields on a smooth manifold.
- Tangent spaces at the identity of Lie groups.

We should often think of a Lie algebra of being one of those (intimately related) concepts.

Example (Witt algebra over $\mathbb{C}$)**.** Let $A = \mathbb{C}[z, z^{-1}]$ be the algebra of Laurent polynomials in one variable and consider

$$\mathrm{Der}(A) = \left\{ f(z)\frac{d}{dz} : f \in \mathbb{C}\left[z, z^{-1}\right] \right\} = \mathrm{span}\left\{ L_j := -z^{j+1}\frac{d}{dz} : j \in \mathbb{Z} \right\}.$$

We compute the structure of $\mathrm{Der}(A)$:

$$\begin{aligned} \left[L_m, L_n\right] f &= \left[-z^{m+1}\frac{d}{dz}, -z^{n+1}\frac{d}{dz}\right] f \\ &= \left((n+1)z^{m+n+1}f' + z^{m+n+2}f''\right) - \left((m+1)z^{m+n+1}f' + z^{m+n+2}f''\right) \\ &= (n-m)z^{m+n+1}\frac{df}{dz} = (m-n)L_{m+n}f \end{aligned}$$

This is called the *Witt algebra* over $\mathbb{C}$, denoted Witt; its basis is $\left\{L_j : j \in \mathbb{Z}\right\}$ and its structure is

$$\left[L_m, L_n\right] = (m-n)L_{m+n}.$$

Example (Vector fields on S^1)**.** Consider the complex vector fields on $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}/\sim\}$:

$$\mathrm{Vect}_{\mathrm{fin}} := \left\{ f\frac{d}{d\theta} : f\colon S^1 \to \mathbb{C} \text{ smooth with finite Fourier expansion} \right\},$$

i.e. we impose that each element is in the finite span of $\cos(n\theta)\frac{d}{d\theta}$ and $\sin(n\theta)\frac{d}{d\theta}$, or equivalently in the finite span of $\{e^{in\theta}\frac{d}{d\theta} : n \in \mathbb{Z}\}$. Without the finiteness assumption, we would have to deal with the Fourier analysis and convergence issues. Thus we have a basis

$$\left\{ L_n := ie^{in\theta}\frac{d}{d\theta} : n \in \mathbb{Z} \right\}.$$

Set $z := e^{i\theta}$, then $L_n = -z^{n+1}\frac{d}{dz}$, and we recover the Witt algebra.

Example (Diffeomorphisms of S^1)**.** Let $G := \mathrm{Diff}_+(S^1)$ be the group of (orientation-preserving) diffeomorphisms of S^1 under composition. It acts on $C^\infty(S^1, \mathbb{C})$ via $(g.f)(z) := f\big(g^{-1}(z)\big)$ for $g \in G$. An element of the tangent space at $1 \in G$ has the form

$$g(z) = z\big(1 + \epsilon(z)\big) = z + \sum_{n=-\infty}^{\infty} \lambda_n \epsilon z^{n+1}.$$

Then

$$g^{-1}(z) = z\big(1 - \epsilon(z)\big) = z - \sum_{n=-\infty}^{\infty} \lambda_n \epsilon z^{n+1}.$$

Thus

$$\begin{aligned} \big(g.f\big)(z) &= f\big(z(1-\epsilon(z))\big) &&= f\Big(z - \sum_{n=-\infty}^{\infty} \lambda_n \epsilon z^{n+1}\Big) \\ &= \Big(1 - \sum_{n=-\infty}^{\infty} \lambda_n \epsilon z^{n+1}\frac{d}{dz}\Big) f(z) &&= \Big(1 + \sum_{n=-\infty}^{\infty} \lambda_n \epsilon L_n\Big) f(z). \end{aligned}$$

The L_n for a topological basis for $\mathrm{Lie}(G)$.

The three preceding examples all give the same Lie algebra structure. The Witt algebra has two further properties:

1. There is an anti-linear Lie algebra involution $\omega(L_n) := L_{-n}$.
2. Thence we can build a real form of the Witt algebra as $\{x \in \mathrm{Witt} : \omega(x) = -x\}$.

2 Central extensions

Definition 2.1. A *central extension* of a Lie algebra $\mathfrak{g}$ is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{z} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0 \tag{2.1}$$

such that $\mathfrak{z} \subseteq Z(\widehat{\mathfrak{g}}) = \{x \in \widehat{\mathfrak{g}} : [x, \widehat{\mathfrak{g}}] = 0\}$.

As vector spaces the sequence (2.1) splits as $\widehat{\mathfrak{g}} = \mathfrak{g}' \oplus \mathfrak{z}$, say via a section $\sigma\colon \mathfrak{g} \to \mathfrak{g}'$. Let

$$\beta\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}, \quad (x, y) \mapsto \big[\sigma(x), \sigma(y)\big] - \sigma\big([x, y]\big).$$

We check that β satisfies

- $\beta(x, y) = -\beta(y, x)$, and

- $\beta\big(x,[y,z]\big)+\beta\big(y,[z,x]\big)+\beta\big(z,[x,y]\big)=0$.

Conversely, given $\beta\in C^2(\mathfrak{g};\mathfrak{z}):=\{\beta\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}:\ \text{satisfying the above}\}$, we can construct

$$\widehat{\mathfrak{g}}_\beta:=\mathfrak{g}\oplus\mathfrak{z}\text{ as vector spaces},\quad \big[(g,z),(g',z')\big]:=\big([g,g'],\beta(g,g')\big)\,.$$

Then $\widehat{\mathfrak{g}}_\beta$ is a central extension. However, there exist isomorphisms of such $\widehat{\mathfrak{g}}_\beta$:

$$\begin{tikzcd}
\mathfrak{z} \arrow[r] \arrow[d, equal] & \widehat{\mathfrak{g}}_\beta=\mathfrak{g}\oplus\mathfrak{z} \arrow[r] \arrow[d, "\theta"', "\cong"] & \mathfrak{g} \arrow[d, equal] \\
\mathfrak{z} \arrow[r] & \widehat{\mathfrak{g}}_\gamma=\mathfrak{g}\oplus\mathfrak{z} \arrow[r] & \mathfrak{g}
\end{tikzcd}$$

We find quickly that $\theta\colon \widehat{\mathfrak{g}}_\beta\to\widehat{\mathfrak{g}}_\gamma$ is of the form $\theta\big((g,z)\big)=\big(g,\theta_2(g)+z\big)$, where $\theta_2\colon \mathfrak{g}\to\mathfrak{z}$; and for θ to be a Lie algebra homomorphism, we require that $\theta_2\big([g,g']\big)=\gamma(g,g')-\beta(g,g')$.

So we find that central extensions are classified by

$$H^2(\mathfrak{g};\mathfrak{z})=\frac{C^2(\mathfrak{g};\mathfrak{z})}{dB^1(\mathfrak{g};\mathfrak{z})}\,,$$

where $B^1=\{\theta\colon\mathfrak{g}\to\mathfrak{z}\}$, and $d\theta(g,g')=\theta\big([g,g']\big)$.

Exercise 2.2. Show that if $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then $H^2(\mathfrak{g};\mathbb{C})=0$.

Proposition 2.3. $H^2(\mathrm{Witt}):=H^2(\mathrm{Witt};\mathbb{C})\cong\mathbb{C}$. *In other words, there is a one-dimensional space of central extensions of the Witt algebra.*

Proof. Let $\beta\colon \mathrm{Witt}\times\mathrm{Witt}\to\mathbb{C}$, $(L_m,L_n)\mapsto\beta(m,n)$ satisfy the two conditions from above, namely

- $\beta(m,n)=-\beta(n,m)$, and
- $(m-n)\beta(l,m+n)+(n-l)\beta(m,l+n)+(l-m)\beta(n,l+m)=0$.

Freedom of choice is provided by $d\theta$ for some

$$\theta\colon\ \mathrm{Witt}\to\mathbb{C}\,,\quad L_m\mapsto\theta(m)\,.$$

Writing $\theta_\beta(m):=\beta(0,m)/m$ for $m\neq 0$, replace β by $\beta+d\theta_\beta$: We see that w.l.o.g. $\beta(0,m)=0$ for all $m\in\mathbb{Z}$. Now with $l=0$, we get $n\beta(m,n)-m\beta(n,m)=0$, i.e. $(m+n)\beta(m,n)=0$. Thus $\beta(m,n)=\delta_{m,-n}\widehat{\beta}(m)$, where $\widehat{\beta}(m)=-\widehat{\beta}(-m)$.

With $l+m+n=0$, we produce a relation which for $n=1$ gives

$$(1-m)\widehat{\beta}(m+1)+(m+2)\widehat{\beta}(m)-(2m+1)\widehat{\beta}(1)=0\,.$$

This recurrence relation has a two-dimensional solution space:

$$\widehat{\beta}(m)=\alpha_1 m+\alpha_2 m^3\ ,\text{ with }\alpha_1,\alpha_2\in\mathbb{C}\,.$$

Now replace β with $\beta+d\theta_\beta$, where $\theta_\beta(m)=\delta_{m,0}\frac{\alpha_1}{2}$, to get $\beta(m,n)=\delta_{m,-n}\alpha_2 m^3$. □

3 The Virasoro algebra

Definition 3.1. The *Virasoro algebra* (denoted by Vir) is the central extension of the Witt algebra with the choice $\widehat{\beta}(m)=\frac{1}{12}(m^3-m)$ in the notation of Proposition 2.3.

The Virasoro Algebra is spanned by $\{L_m: m\in\mathbb{Z}\}\cup\{c\}$ such that c is central, i.e. $[c,L_m]=0$ for all $m\in\mathbb{Z}$, and

$$\big[L_m,L_n\big]=(m-n)L_{m+n}+\delta_{m,-n}\frac{1}{12}(m^3-m)c\,.$$

The number $1/12$ is chosen so that c acts as the scalar 1 on a natural irreducible representation that we are about to see.

4 The Heisenberg algebra

The Witt algebra was a simple point of departure for infinite dimensional Lie algebras. Another simple case is given by constructing loops on a (1-dimensional) abelian Lie algebra, and again making a central extension. This leads to

Definition 4.1. The *Heisenberg algebra* $\mathscr{H}$ is spanned by $\{a_n : n \in \mathbb{Z}\} \cup \{h\}$, where h is central and $[a_m, a_n] = m\delta_{m,-n}h$. (We could even rescale to get rid of m.)

This algebra has a representation as an algebra of operators on $B(\mu, h) := \mathbb{C}[x_1, x_2, \ldots]$, on which the $\mathscr{H}$-module structure is given by $\rho\colon \mathscr{H} \to \mathfrak{gl}(B(\mu, h))$ as follows:

$$\rho\colon a_n \mapsto \rho(a_n) = \begin{cases} \frac{\partial}{\partial x_n} & n > 0\,, \\ -hnx_{-n} & n < 0\,, \\ \mu & n = 0\,, \end{cases} \qquad \text{and} \qquad \rho\colon h \mapsto \rho(h) = (-\times h)\,. \tag{4.1}$$

Exercise 4.2. If $h \neq 0$, then $B(\mu, h)$ is irreducible, while if $h = 0$ it is not (e.g. the subspace of constants $\mathbb{C} < B(\mu, h)$ is invariant).

There exists an involution $\omega(a_n) = a_{-n}$, $\omega(h) = h$.

Link to Virasoro algebras. Define

$$L_k := \frac{1}{2} \sum_{j \in \mathbb{Z}} :a_{-j}a_{j+k}: \text{ for } k \in \mathbb{Z}\,.$$

The notation "$:a_{-j}a_{j+k}:$" means "normal ordering", i.e.

$$:a_{-j}a_{j+k}: \quad := \quad \begin{cases} a_{-j}a_{j+k} & \text{if } -j \leq j+k, \\ a_{j+k}a_{-j} & \text{if } -j \geq j+k. \end{cases}$$

Explicitly, we have

$$L_k = \begin{cases} \frac{1}{2}a_n^2 + \sum_{j>0} a_{n-j}\, a_{n+j} & \text{if } k = 2n, \\ \sum_{j>0} a_{n+1-j}\, a_{n+j} & \text{if } k = 2n+1. \end{cases}$$

The operator L_k is an infinite sum, but it is well-defined when acting on any element of $B(\mu, h)$ because of Property (2) below: On $B(\mu, h)$ we have that

1. a_0 and h act diagonally,
2. on a given element of $B(\mu, h)$ a_n acts as zero for $n \gg 0$, and
3. a_n acts locally nilpotently for $n > 0$.

Theorem 4.3. *With L_k defined as above,*

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{1}{12}(m^3 - m)$$

as operators on $B(\mu, 1)$, i.e. as a representation of the Virasoro algebra. (But not of the Witt algebra!)

Sketch of proof. Define a function

$$\psi\colon \mathbb{R} \to \mathbb{R}\,, \quad x \mapsto \psi(x) := \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

and for $\epsilon > 0$, define $L_k(\epsilon) := \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+k} : \psi(\epsilon j)$. This has only finitely many non-zero terms, and $L_k(\epsilon) \to L_k$ as $\epsilon \to 0$. Now calculate: $\left[a_n, L_k\right] = n a_{k+n}$ (using $L_k(\epsilon)$), and hence

$$\left[L_m(\epsilon), L_n\right] = \frac{1}{2}\sum_{j\in\mathbb{Z}} (j+m) : a_{-j}a_{j+m+n} : \psi(\epsilon j) + \\ \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{n-j}a_{j+m} : \psi(\epsilon j) - \delta_{m,-n}\frac{1}{2}\sum_{j=-1}^{-m} j(j+m)\psi(\epsilon j)\,,$$

and let $\epsilon \to 0$. □

Exercise 4.4. Is $B(\mu, 1)$ irreducible as a Virasoro module?

5 Representations of the Virasoro algebra

Recall that we have an anti-linear involution $\omega(L_n) = L_{-n}$, $\omega(c) = c$.

Definition 5.1 (Unitarity)**.** A representation $\rho\colon \mathrm{Vir} \to \mathfrak{gl}(V)$ of the Virasoro algebra is *unitary* if there exists a non-degenerate hermitian form $\langle -, - \rangle$ on V such that

1. $\langle v, v \rangle \ge 0$ for all $v \in V$, with equality if and only if $v = 0$, and
2. $\langle \rho(x).v_1, v_2 \rangle = \langle v_1, \rho(\omega(x)).v_2 \rangle$ for all $x \in \mathrm{Vir}$, $v_1, v_2 \in V$, i.e. $x^\dagger = \omega(x)$.

5.1 Family 1: Chargeless representations

Let

$$V = V_{\alpha,\beta} = P(z)z^\alpha (dz)^\beta \cong \mathbb{C}[z, z^{-1}] = \operatorname{span}\left\{v_n : n \in \mathbb{Z}\right\}\,, \tag{5.1}$$

where $\alpha, \beta \in \mathbb{C}$ and $P \in \mathbb{C}[z, z^{-1}]$. Define a representation of the Virasoro algebra by the following action:

$$c \mapsto 0 \qquad \text{and} \qquad L_n(v_k) = -\big(k + \alpha + \beta(n+1)\big)v_{n+k}\,. \tag{5.2}$$

This is the representation one would discover by using the definition $L_n := -z^{n+1}\frac{d}{dz}$ on the elements $v_k := z^{k+\alpha}(dz)^\beta$.

Now we start to use the structure of Vir. Set

$$\mathfrak{h} := \mathbb{C}c \oplus \mathbb{C}L_0 \text{ (abelian);} \quad \mathrm{Vir}_\pm := \operatorname{span}\left\{L_k : k \in \mathbb{Z}_\pm\right\}.$$

We have the so-called *triangular decomposition*

$$\mathrm{Vir} = \mathrm{Vir}_- \oplus \mathfrak{h} \oplus \mathrm{Vir}_+\,. \tag{5.3}$$

Exercise 5.2. Check that $\mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}L_0$ is a Cartan subalgebra of Vir, i.e. that it is nilpotent and self-normalising.

Remark 5.3. In the representation $V_{\alpha,\beta}$, $\mathfrak{h}$ acts by scalars on each v_k: $L_0 v_k = -(k+\alpha+\beta)v_k$.

For a general representation $\rho\colon \text{Vir} \to \mathfrak{gl}(V)$, the decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

is called a *weight space decomposition*, where $V_\lambda := \{v \in V : \rho(z).v = \lambda(z)v \text{ for all } z \in \mathfrak{h}\}$. Obviously, the existence of a weight space decomposition is a restriction on the representation.

Lemma 5.4. *Let V be a representation of an abelian Lie algebra A, such that $V = \bigoplus_{\lambda \in A^*} V_\lambda$ and such that each V_λ is finite-dimensional. Then any subrepresentation $U \le V$ decomposes as $U = \bigoplus_\lambda (U \cap V_\lambda)$.*

Proof. Any $v \in V$ can be written as $v = \sum_{j=1}^m v_j$ with $v_j \in V_{\lambda_j}$ for $j = 1, \ldots, m$, i.e. $a.v_j = \lambda_j(a)v_j$ for all $a \in A$. Suppose $v \in U$. We need to show that $v_j \in U$ for each j. Since $\lambda_j \neq \lambda_k$ for $j \neq k$, we can find $a \in A$ such that $\lambda_j(a) \neq \lambda_k(a)$ if $j \neq k$. Then

$$\begin{aligned} v &= v_1 + \cdots + v_m \\ a.v &= \lambda_1(a)v_1 + \cdots + \lambda_m(a)v_m \\ &\vdots \\ a^{m-1}.v &= \lambda_1(a)^{m-1}v_1 + \cdots + \lambda_m(a)^{m-1}v_m\,. \end{aligned}$$

Since each $a^j.v \in U$, we see that $[\lambda_i(a)^j](v_1, \ldots, v_m)^T \in U^m$, where $1 \le i \le m$ and $0 \le j \le m-1$. But the matrix $[\lambda_i(a)^j]$ is a Vandermonde matrix and thus, since the $\lambda_i(a)$ are pairwise distinct, it has non-zero determinant, and hence $(v_1, \ldots, v_m)^T \in U^m$, as required. □

Theorem 5.5.

1. *$V_{\alpha,\beta}$ is reducible if and only if either $\alpha \in \mathbb{Z}$ and $\beta = 0$, or $\alpha \in \mathbb{Z}$ and $\beta = 1$. In those cases, $V_{\alpha,\beta}$ has a composition series whose irreducible sections are a trivial representation and an irreducible representation $V'_{\alpha,\beta}$ of codimension one in $V_{\alpha,\beta}$. (As vector spaces we may consider these to be* $\operatorname{span}\{v_{-(\alpha+\beta)}\}$ *and* $\operatorname{span}\{v_k : k \neq -\alpha-\beta\}$, *respectively.)*

   (If $V_{\alpha,\beta}$ is irreducible, then set $V'_{\alpha,\beta} = V_{\alpha,\beta}$.)

2. *$V'_{\alpha,\beta}$ is unitary if and only if $\beta + \overline{\beta} = 1$ and $\alpha + \beta = \overline{\alpha} + \overline{\beta}$.*

Proof. Part (1): By Lemma 5.4, any subrepresentation U decomposes as a direct sum of its weight spaces, so if U is a non-zero subrepresentation, then there exists k with $v_k \in U$. Now given $v_k \in U$, we have $L_n(v_k) \in U$ for all n, and so we get scalar conditions from Equation (5.2), i.e.

$$v_{n+k} \notin U \Rightarrow k + \alpha + \beta(n+1) = 0\,.$$

If $v_{n+k}, v_{m+k} \notin U$, then $\beta = 0$ and $\alpha = -k$, and hence $\mathbb{C}v_k$ is a subrepresentation, and $V_{\alpha,\beta}/\mathbb{C}v_k$ is irreducible (because $L_n(v_t) = (t-k)v_{t+n}$ for $t \neq k$).

Now assume that only one vector, say v_m, does not belong to U. Then for $l \neq k, m$ we must have $L_{n-k}(v_k) = 0 = L_{m-l}(v_l)$, whence $\beta = 1$, $a = -(m+1)$ and $U = \operatorname{span}\{v_t : t \neq m\}$.

Part (2): Unitarity implies $\langle v_k, v_k\rangle > 0$ for all relevant k. Hence

- $\langle L_0(v_k), v_k\rangle = \langle v_k, L_0(v_k)\rangle$, whence $\alpha + \beta \in \mathbb{R}$, and

- $\langle L_{-1}(v_k), v_{k-1}\rangle = \langle v_k, L_1(v_{k-1})\rangle$ and $\langle L_{-2}(v_k), v_{k-2}\rangle = \langle v_k, L_2(v_{k-2})\rangle$. Using this to produce two different looking relations between $\langle v_k, v_k\rangle$ and $\langle v_{k-2}, v_{k-2}\rangle$ for infinitely many different integers k, we find an equality of polynomials

$$\left(X-\overline{\beta}\right)\left(X-(1+\overline{\beta})\right)\left(X-2(1+\beta)\right) = \left(X-(1-\beta)\right)\left(X-(2-\beta)\right)\left(X-2\overline{\beta}\right),$$

whence $\beta+\overline{\beta}=1$.

Conversely, $\langle v_k, v_m\rangle = \delta_{km}$ is a well-defined positive-definite hermitian form on $V'_{\alpha,\beta}$ because of the conditions on α, β. □

Exercise 5.6. Is $V_{\alpha,\beta}$ completely reducible? That is, if it is not irreducible, is $V_{\alpha,\beta} = \mathbb{C}\oplus V'_{\alpha,\beta}$ a decomposition of Lie algebra representations?

5.2 Family 2: Highest-weight representations

The Virasoro algebra has a representation on $B(\mu,1) = \mathbb{C}[x_1, x_2, \ldots]$ as in Equation (4.1) above. Recall that

$$L_0 := \frac{1}{2}a_0^2 + \sum_{j>0} a_{-j}a_j = \frac{\mu^2}{2} + \sum_{j>0} a_{-j}a_j\,.$$

On a monomial $\prod_i x_i^{n_i}$, the term $a_{-j}a_j$ acts by multiplication by jn_j, and so we see that $\sum_{j>0} a_{-j}a_j$ picks out the "degree" of a homogeneous polynomial, where $\deg(x_j) = j$, i.e.

$$B(\mu,1) = \sum_{\lambda\in\mathfrak{h}^*} B(\mu,1)_\lambda\,,$$

where

$$B(\mu,1)_\lambda = \begin{cases}\text{homogeneous polynomials of degree } N & \lambda(c)=1,\ \lambda(L_0) = \mu^2/2+N\,,\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $B(\mu,1)$ for $\mu\in\mathbb{R}$ is *unitary*, and in fact it is induced from a unitary representation of the Heisenberg algebra $\mathscr{H}$. The monomials form an orthonormal basis:

$$\left\langle x_1^{k_1}\cdots x_n^{k_n},\, x_1^{k_1}\cdots x_n^{k_n}\right\rangle = \frac{\prod_{j=1}^n k_j!}{\prod_{j=1}^n j^{2k_j}}\,.$$

(This just says that $\langle P,Q\rangle$ is the constant coefficient of $\omega(P)Q.1$, where P and Q are polynomials in the (commuting) variables a_k, for $k<0$.)

Proposition 5.7. *Let V be a unitary representation of the Virasoro algebra (with c acting as a scalar) such that*

$$V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda \text{ with } \dim V_\lambda < \infty\,.$$

Then V is completely reducible.

Proof. Let $U<V$ be an invariant subspace. Observe that $\omega(L_0) = L_0$. Unitarity forces $\lambda\in\mathfrak{h}^*_{\mathbb{R}}$ by considering the effect of L_0 on $\langle v,v\rangle>0$. It then follows that $\langle V_\lambda, V_\mu\rangle = 0$ if $\lambda\neq\mu$. Set $U_\lambda := U\cap V_\lambda$, and define $U_\lambda^\perp\subset V_\lambda$ as the orthogonal complement to U_λ in the form restricted to V_λ. Then $V_\lambda = U_\lambda\oplus U_\lambda^\perp$, and $\bigoplus_\lambda U_\lambda^\perp =: U^\perp$ is invariant under the Virasoro algebra. So $V = U\oplus U^\perp$. □

By our previous observation we conclude that $B(\mu,1)$ is a completely reducible Virasoro module. Write $B'(\mu,1) \le B(\mu,1)$ for the submodule generated by 1. It is unitary, hence completely reducible, and if $B'(\mu,1) = U \oplus U^{\perp}$, then look at

$$U_{\mu^2/2} \oplus U^{\perp}_{\mu^2/2} = B'(\mu,1)_{\mu^2/2} = \mathbb{C}.1$$

to see that

$$1 \in U_{\mu^2/2} \Rightarrow B'(\mu,1) = U\ ,\ \text{ i.e. } B'(\mu,1) \text{ is irreducible.}$$

At this point we do not know whether $B'(\mu,1) = B(\mu,1)$.

Definition 5.8 (Highest-weight representations)**.** A *highest-weight representation* of the Virasoro algebra is a representation V such that there exists a $v \in V$ satisfying

1. $c_{\mathrm{Vir}}.v = cv$ (here c is a complex number on the right hand side),
2. $L_0(v) = hv$ for $h \in \mathbb{C}$,
3. $V = \operatorname{span}\{(L_{-i_k} \cdots L_{-i_1})(v) : 0 \le i_1 \le \cdots \le i_k\}$.

Remark 5.9.

- c_{Vir} acts as the complex number c on all of V.
- $L_0\big(L_{-i_k} \cdots L_{-i_1}(v)\big) = \big(h + i_k + \cdots + i_1\big)\big(L_{-i_k} \cdots L_{-i_1}\big)(v)$ thanks to $\big[L_0, L_{-i}\big] = iL_{-i}$. So we get a weight space decomposition
$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda\ ,$$
where $V_\lambda \neq 0$ only if $\lambda(c_{\mathrm{Vir}}) = c$ and $\lambda(L_0) = h + \mathbb{Z}_{\ge 0}$.
- $L_k(v) = 0$ for $k > 0$ (because $V_\lambda = 0$ for $\lambda(L_0) = h - k$).

The (c,h) are called the *highest weights* of V.

Remark 5.10. $B'(\mu,1)$ is a highest-weight representation with weight $\big(1, \mu^2/2\big)$.

Definition 5.11. If $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, then the *formal character* of V is

$$\mathrm{ch}_V(q,t) := \sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda)\, q^{\lambda(L_0)}\, t^{\lambda(c)}\ .$$

Key idea: There exists a universal highest-weight representation for any $(c,h) \in \mathbb{C}^2$. To define this we introduce first the universal enveloping algebra of a Lie algebra.

6 Enveloping algebras

Note that any associative algebra A is a Lie algebra under the commutator bracket $[x,y] := xy - yx$. We write A_{ad} for this Lie algebra structure.

Definition 6.1. Let $\mathfrak{g}$ be an arbitrary Lie algebra over some field $\Bbbk$. The *universal enveloping algebra* of $\mathfrak{g}$ is an associative algebra $U(\mathfrak{g})$ over $\Bbbk$ with unit and a Lie algebra homomorphism $\iota\colon \mathfrak{g} \to U(\mathfrak{g})_{\mathrm{ad}}$ satisfying the following universal property:

For every arbitrary associative algebra with unit A over $\Bbbk$ and a Lie algebra homomorphism $j\colon \mathfrak{g} \to A_{\mathrm{ad}}$, there exists a unique homomorphism of associative algebras $\phi\colon U(g) \to A$ such that $j = \phi \circ \iota$.

Exercise 6.2. Show that if $U(\mathfrak{g})$ exists, then it is unique.

Exercise 6.3. Show that any Lie algebra representation of $\mathfrak{g}$ is a (left) $U(\mathfrak{g})$-module and vice-versa. There is an equivalence of categories between representations of $\mathfrak{g}$ and left $U(\mathfrak{g})$-modules.

The universal enveloping algebra $U(\mathfrak{g})$ exists since we can construct it explicitly as

$$U(\mathfrak{g}) := T(\mathfrak{g})\big/\langle x\otimes y - y\otimes x - [x,y] : x,y\in\mathfrak{g}\rangle\ , \tag{6.1}$$

where $T(\mathfrak{g})$ denotes the tensor algebra of (the vector space) $\mathfrak{g}$, i.e. $T(\mathfrak{g}) := \bigoplus_{i\geq 0}\mathfrak{g}^{\otimes i}$. If $\{x_b : b\in B\}$ is a basis for $\mathfrak{g}$, the $T(\mathfrak{g})$ is just the free algebra on the variables x_b.

Exercise 6.4. Show that $U(\mathfrak{g})$ as defined in Equation (6.1) satisfies the universal property from Definition 6.1.

Theorem 6.5 (Poincaré, Birkhoff, Witt)**.** *Let $\{x_b : b\in B\}$ be a basis for $\mathfrak{g}$, where B is ordered. Then the set of ordered monomials*

$$\left\{x_{b_1}^{i_1}x_{b_2}^{i_2}x_{b_3}^{i_3}\cdots : b_1<b_2<b_3<\cdots, i_j\in\mathbb{Z}_{\geq 0}\right\}$$

is a basis for $U(\mathfrak{g})$.

7 The universal highest-weight representations of Vir

Recall from Remark 5.9 that a highest-weight representation V of the Virasoro algebra has a weight space decomposition $V=\bigoplus_{\lambda\in\mathfrak{h}^*}V_\lambda$. The Virasoro algebra has a sub-Lie-algebra $\mathrm{Vir}_{\geq} := \mathrm{Vir}_+\oplus\mathfrak{h}$ (cf. Equation (5.3); but be careful, this is only a vector space direct sum, not a Lie algebra direct sum). We will now construct a *universal highest-weight representation* of the Virasoro algebra. To this end, define

$$M(c,h) := U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_{\geq})}\mathbb{C}(c,h)\ , \tag{7.1}$$

where the action of $U(\mathrm{Vir}_{\geq})$ on $U(\mathrm{Vir})$ is by right multiplication, and on the one-dimensional space $\mathbb{C}(c,h)$ on the right-hand side it is as follows:

$$L_k.\lambda = 0 \text{ for } k>0, \qquad L_0.\lambda = h\lambda\ , \qquad c_{\mathrm{Vir}}.\lambda = c\lambda\ , \qquad \text{for all } \lambda\in\mathbb{C}.$$

We call this $M(c,h)$ the *Verma module.*

Exercise 7.1. Check that the Verma module $M(c,h)$ is a well-defined representation of the Virasoro algebra, and moreover that it is a highest-weight representation with highest weight (c,h). (See also Proposition 7.2 (1).)

Proposition 7.2.

1. *$M(c,h)=\bigoplus_{s\in\mathbb{Z}_{\geq 0}}M(c,h)_{h+s}$, where $M(c,h)_{h+s}$ has a basis of vectors $\{(L_{-i_k}\cdots L_{-i_1})\otimes 1\}$, where $0<i_1\leq i_2\leq\cdots\leq i_k$ and $\sum_{j=1}^k i_j = s$.*

   (This has dimension $P(s)$ = the number of partitions of s; thus

   $$\mathrm{ch}_{M(c,h)}(q,t) = \frac{t^h}{\phi(q)}\ , \text{ where } \ \phi(q)=\prod_{j\geq 1}(1-q^j)\ .$$

   Note that $L_0.(1\otimes 1) := L_0\otimes 1 = 1\otimes(L_0.1) = h(1\otimes 1)$, and $c.(1\otimes 1) = c(1\otimes 1)$, which proves that this is a highest-weight representation.)

2. *(Universality.)* $M(c,h)$ *is indecomposable, and any highest-weight representation of highest weight* (c,h) *is a quotient of* $M(c,h)$.

3. $M(c,h)$ *has a unique irreducible quotient* $V(c,h)$.

Proof. Part (1) is clear, since by Theorem 6.5 $U(\mathrm{Vir})$ is free over $U(\mathrm{Vir}_{\geq})$ with basis $\{L_{-i_k}\cdots L_{-i_1} : 0 < i_1 \leq \cdots \leq i_k\}$.

Part (2). If $M(c,h) = V \oplus W$, this respects the weight space decomposition by Lemma 5.4. Hence $M(c,h)_h = \mathbb{C}$, so without loss of generality, $V_h = \mathbb{C}$ and $W_h = 0$. Then $1 \otimes 1 \in V$. But $1 \otimes 1$ generates $M(c,h)$ as an algebra, so $M(c,h) = V$.

If N is a highest-weight representation with highest weight (c,h), then there exists $v \in N$ such that $c.v = cv$, $L_0.v = hv$ and $L_k.v = 0$ for all $k > 0$. Hence $\mathbb{C}v = \mathbb{C}(c,h)$. The evaluation map $\mathrm{ev}\colon M(c,h) := U(\mathrm{Vir}) \otimes \mathbb{C}(c,h) \twoheadrightarrow N$, $\rho \otimes 1 \mapsto \rho.v$ thus factors through the tensor product over $U(\mathrm{Vir}_{\geq})$ and so is the desired quotient.

Part (3). This is equivalent to a unique maximal proper submodule. Any submodule inherits a weight space decomposition, so any proper submodule P has $P_h = 0$. (Note that all proper submodules lie in the complement of the top weight space.) Thus the sum of all proper submodules is again proper, and clearly maximal. □

Theorem 7.3 (Mathieu; Chari, Pressley (unitary case))**.** *An irreducible representation* V *of the Virasoro algebra with* $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ *is precisely one of the following three:*

- $V'_{\alpha,\beta}$ *from Equation* (5.1) *and Theorem 5.5,*
- $V(c,h)$, *the irreducible quotient of* $M(c,h)$ *from Proposition 7.2 (3), or*
- $V(c,h)^*$, *the* restricted dual representation *of* $V(c,h)$, *which is defined as*

$$V(c,h)^* := \bigoplus_t \left[V(c,h)_t\right]^* .$$

It is acted on in the usual way by $(X.\theta)(v) := \theta(-X.v)$ *for* $X \in \mathrm{Vir}$, $\theta \in V(c,h)^*$ *and* $v \in V(c,h)$*; it is a* lowest-weight *representation.* □

The proof of Theorem 7.3 is tricky; it proceeds by an analysis in positive characteristic.

8 Irreducibilty and unitarity of Virasoro representations

First note that the involution ω on Vir extends to $U(\mathrm{Vir})$. Define a form on $M(c,h)$ as follows: For any $P = L_{-i_k}\cdots L_{-i_1}$ and $Q = L_{-j_l}\cdots L_{-j_1}$, and for $v = 1 \otimes 1$, let

$$\langle P(v), Q(v)\rangle := \big((\omega(P)Q).v\big)_{\text{coeff } v}$$

be the coefficient of v in the weight space decomposition, i.e. the coefficient of v in the weight decomposition of $L_{-i_k}\cdots L_{-i_1} L_{-j_l}\cdots L_{-j_1} v$.

Exercise 8.1. Check that this form is contravariant and sesquilinear (though not necessarily non-degenerate). Check that $M(c,h)_\lambda$ and $M(c,h)_\mu$ are pairwise orthogonal if $\lambda \neq \mu$.

Now let $M'(c,h) \leq M(c,h)$ be the maximal proper submodule, which we know to exist.

Lemma 8.2. $M'(c,h) = \ker\big(\langle -,-\rangle\big) := \big\{k : \langle k,-\rangle = 0\big\}$.

Proof. The kernel is a submodule since $\langle x.k, -\rangle = \langle k, \omega(x).-\rangle = 0$ for all $k \in \ker\big(\langle -,-\rangle\big)$ and $x \in \mathrm{Vir}$. Note that v is not in the kernel since $\langle v, v\rangle \neq 0$, so the kernel is indeed a proper submodule.

Now suppose $P.v \in M'(c,h)$, and thus $\omega(Q)P.v \in M'(c,h)$ for any Q. Hence $\langle Q(v), P(v)\rangle = 0$, and thus $P.v$ is in the kernel. □

To understand irreducibility or unitarity properties of $M(c,h)$ we need to understand the form $\langle -,-\rangle$. Define $\langle -,-\rangle_N$ to be the restriction of $\langle -,-\rangle$ to $M(c,h)_{h+N}$. This is a form on a finite-dimensional vector space; let $\det_N(c,h)$ be its determinant.

We compute the first two values: $\det_1(c,h) = \langle L_{-1}.v, L_{-1}.v\rangle$, which is the $(L_1L_{-1}.v)$-coefficient of v, so $\det_1(c,h) = 2h$. Next,

$$\det{}_2(c,h) = \begin{vmatrix} \langle L_{-2}.v, L_{-2}.v\rangle & \langle L_{-2}.v, L_{-1}L_{-1}.v\rangle \\ \langle L_{-1}L_{-1}.v, L_{-2}.v\rangle & \langle L_{-1}L_{-1}.v, L_{-1}L_{-1}.v\rangle \end{vmatrix} .$$

But

$$\begin{aligned} \langle L_{-2}.v, L_{-2}.v\rangle &= \big(L_2L_{-2}.v\big)_{\text{coeff } v} \text{, working in the Verma module,} \\ &= \Big(\big([L_2, L_{-2}] + L_{-2}L_2\big).v\Big)_{\text{coeff } v} \\ &= \Big(\big(4L_0 + \tfrac{1}{12}(8-2)c\big).v + 0\Big)_{\text{coeff } v} \\ &= 4h + \tfrac{c}{2} . \end{aligned}$$

With similar calculations we get that $\det_2(c,h) = 2h(16h^2 + 2hc - 10h + c)$.

Theorem 8.3 (Kac determinant formula)**.**

$$\det{}_n(c,h) = K \prod_{\substack{r,s\in\mathbb{N} \\ 1\le r,s\le n}} \big(h - h_{r,s}(c)\big)^{P(n-rs)} ,$$

where $P(m)$ is the number of partitions of m,

$$K = \prod_{\substack{r,s\in\mathbb{N} \\ 1\le r,s\le n}} \big((2r)^s s!\big)^{P(n-rs)-P(n-r(s+1))} ,$$

and

$$h_{r,s}(c) = \tfrac{1}{48}\big((13-c)(r^2+s^2) + \sqrt{(c-1)(c-25)}(r^2-s^2) - 24rs - 2 + 2c\big) .$$

Examples.

- $V(1,h) = M(1,h)$ if and only if $h \neq \frac{m^2}{4}$, $m \in \mathbb{Z}$.
- $V(0,h) = M(0,h)$ if and only if $h \neq \frac{m^2-1}{24}$, $m \in \mathbb{Z}$.

Exercise 8.4 (Unitarity)**.** Show that $\det_n(c,h) > 0$ for all n if $c > 1$ and $h > 0$.

Note that $\langle L_{-n}.v, L_{-n}.v\rangle = 2nh + c\frac{n^3-n}{12}$, so considering large enough n we see that unitarity implies that $c \geq 0$.

We saw that unitarity implies $c \geq 0$ and $h \geq 0$. If we restrict to the region $c \geq 1$ and $h \geq 0$, then, since $\det_n(c,h) > 0$, the form is positive semi-definite in this region (i.e. it is positive-definite for $V(c,h)$) if it is so *at least once* in this region. But we already found one unitary irreducible highest-weight representation of Vir with $h = \frac{\mu^2}{4}$, $c = 1$, namely $B'(\mu, 1)$.

For $c = 0$, it can be shown that the only unitary representation is $V(0,0) \cong \mathbb{C}$. We want to understand what happens in the range $0 \le c < 1$. Reparametrise as follows:

$$c(m) := 1 - \frac{6}{(m+2)(m+3)} \quad \text{for } m \ge 0.$$

Then

$$h_{r,s}\big(c(m)\big) \equiv h_{r,s}(m) = \frac{\big((m+3)r - (m+2)s\big)^2 - 1}{4(m+2)(m+3)} \, .$$

Theorem 8.5 (Friedan, Qiu, Shenker; Goddard, Kent, Olive)**.** *In the region* $0 \le c < 1$, *unitary representations occur precisely at*

$$\big((c(m), h_{r,s}(m)\big) \;\; \textit{for} \;\; m, r, s \in \mathbb{Z}_{\ge 0} \;\; \textit{such that} \;\; 1 \le s \le r \le m+1 \, .$$

9 A little infinite-dimensional surprise

9.1 Fermionic Fock

Let $V = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} v_n$. Then $\mathfrak{gl}(V) =: \mathfrak{gl}_\infty$ is the Lie algebra of matrices with a finite number of non-zero entries. It is spanned by the $E_{i,j} \colon v_j \mapsto v_i$. Let

$$F^{(0)} := \Lambda^{\infty}_{(0)} V = \operatorname{span}\Big\{ v_{\mathbf{i}} := v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge v_{i_{m-1}} \wedge v_{-m} \wedge v_{-m-1} \wedge \cdots \,:\, i_0 > i_1 > \cdots i_{m-1} > -m \Big\} \, .$$

Then $\mathfrak{gl}_\infty$ acts on $F^{(0)}$ via

$$A.v_{\mathbf{i}} = A.v_{i_0} \wedge v_{i_2} \wedge \cdots + v_{i_1} \wedge A.v_{i_2} \wedge \cdots + \cdots \, ,$$

(note that this action preserves tails). In detail,

$$\text{if } i \neq j, \text{ then } E_{i,j}.v_{\mathbf{i}} = \begin{cases} 0 & \text{if } j \text{ does not appear in } \mathbf{i} \text{ or if } i \text{ appears in } \mathbf{i}, \\ \text{replace } v_j \text{ with } v_i \text{ in } v_{\mathbf{i}} & \text{otherwise,} \end{cases}$$

$$\text{and } E_{j,j}.v_{\mathbf{i}} = \begin{cases} v_{\mathbf{i}} & \text{if } j \text{ appears in } \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases}$$

The space $F^{(0)}$ has a basis labelled by partitions $\lambda \vdash n$, where $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots) \in \mathbb{Z}^{\infty}_{\ge 0}$ with $\sum_i \lambda_i = n$. To see this, map

$$\mathbf{i} = \big(i_0, \dots, i_m, \dots\big) \mapsto \lambda_{\mathbf{i}} := \big(i_0, i_1 + 1, i_2 + 2, \dots, i_m + m, \dots\big) \, ,$$

and this map is clearly reversible. It follows that $F^{(0)} = \operatorname{span}\{v_\lambda\}$.

Define $\deg(v_\lambda) := |\lambda| = n$ if $\lambda \vdash n$, so that $|\lambda|$ is the number of which λ is a partition; i.e. $\deg(v_{\mathbf{i}}) = \sum_{s\ge 0}(i_s + s)$.

More generally, we have $F^{(n)}$, which is defined as for $F^{(0)}$, but with the basic vector being $v_n \wedge v_{n-1} \wedge \cdots$ instead of $v_0 \wedge v_{-1} \wedge \cdots$. The Lie algebra $\mathfrak{gl}_\infty$ acts naturally on it.

Remark 9.1 (Etymology)**.** The notation $F^{(n)}$ comes from the term “Fermionic Fock”. The earlier notation $B(\mu, h)$ for Heisenberg representations stands for “Bosonic”.

9.2 Central extensions of $\mathfrak{gl}_\infty$

Let $\mathfrak{a}_\infty \supset \mathfrak{gl}_\infty$ be the algebra of matrices with a finite number of non-zero "diagonals". This contains a big abelian subalgebra spanned by Λ_k for $k \in \mathbb{Z}$, where $\Lambda_k := \sum_{i\in\mathbb{Z}} E_{i,i+k}$, so that $\Lambda_k.v_j = v_{j-k}$ for all j.

Exercise 9.2. Show that the elements Λ_k commute with each other.

Exercise 9.3. Consider the representation $V_{\alpha,\beta}$ of the Witt algebra. Show that the Witt algebra embeds in $\mathfrak{a}_\infty$ using this.

A typical element of $\mathfrak{a}_\infty$ is a finite linear combination of terms $\sum_{i\in\mathbb{Z}} \lambda_i E_{i,i+k}$. If $k \neq 0$, then $\sum_{i\in\mathbb{Z}} \lambda_i E_{i,i+k}$ acts on $F^{(0)}$, because for $i \gg 0$ or $i \ll 0$ we have an action by 0 for $E_{i,i+k}$ on $v_{\mathbf{i}}$.

However, if $k = 0$, then we would have $\sum_{i\in\mathbb{Z}} \lambda_i E_{i,i}.v_{\mathbf{j}} = \sum_{s\geq 0} \lambda_{j_s} v_{\mathbf{j}}$, which is not well defined! To remedy the situation, we adjust the "action" as follows:

- $\widehat{E}_{i,j}$ acts as $E_{i,j}$ for $i \neq j$.
- $\widehat{E}_{i,i}$ acts as $E_{i,i}$ for $i \geq 0$, and as $E_{i,i} - 1$ for $i < 0$.

For example,

$$\widehat{E}_{i,i}.v_{\mathbf{j}} = \begin{cases} v_{\mathbf{j}} & \text{if } i \geq 0 \text{ and } j_s = i \text{ for some } s. \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 9.4. Show that the $\widehat{E}_{i,j}$ differ from $E_{i,j}$ only by a multiple of I, so commutators are unchanged. Show further that

$$\begin{aligned} &[\widehat{E}_{i,j}, \widehat{E}_{k,l}] = 0 \text{ for } j \neq k,\ i \neq l, && [\widehat{E}_{i,j}, \widehat{E}_{j,l}] = \widehat{E}_{i,l} \text{ for } i \neq l, \\ &[\widehat{E}_{i,j}, \widehat{E}_{k,i}] = -\widehat{E}_{k,j} \text{ for } k \neq j, && [\widehat{E}_{i,j}, \widehat{E}_{j,i}] = \widehat{E}_{i,i} - \widehat{E}_{j,j} + \alpha(\widehat{E}_{i,i}, \widehat{E}_{j,j})I\,, \end{aligned}$$

where

$$\alpha(\widehat{E}_{i,i}, \widehat{E}_{j,j}) := \begin{cases} -1 & \text{if } i \geq 0,\ j < 0, \\ +1 & \text{if } i < 0,\ j \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can say this in two ways: We either get a projective representation of $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$, or a genuine representation for a central extension of $\mathfrak{gl}_\infty$ and $\mathfrak{a}_\infty$:

$$\widehat{\mathfrak{a}_\infty} := \mathfrak{a}_\infty \oplus \mathbb{C}c\,,$$

a central extension with α as the defining 2-cocycle. As before, we define the diagonal elements $\widehat{\Lambda}_k := \sum_{i\in\mathbb{Z}} \widehat{E}_{i,i+k}$, and we get an action on $F^{(n)}$ via

$$\widehat{\Lambda}_0.v_{\mathbf{j}} = n v_{\mathbf{j}} \qquad \text{and} \qquad [\widehat{\Lambda}_k, \widehat{\Lambda}_l] = \delta_{k,-l} k I\,,$$

i.e. we get a representation of the Heisenberg algebra with a_0 acting as n and h acting as 1.

Lemma 9.5. *The representation $B(n,1)$ of the Heisenberg algebra is isomorphic to $F^{(n)}$ as graded $\mathscr{H}$-modules (but obviously* not *as algebras).*

Proof. Let $\phi(P(x_1, x_2, \ldots)) := P(\widehat{\Lambda}_{-1}, \widehat{\Lambda}_{-2}, \ldots).v_n \wedge v_{n-1} \wedge \cdots$, and check that

- this is an $\mathscr{H}$-map (using that $B(n,1)$ is irreducible to see injectivity), and that
- $P(\widehat{\Lambda}_{-1}, \widehat{\Lambda}_{-2}, \ldots).v_n \wedge v_{n-1} \wedge \cdots$ spans $F^{(n)}$. (Count dimensions of the homogeneous components on both sides.) □

Exercise 9.6. Recall that Witt $\hookrightarrow \mathfrak{a}_\infty$. Show that this embedding extends to an embedding Vir $\hookrightarrow \widehat{\mathfrak{a}_\infty}$. This depends on α and β; what is the central charge?

The isomorphism from Lemma 9.5 allows us to consider the module structure of one side acting on the other side. For instance, we see that the Virasoro algebra acts on $F^{(n)}$ as in Theorem 4.3. In particular, we have a distinguished isomorphism between $B(0,1) = \mathbb{C}[x_1, x_2, \dots]$ and $F^{(0)}$.

Exercise 9.7. Let $k \in \mathbb{Z}_{\geq 0}$ and $f_k = v_k \wedge v_{k-1} \wedge \cdots \wedge v_1 \wedge v_{-k} \wedge v_{-k-1} \wedge \cdots$. Show that $L_0.f_k = k^2 f_k$ and $L_j.f_k = 0$ for all $j > 0$. Deduce that $B(0,1)$ is not irreducible as a Virasoro module, and that it is in fact a direct sum of infinitely many unitary irreducible representations.

10 Hands-on loop and affine algebras

Let $\mathfrak{g}$ be any complex Lie algebra and and R any commutative algebra over $\mathbb{C}$. Then the vector space $\mathfrak{g} \otimes_{\mathbb{C}} R$ is a Lie algebra via

$$\big[X \otimes r, Y \otimes s\big] := \big[X, Y\big] \otimes rs\,.$$

Thus we have a rich source of infinite-dimensional Lie algebras. (We could also try to consider a sheafified version by replacing R with some $\mathscr{O}_X$-algebra.)

In the case $R = \mathbb{C}\big[t, t^{-1}\big]$, the Laurent polynomials in one variable, we get $\mathscr{L}\mathfrak{g} := \mathfrak{g} \otimes R$, the *loop algebra of* $\mathfrak{g}$. This has a description as $C_{\mathrm{alg}}\big(\mathbb{C}^\times, \mathfrak{g}\big)$ or as $C_{\mathrm{sf}}\big(S^1, \mathfrak{g}\big)$, where "$C_{\mathrm{sf}}$" stands for the space of smooth maps with finite Fourier expansion, and the correspondence is between $\psi\colon S^1 \to \mathfrak{g}$ and its Fourier coefficients in $\psi = \sum_{n\in\mathbb{Z}} e^{2\pi i n\theta} X_n$. Either way, a basis $\{X_a\}$ for $\mathfrak{g}$ determines a basis

$$\big\{X_a(n) := X_a \otimes t^n\big\}$$

for $\mathscr{L}\mathfrak{g}$, and the Lie algebra structure is

$$\big[X_a(n), X_b(m)\big] = \big[X_a, X_b\big](n+m)\,.$$

Review. A finite-dimensional Lie algebra $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and $\mathfrak{g}$ has no proper non-zero ideals (i.e. $0 \neq I \subsetneq \mathfrak{g}$ such that $\big[\mathfrak{g}, I\big] \subseteq I$). Recall that $\mathfrak{g}$ acts on itself via the *adjoint action* $\mathrm{ad}(x).y := [x,y]$ for all $x, y \in \mathfrak{g}$. Then $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is itself irreducible under the adjoint representation.

Lemma 10.1. *If $\mathfrak{g}$ is simple, then there exists an invariant bilinear form on $\mathfrak{g}$, unique up to scaling. Here invariance means*

$$B\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}\,,\ \text{such that}\ B\big([x,y]\otimes z\big) = B\big(x \otimes [y,z]\big)\,, \tag{10.1}$$

which is just the infinitesimal (i.e. Lie algebra) version of group invariance.

Proof. Let M, N be representations of $\mathfrak{g}$; then $M \otimes N$ is a representation via

$$x.(m\otimes n) := x.m \otimes n + m \otimes x.n\,,$$

and $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is a representation via

$$(x.f)(m) := f(-x.m)\ \text{ for } f \in M^*\,,\ m \in M\,.$$

Define

$$M^{\mathfrak{g}} := \big\{m \in M : x.m = 0 \text{ for all } x \in \mathfrak{g}\big\}\,.$$

The adjoint representation induces via tensor product a representation on $\mathfrak{g}\otimes\mathfrak{g}$, and then by taking duals on $(\mathfrak{g}\otimes\mathfrak{g})^*$, the space of bilinear maps on $\mathfrak{g}$. Then Equation (10.1) states $B\big([y,x]\otimes z\big)+B\big(x\otimes[y,z]\big)=0$ for all $x,y,z\in\mathfrak{g}$, which is equivalent to $y.B=0$ for all y. Hence

$$B\in\big((\mathfrak{g}\otimes\mathfrak{g})^*\big)^{\mathfrak{g}}\cong\operatorname{Hom}_{\mathfrak{g}}\big(\mathfrak{g}\otimes\mathfrak{g},\mathbb{C}_{\text{triv.}}\big)\cong\operatorname{Hom}_{\mathfrak{g}}\big(\mathfrak{g},\mathfrak{g}^*\big)\cong\begin{cases}\mathbb{C} & \text{if } \mathfrak{g}\cong\mathfrak{g}^* \text{ by Schur's Lemma,}\\ 0 & \text{otherwise.}\end{cases}\qquad\square$$

Now here is an explicit form:

$$K(x,y)=\operatorname{tr}\big(\operatorname{ad}x\circ\operatorname{ad}y\big),\text{ the \textit{Killing form}.}$$

The Killing form in invariant by the Jacobi identity, symmetric by definition, and non-zero (by some theory which boils down to the fact that simple Lie algebras are not nilpotent).

Theorem 10.2 (Garland – I think)**.**
$$H^2\big(\mathscr{L}\mathfrak{g};\mathbb{C}\big)=\mathbb{C}\,.$$

Exercise 10.3. Prove the theorem.

Let $d\in\operatorname{Der}(R)$, i.e. $d(rs)=d(r)s+rd(s)$. This extends to $\mathfrak{g}\otimes R$ via $d(X\otimes r):=X\otimes d(r)$, and thus

$$d\big([X\otimes r,Y\otimes S]\big)=\big[d(X\otimes r),Y\otimes s\big]+\big[X\otimes r,d(Y\otimes s)\big]=\big[X\otimes dr,Y\otimes s\big]+\big[X\otimes r,Y\otimes ds\big]\,.$$

Hence we get a new Lie algebra

$$(\mathfrak{g}\otimes R)\rtimes\mathbb{C}d\,,\text{ where }\big[X\otimes r+\mu d,Y\otimes s+\lambda d\big]=\big[X,Y\big]\otimes rs+\mu Y\otimes ds-\lambda X\otimes dr\,.$$

By applying this to the loop algebra, we get the new algebra

$$\overline{\mathscr{L}\mathfrak{g}}:=\mathscr{L}\mathfrak{g}\rtimes\mathbb{C}d\,,$$

where $d:=t\frac{d}{dt}$. (The derivation tells us about degrees.)

Lemma 10.4. *$H^2\big(\overline{\mathscr{L}\mathfrak{g}};\mathbb{C}\big)=\mathbb{C}$, so there exists a unique (up to scalar) central extension of $\overline{\mathscr{L}\mathfrak{g}}$. It is given by vanishing on d and*

$$\beta(X\otimes f,Y\otimes g)=\operatorname{Res}_{t=0}\big(f\,dg\big)\,K(X,Y)\,.$$

Remark 10.5. This produces a central extension which restricts to the central extension arising from Theorem 10.2.

Proof. We need $\beta\colon\overline{\mathscr{L}\mathfrak{g}}\otimes\overline{\mathscr{L}\mathfrak{g}}\to\mathbb{C}$ which satisfies anti-symmetry and the Jacobi relations, modulo linear functionals.

We know that $\overline{\mathscr{L}\mathfrak{g}}$ acts on $(\overline{\mathscr{L}\mathfrak{g}}\otimes\overline{\mathscr{L}\mathfrak{g}})^*$, and a check shows that this descends to an action of $\overline{\mathscr{L}\mathfrak{g}}$ on $H^2\big(\overline{\mathscr{L}\mathfrak{g}};\mathbb{C}\big)$. We get decompositions into d-eigenspaces

$$\overline{\mathscr{L}\mathfrak{g}}=\bigoplus_{i\in\mathbb{Z}}\overline{\mathscr{L}\mathfrak{g}}_i\,,\text{ where }\overline{\mathscr{L}\mathfrak{g}}_i=\big\{X\in\overline{\mathscr{L}\mathfrak{g}}:[d,X]=iX\big\}\,,\text{ and hence}$$
$$H^2\big(\overline{\mathscr{L}\mathfrak{g}};\mathbb{C}\big)=\bigoplus_{i\in\mathbb{Z}}H^2_i\,.$$

Claim. $H^2_i=0$ if $i\neq0$. (In fact, generalising what we're about to do, it is simply better to see that $\overline{\mathscr{L}\mathfrak{g}}$ acts trivially on H^2.) To see this, note that

$$\beta\big(d,[x(i),y(j)]\big)=-\beta\big(y(j),[d,x(i)]\big)-\beta\big(x(i),[y(j),d]\big)=(i+j)\beta\big(x(i),y(j)\big)\,.$$

So for $i \neq 0$, setting $F\big(a(i)\big) = \frac{1}{i}\beta\big(d, a(i)\big)$ ensures that $H_i^2 = 0$. //

Now $(\overline{\mathscr{L}\mathfrak{g}})_0 = \mathfrak{g} \otimes 1 \oplus \mathbb{C}d$, hence $H^2\big((\overline{\mathscr{L}\mathfrak{g}})_0; \mathbb{C}\big) = 0$ by (a generalisation of) Exercise 2.2. So we can ensure that $\beta\big((\overline{\mathscr{L}\mathfrak{g}})_0, (\overline{\mathscr{L}\mathfrak{g}})_0\big) = 0$.

Define $\beta_i \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ by $\beta_i(x, y) = \beta\big(x(i), y(-i)\big)$. Then $\mathfrak{g} \otimes t^n$ is a $(\mathfrak{g} \otimes 1)$-representation (via the adjoint representation). The Jacobi identity for β implies that β_i is invariant. Therefore $\beta_i = \lambda_i K$, and by anti-symmetry $\lambda_i = -\lambda_{-i}$. The Jacobi identity again implies that $\lambda_{i+1} = \lambda_i + \lambda_1$. Therefore

$$\beta\big(x(i), y(-i)\big) = -\lambda i K(x, y)$$

for some scalar λ, which gives the desired unique central extension. □

Definition 10.6. We call $\mathscr{L}\mathfrak{g} \oplus \mathbb{C}c$ the *affine Lie algebra*, and $\mathscr{L}\mathfrak{g} \oplus \mathbb{C}d \oplus \mathbb{C}c$ the *affine Kac-Moody Lie algebra*.

Remark 10.7 (Relation to the Virasoro algebra)**.** In the central extension $\overline{\mathscr{L}\mathfrak{g}}$ we have the constituent algebra $\mathbb{C}\big[t^{\pm 1}\big]$. However, natural constructions should be independent of the parameter t, so we expect an action of the Virasoro algebra on $\overline{\mathscr{L}\mathfrak{g}}$.

11 Simple Lie algebras

Consider the algebra

$$\mathfrak{sl}_{n+1} \equiv \mathfrak{sl}(n+1; \mathbb{C}) := \Big\{ X \in \mathsf{Mat}_{n+1}(\mathbb{C}) : \operatorname{tr} X = 0 \Big\} .$$

In the classification of simple Lie algebras, this is of type A_n. This Lie algebra has Lie subalgebras $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ of diagonal, strictly upper-triangular and strictly lower-triangular matrices, respectively, and there is a decomposition $\mathfrak{sl}_{n+1} \cong \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a direct sum of vector spaces, where each summand is actually a subalgebra.

Note that for $i = 1, \dots, n$ there are elements

$$\begin{aligned} e_i &= E_{i,i+1} , \\ h_i &= E_{i,i} - E_{i+1,i+1} , \\ f_i &= E_{i+1,i} , \end{aligned}$$

which generate the Lie algebra (e.g. $E_{i,j} = [\cdots[[e_i, e_{i+1}], e_{i+2}]\cdots, e_{j-1}]$ for $i < j$). Observe that the following relations are obviously satisfied:

$$\begin{gathered} \text{For all } i, j, \quad [h_i, h_j] = 0 \ , \quad [h_j, e_i] = \alpha_i(h_j) e_i \ , \text{ and } \ [h_j, f_i] = -\alpha_i(h_j) f_i \ , \\ \text{and} \quad [e_i, f_j] = 0 \text{ for } i \neq j, \quad [e_i, f_i] = h_i \text{ for all } i, \end{gathered} \tag{11.1}$$

where

$$\alpha_i(h_j) = \begin{cases} 2 & i = j , \\ -1 & |i - j| = 1 , \\ 0 & \text{otherwise.} \end{cases}$$

Less obviously, we have

$$\big[e_i, e_j\big] = 0 \text{ if } \big|i - j\big| \neq 1 \ , \quad \big[e_i, [e_i, e_{i+1}]\big] = 0 = \big[e_{i+1}, [e_{i+1}, e_i]\big] \text{ for all } i.$$

We write this concisely as $\big(\operatorname{ad}(e_i)\big)^{1-\alpha_i(h_j)}(e_j) = 0$ (for $i \neq j$). There are completely similar relations for the f_i.

Exercise 11.1. Find similar generators and relations for the Lie algebras

$$\mathfrak{g}(M) := \left\{X \in \mathsf{Mat}_k(\mathbb{C}) : X^T M + MX = 0\right\}, \text{ where}$$

1. $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ for $k = 2n+1$,

2. $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ for $k = 2n$, and

3. $M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ for $k = 2n$.

Definition 11.2. The matrix $A := \left(\alpha_i(h_j)\right)_{ij}$ is called the *Cartan matrix*, and it determines finite-dimensional simple Lie algebras completely.

It is known from the theory of finite-dimensional simple complex Lie algebras that the Cartan matrix satisfies the following properties:

- $A \in \mathsf{Mat}_n(\mathbb{Z})$ and A is indecomposable (i.e. not block-diagonal with more than one block);
- $A_{ii} = 2$ for all i;
- $A_{ij} \in \{0, -1, -2, -3\}$ for all $i \neq j$;
- $A_{ij} = 0$ if and only if $A_{ji} = 0$;
- if $A_{ij} = -2$ or -3, then $A_{ji} = -1$.

These conditions imply that A is "non-degenerate" in appropriate sense which we will meet soon; in particular this includes $\det A \neq 0$.

12 Kac-Moody Lie algebras

Definition 12.1. We relax the conditions on A slightly and say that $A \in \mathsf{Mat}_n(\mathbb{Z})$ is a *generalised Cartan matrix* (GCM) if

- $A_{ii} = 2$,
- $A_{ij} \leq 0$ if $i \neq j$, and
- $A_{ij} = 0$ if and only if $A_{ji} = 0$.

From the data of a GCM we now construct a complex Lie algebra in three steps. If the GCM is actually a Cartan matrix, this will produce the associated finite-dimensional simple Lie algebra.

12.1 Step I: Realisations

Definition 12.2. Let $A \in \mathsf{Mat}_n(\mathbb{C})$. A *realisation* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where

- $\mathfrak{h}$ is a finite-dimensional vector space over $\mathbb{C}$,
- $\Pi^\vee = \{h_1, \ldots, h_n\} \subset \mathfrak{h}$ is a linearly independent subset, and
- $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ is a linearly independent subset,

such that $\alpha_j(h_i) = A_{ij}$.

Lemma 12.3. *If* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *is a realisation of* A*, then* $\dim \mathfrak{h} \geq 2n - \operatorname{rk} A$.

Proof. Let $r = \operatorname{rk} A$ and $\dim \mathfrak{h} = m$. Extend Π and $\Pi^\vee$ to bases $\{\alpha_1, \dots, \alpha_m\}$ and $\{h_1, \dots, h_m\}$ respectively, producing an invertible matrix

$$\big(\alpha_j(h_i)\big) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The submatrix of the first n rows, $\big(A \mid B\big)$, has rank n since the whole matrix is invertible, and A already has r linearly independent columns. So $\operatorname{rk} B \geq n - r$. But B has $m - n$ columns, so $m - n \geq n - r$. $\square$

Definition 12.4. A *minimal realisation of* A is a realisation of A such that $\dim \mathfrak{h} = 2n - \operatorname{rk} A$.

Example. Let $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Then $\operatorname{rk} A = 1$, so a minimal realisation must have $\dim \mathfrak{h} = 3$. So let $\mathfrak{h} = \mathbb{C}^3$ and $\mathfrak{h}^* = \mathbb{C}^3$, with respective bases $\{h_0, h_1, d\}$ and $\{\alpha_0, \alpha_1, \gamma\}$:

$$\begin{aligned} h_0 &= (0, 2, 0) & \qquad \alpha_0 &= (-1, 1, 1) \\ h_1 &= (1, -1, 0) & \alpha_1 &= (1, -1, 0) \\ d &= (0, 0, 1) & \gamma &= (\tfrac{1}{2}, \tfrac{1}{2}, 0) \end{aligned}$$

(This is associated to the Lie algebra $\widehat{\mathfrak{sl}}_2$, to which we will return later.)

Proposition 12.5. *Any square matrix has a minimal realisation, and any two minimal realisations are isomorphic.*

Remark 12.6. An isomorphism of realisations is $\Phi\colon (\mathfrak{h}, \Pi, \Pi^\vee) \to (\mathfrak{h}', \Pi', \Pi'^\vee)$, where $\Phi\colon \mathfrak{h} \to \mathfrak{h}'$ is an isomorphism such that $\Phi(h_i) = h_i'$ and $\Phi^*(\alpha_i') = \alpha_i$ for all i.

Proof of 12.5. Let $r = \operatorname{rk} A$, so that possibly after re-ordering

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with A_{11} non-singular. Extend this to a $(2n - r) \times (2n - r)$-matrix

$$C = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-r} \\ 0 & I_{n-r} & 0 \end{pmatrix}.$$

This is non-singular, e.g. $\det C = \pm \det A_{11}$. Let $\alpha_1, \dots, \alpha_n$ be the first n coordinate functions, and let $h_1, \dots, h_n$ be the first n rows of C. This is a minimal realisation. The proof of uniqueness is left as an exercise. $\square$

Exercise 12.7. Complete the proof of Proposition 12.5, i.e. show that any two minimal realisations of a square matrix A are isomorphic.

12.2 Step II: A big Lie algebra

Let $A \in \mathsf{Mat}_n(\mathbb{Z})$ be a generalised Cartan matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ a minimal realisation of A, where $\Pi^\vee = \left(\alpha_1, \ldots, \alpha_n\right)$. Let

$$X := \left\{e_1, \ldots, e_n, f_1, \ldots, f_n, \widetilde{x} : x \in \mathfrak{h}\right\},$$

The *free Lie algebra $L(X)$ generated by X* is defined as follows: Let

$$\mathbb{C}\langle X \rangle := \mathbb{C}\langle e_1, \ldots, e_n, f_1, \ldots, f_n, \widetilde{x} \rangle$$

be the free associative algebra generated by X. Then $\mathbb{C}\langle X\rangle_{\mathrm{ad}}$ is a Lie algebra (under commutators), and $L(X)$ is the Lie subalgebra of $\mathbb{C}\langle X\rangle_{\mathrm{ad}}$ generated by the elements of X. Its elements are called *Lie words*. (More generally, any Lie algebra is spanned by Lie words in a set of generators.)

Definition 12.8. Let $\widetilde{L}(A) := L(X)/\langle R\rangle$, where R is the set of relations modelled on those in (11.1). Specifically, R consists of

- $\widetilde{x} - \lambda\widetilde{y} - \mu\widetilde{z}$ whenever $x = \lambda y + \mu z$ in $\mathfrak{h}$, for $x, y, z \in \mathfrak{h}$,
- $\left[\widetilde{x}, \widetilde{y}\right]$ whenever $x, y \in \mathfrak{h}$,
- $\left[e_i, f_j\right]$ for $i \neq j$, $\left[e_i, f_i\right] - \widetilde{h}_i$ for all $i = 1, \ldots, n$,
- $\left[\widetilde{x}, e_i\right] - \alpha_i(x)e_i$ and $\left[\widetilde{x}, f_i\right] + \alpha_i(x)f_i$ for all $i = 1, \ldots, n$.

Remark 12.9. The Lie algebra $\widetilde{L}(A)$ is independent of the choice of minimal realisation of A thanks to Proposition 12.5.

Theorem 12.10 (Structure of $\widetilde{L}(A)$)**.**

1. *$\widetilde{L}(A) = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+$, where $\widetilde{\mathfrak{h}} \cong \mathfrak{h}$ is abelian and $\widetilde{\mathfrak{n}}_-$, $\widetilde{\mathfrak{n}}_+$ are **freely** generated by $f_1, \ldots, f_n$, $e_1, \ldots, e_n$, respectively.*
2. *$\widetilde{L}(A) = \bigoplus_{\alpha \in Q} \widetilde{L}_\alpha$, where*
$$\widetilde{L}_\alpha := \left\{x \in \widetilde{L}(A) : \left[\widetilde{h}, x\right] = \alpha\left(\widetilde{h}\right)x \text{ for all } \widetilde{h} \in \widetilde{\mathfrak{h}}\right\}$$
and $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n \subset \mathfrak{h}^*$.
3. *$\widetilde{L}_0 = \widetilde{\mathfrak{h}}$.*
4. *$\widetilde{L}_\alpha = 0$ unless $\alpha \in Q_+$ or $\alpha \in Q_-$, where $Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n$, and analogously for Q_-.*
5. *$\left[\widetilde{L}_\alpha, \widetilde{L}_\beta\right] \subseteq \widetilde{L}_{\alpha+\beta}$.*

Proof. Define $\widetilde{\omega}\colon e_i \leftrightarrow -f_i$ and $\widetilde{x} \leftrightarrow -\widetilde{x}$. Then $\widetilde{\omega}$ induces a Lie algebra involution on $\widetilde{L}(A)$ (just check it preserves the relations). Let $\widetilde{\mathfrak{n}}_\pm$ be the space generated by respectively e_i and f_i for $i = 1, \ldots, n$, and $\widetilde{\mathfrak{h}}$ the space spanned by elements $\widetilde{x}$ for all $x \in \mathfrak{h}$. The involution swaps $\widetilde{\mathfrak{n}}_+$ and $\widetilde{\mathfrak{n}}_-$.

For $\lambda \in \mathfrak{h}^*$, let

$$\theta_\lambda\colon X \to \mathrm{End}\big(T(V)\big), \text{ where } V = \mathrm{span}\left\{v_1, \ldots, v_n\right\},$$

be given by

$$\begin{aligned}
\widetilde{x} \quad &\mapsto \quad \left(v_{i_1}\cdots v_{i_s} \mapsto (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_s})(x)v_{i_1}\cdots v_{i_s}\right), \\
f_j \quad &\mapsto \quad \text{multiplication by } v_j, \\
e_j \quad &\mapsto \quad \begin{cases} 1 & \mapsto 0 \\ v_i & \mapsto \delta_{ij}\lambda(h_j).1 \\ v_{i_1}\cdots v_{i_s} & \mapsto \delta_{i_1 j}(\lambda - \alpha_{i_2} - \cdots - \alpha_{i_s})(h_j)v_{i_2}\cdots v_{i_s} + v_{i_1}\big(\theta_\lambda(e_j)(v_{i_2}\cdots v_{i_s})\big) \end{cases}
\end{aligned}$$

It is an exercise to check that this induces a Lie algebra representation of $\widetilde{L}(A)$ (again, it induces a representation of the free Lie algebra generated by X; now check the relations). So in fact we get an (abusively denoted) map $\theta_\lambda \colon \widetilde{L}(A) \to \operatorname{End}\big(T(V)\big)$.

We can now prove part (1) of the theorem.

- $\mathfrak{h} \cong \widetilde{\mathfrak{h}}$: There is a natural surjective Lie algebra homomorphism $x \mapsto \tilde{x}$. If $\tilde{x} = 0$ in $\widetilde{L}(A)$, we would have $\theta_\lambda(\tilde{x}) = 0 \in \operatorname{End}\big(T(V)\big)$ for all $\lambda \in \mathfrak{h}^*$; but $\theta_\lambda(\tilde{x})(1) = \lambda(x)$. This can be 0 for all λ only if $x = 0$.

- $\tilde{\mathfrak{n}}_-$ is free: Let $\phi(w) = \theta_\lambda(w).1$ (by construction this is independent of λ), i.e.

  $$\phi\big(w(f_1,\dots,f_n)\big) = w(v_1 \cdots v_n)\ .$$

  So we get a surjective Lie algebra homomorphism $\phi \colon \tilde{\mathfrak{n}}_- \to L(V)$ onto the free Lie algebra generated by V; it has an inverse which sends v_i to f_i.

- $\tilde{\mathfrak{n}}_+$ is free: Apply $\tilde{\omega}$ to $\tilde{\mathfrak{n}}_-$ to see that $\tilde{\mathfrak{n}}_+$ is free.

- $\tilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}} + \tilde{\mathfrak{n}}_+$ is a direct sum: Let $w_- + \tilde{x} + w_+ = 0$. Then $\theta_\lambda(w_-) + \theta_\lambda(\tilde{x}) + \theta_\lambda(w_+) = 0$ for all $\lambda \in \mathfrak{h}^*$. Evaluating this at $1 \in T(V)$ gives $\phi(w_-) + \lambda(x) = 0$, where $\phi(w_-) \in T(V)_{>0}$ and $\lambda(x) \in T(V)_0$. Thus $\lambda(x) = 0$ and $\phi(w_-) = 0$, which by the arguments above implies that $\tilde{x} = 0$ and $w_- = 0$. It then follows that $w_+ = 0$, too.

- So $\tilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+ \subseteq \widetilde{L}(A)$. To see that it is all of $\widetilde{L}(A)$, it is enough to check that it is closed under taking brackets since it contains the generating set X. In other words, we need

  $$\operatorname{ad}(e_i)\big(\tilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+\big) \subset \tilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+\ ,$$

  and similarly for f_i and $\tilde{x}$. We deal only with the e_i claim, the rest are similar. It is easy to see for the direct summands $\widetilde{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}_+$. For $\tilde{\mathfrak{n}}_-$ we have

  $$\operatorname{ad}(e_i)(f_j) = \delta_{ij}\widetilde{h}_j \in \widetilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_-\ .$$

  By induction,

  $$\operatorname{ad}(e_i)\big[w_1(f_1,\dots,f_n), w_2(f_1,\dots,f_n)\big] = \big[\operatorname{ad}(e_i)(w_1), w_2\big] + \big[w_1, \operatorname{ad}(e_i)w_2\big]\ ,$$

  which completes part (1).

The further parts (2)–(5) are straight-forward: $\tilde{\mathfrak{n}}_+$ consists of Lie words in the e_i's, and so $\tilde{x}$ acts on $\big[e_{i_1},[e_{i_2},\dots]\big]$ by multiplication by $\big(\alpha_{i_1} + \alpha_{i_2} + \cdots\big)(x)$. Thus

$$\tilde{\mathfrak{n}}_+, \tilde{\mathfrak{n}}_-, \widetilde{\mathfrak{h}} \subseteq \sum_{\alpha \in Q} \widetilde{L}_\alpha\ \text{, and so}\ \sum_{\alpha \in Q} \widetilde{L}_\alpha = \widetilde{L}(A)\ .$$

Parts (3)–(5) follow immediately. □

12.3 Step III: A smaller Lie algebra

We want to get close to a simple without losing the information carried in A. This means that we would like to factor out ideals, but preserve $\mathfrak{h} = \widetilde{L}_0$.

Lemma 12.11. *$\widetilde{L}(A)$ contains a unique ideal I that is maximal with respect to the condition $I \cap \mathfrak{h} = 0$.*

Proof. Let $\mathscr{J} = \{I \lhd \widetilde{L}(A) : I \cap \mathfrak{h} = 0\}$. Then for $I', I'' \in \mathscr{J}$, I' and I'' inherit the weight space decomposition by Lemma 5.4, so

$$I^{(\prime,\prime\prime)} = \bigoplus_{0 \neq \alpha \in Q} I_\alpha^{(\prime,\prime\prime)} .$$

Therefore $(I' + I'')_0 = 0$. So $\sum_{I \in \mathscr{J}} I$ is the desired ideal. □

Definition 12.12. Let $L(A) := \widetilde{L}(A)/I$, where I is the ideal from Lemma 12.11. We call $L(A)$ the *Kac-Moody Lie algebra associated to A.*

It is immediate that we have a decomposition (the so-called *triangular decomposition*

$$L(A) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- .$$

We call $\mathfrak{h}$ the *Cartan subalgebra.* It is also clear that $\widetilde{\omega}$ induces an involution ω on $L(A)$ which exchanges $\mathfrak{n}_+$ and $\mathfrak{n}_-$. Finally, $L(A)$ inherits the weight space decomposition

$$L(A) = \bigoplus_{\alpha \in Q} L_\alpha ,$$

where $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$ is the weight lattice. We collect a few obvious results:

Proposition 12.13.

1. *$\alpha_1, \ldots, \alpha_n$ are positive roots.*
2. $\dim L_{\alpha_i} = 1$.
3. *$k\alpha_i$ is a root if and only if $k = \pm 1$.* □

13 Classification of generalised Cartan matrices

For the duration of this section, let A denote a generalised Cartan matrix of size $n \times n$. To simplify matters, we shall only consider matrices up to permutation by S_n, where $A \sim A'$ if and only if $A_{ij} = A'_{\pi(i),\pi(j)}$ for $\pi \in S_n$. Furthermore we assume that A is indecomposable, i.e. if $A = A_1 \oplus A_2$, then at least one of A_1 and A_2 is zero.

Definition 13.1. Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. We say $v \geq 0$ if $v_i \geq 0$ for all i, and we say $v > 0$ if $v \geq 0$ and $v \neq 0$. We define $v \leq 0$ and $v < 0$ similarly.

13.1 Types of generalised Cartan matrices

Definition 13.2. Let A be a generalised Cartan matrix.

1. A has *finite type* if
   - $\det A \neq 0$,
   - there exists $v \geq 0$ such that $Av > 0$, and
   - if $Av \geq 0$, then $v \geq 0$.
2. A has *affine type* if
   - $\operatorname{corank}(A) = 1$,
   - there exists $v > 0$ such that $Av = 0$, and
   - if $Aw \geq 0$, then $w = \lambda v$ for some $\lambda \in \mathbb{C}$.

(Note that this says that there does not exist any w such that $Aw > 0$.)

3. A has *indefinite type* if

   - there exists $v > 0$ such that $Av < 0$, and
   - if $Av \geq 0$ and $v \geq 0$, then $v = 0$.

Theorem 13.3 (see Kac or Carter)**.** *An indecomposable generalised Cartan matrix A is either finite, affine or indefinite. Moreover,*

- *A is finite if and only if there exists $v > 0$ such that $Av > 0$,*
- *A is affine if and only if there exists $v > 0$ such that $Av = 0$, and*
- *A is indefinite if and only if there exists $v > 0$ such that $Av < 0$.* □

The conclusions of Theorem 13.3 hold under weaker assumptions. Let $A \in \mathsf{Mat}_n(\mathbb{R})$ such that $A_{ii} \geq 2$ for all i. For $J \subseteq \{1, \ldots, n\}$ let A_J be the submatrix of A whose entries are labelled by $J \times J$.

Lemma 13.4. *Assume that A is indecomposable. Then*

- *if A is finite, then A_J is finite, and*
- *if A is affine, then A_J is finite (for proper J).*

Proof. Let $J \subseteq \{1, \ldots, n\}$, so $J = (1, \ldots, m)$ after reordering. Let $P, Q \in \mathsf{Mat}(\mathbb{Z}_{\leq 0})$ such that we can write in block form

$$A = \begin{pmatrix} A_J & P \\ Q & R \end{pmatrix} .$$

If A is finite, then by definition there exists a $v > 0$ such that $Av > 0$, so

$$\begin{pmatrix} A \mid P \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \geq 0 .$$

But

$$P \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \leq 0 \ , \ \text{so} \ \ A_J \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} > 0 \ ,$$

and so A_J is finite. The proof for the affine case uses the same strategy. □

13.2 Symmetric generalised Cartan matrices

We continue to assume that all generalised Cartan matrices are indecomposable. In this subsection we consider the special case in which A is a symmetric matrix, i.e. $A^T = A$.

Proposition 13.5. *Let A be a symmetric generalised Cartan matrix.*

- *A is finite if and only if A is positive definite.*
- *A is affine if and only if A is positive semi-definite and of corank* 1.
- *A is indefinite precisely if it is neither finite nor affine.*

Proof. If A is finite, then by definition there exists $v > 0$ such that $Av > 0$, and so for all $\lambda \leq 0$, we have $(A - \lambda I)v > 0$. By the remark in Definition 13.2 (2), $(A - \lambda I)$ has finite type. **shouldn't it be affine type?** So $\det(A - \lambda I) \neq 0$, and so λ is not an eigenvalue of A. So A is a real, symmetric matrix all of whose eigenvalues are strictly positive, and hence A is positive definite.

Conversely, if A is positive definite, then $\det A \neq 0$, and so A is not affine. If A were indefinite, then there would exist $v > 0$ such that $Av < 0$, so that $v^T Av \leq 0$, contradicting positive definiteness. Hence A is finite.

For affinity, we can use a similar argument (exercise) to deduce positive semi-definiteness. The converse proof that a positive semi-definite matrix is affine is essentially the same as above. □

Definition 13.6. An $(n \times n)$-matrix A is *symmetrisable* if there exists a non-singular matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ and a symmetric matrix B such that $A = DB$.

Exercise 13.7. Find the smallest possible non-symmetric generalised Cartan matrix.

Lemma 13.8. *A generalised Cartan matrix A is symmetrisable if and only if*

$$A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{k-1} i_k} A_{i_k i_1} = A_{i_2 i_1} A_{i_3 i_2} \cdots A_{i_k i_{k-1}} A_{i_1 i_k}$$

for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

Proof. The "only if" direction is trivial. For the "if" direction, recall that A is assumed indecomposable. For each j, choose $1 = j_1, j_2, \ldots, j_t = j$ such that $A_{j_k j_{k+1}} \neq 0$ for all $k = 1, \ldots, t-1$. Let $0 \neq d_1 \in \mathbb{R}$ and define

$$d_j = \frac{A_{j_t j_{t-1}} \cdots A_{j_2 j_1}}{A_{j_1 j_2} \cdots A_{j_{t+1} j_t}} d_1 .$$

(**Exercise:** Check that this is well-defined, i.e. independent of the route from 1 to j.) Let $D := \operatorname{diag}(d_1, \ldots, d_n)$ and $B_{ij} := d_i^{-1} A_{ij}$. We need $B_{ij} = B_{ji}$, i.e.

$$\frac{A_{ij}}{d_i} = \frac{A_{ji}}{d_j} .$$

which is obvious if $A_{ij} = 0$. Assume thus that $A_{ij} \neq 0$, choose a sequence from 1 to i, i.e. $1 = j_1, \ldots, j_t = i$, and augment it by $j_{t+1} = j$. Then

$$d_j = \frac{A_{j j_t}\left(A_{j_t j_{t-1}} \cdots A_{j_2 j_1}\right)}{\left(A_{j_1 j_2} \cdots A_{j_{t-1} j_t}\right) A_{j_t j}} d_1 = \frac{A_{ji}}{A_{ij}} d_1 \qquad \text{and} \qquad d_1 = \frac{A_{j j_t}}{A_{j_t j}} d_i . \qquad \square$$

the indexes seem wrong...

Remark 13.9. The proof of Lemma 13.8 shows that without loss of generality, $d_i > 0$ for all i, and that $B \in \mathsf{Mat}_n(\mathbb{Q})$.

Proposition 13.10. *If A is indecomposable and $v > 0$ is such that $Av > 0$, then $v_i > 0$ for all i.* ***This proposition should move up, but where?***

Proof. After reordering, $v = (v_1, \ldots, v_m, 0, \ldots, 0)$ and $v_i > 0$ for $i = 1, \ldots, m$. So if $Av > 0$, then $\sum_{j=1}^n A_{ij} v_j \geq 0$ for all i (recall that $A_{ij} < 0$ if $i \neq j$). □

Proposition 13.11. *If A is a finite or affine generalised Cartan matrix, then A is symmetrisable.*

Proof. Suppose there exist $A_{i_1 i_2} \neq 0$, $A_{i_2 i_3} \neq 0$, $\ldots$, $A_{i_{k-1} i_k} \neq 0$, $A_{i_k i_1} \neq 0$ (where $k \geq 3$ for non-triviality). Minimality means that

$$A_{i_s i_t} = 0 \text{ for all } (s,t) \notin \left\{(1,2),(2,3),\ldots,(k,1),(2,1),(3,2),\ldots,(1,k)\right\} .$$

For $J = \{i_1,\ldots,i_k\}$, A_J has the form

$$A_J = \begin{pmatrix} 2 & -r_1 & \ldots & -s_k \\ -s_1 & \ddots & 0 & \vdots \\ \vdots & 0 & 2 & -r_{k-1} \\ -r_k & \ldots & -s_{k-1} & 2 \end{pmatrix} ,$$

where $r_i, s_i > 0$, and this matrix is either finite or affine, i.e. there exists $v = (v_1,\ldots,v_k) > 0$ such that $A_J v \geq 0$. Let

$$M := \mathrm{diag}(v)^{-1} A_J \,\mathrm{diag}(v) = \begin{pmatrix} 2 & -r'_1 & \ldots & -s'_k \\ -s'_1 & \ddots & 0 & \vdots \\ \vdots & 0 & 2 & -r'_{k-1} \\ -r'_k & \ldots & -s'_{k-1} & 2 \end{pmatrix}$$

where

$$\begin{aligned} r'_i &= v_i^{-1} r_i v_{i+1} > 0 \\ s'_i &= v_{i+1}^{-1} s_i v_i > 0 \\ r'_i s'_i &= r_i s_i \in \mathbb{Z} \end{aligned}$$

and

$$\sum_j M_{ij} = \sum_{j=1}^{n} v_i^{-1} (A_J)_{ij} v_j = v_i^{-1} A_J v \geq 0 .$$

So $\sum_{i,j} M_{ij} \geq 0$, but

$$\sum_{i,j} M_{ij} = 2k - (r'_1 + s'_1) - \cdots - (r'_k + s'_k) .$$

So

$$\frac{r'_i + s'_i}{2} \geq \sqrt{r'_i s'_i} = \sqrt{r_i s_i} \geq 1 .$$

So

$$\begin{aligned} r'_i + s'_i \geq 2 \quad &\Rightarrow \quad r'_i + s'_i = 2 \\ &\Rightarrow \quad r_i s_i = 1 , \end{aligned}$$

and so

$$A_J = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 2 & -1 & & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ -1 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix} \tag{13.1}$$

This is symmetric; $v = (1,\ldots,1) > 0$, and $A_J v = 0$, so A_J is affine, and so $A_J = A$. $\square$

Corollary 13.12. *If A is an indecomposable, symmetrisable generalised Cartan matrix, then*

1. *A is finite if and only if* $\det A_J > 0$ *for all J,*
2. *A is affine if and only if* $\det A = 0$ *and* $\det A_J > 0$ *for all proper J, and*
3. *A is indefinite precisely if it is neither finite nor affine.*

Proof. For (1) and (2), note that if A is finite or affine, then by Proposition 13.11 it is symmetrisable. So $A = DB$ with $d_i > 0$ and B symmetric, and B is of the same type as A. The conclusions of (1) and (2) follow by applying Proposition 13.5 to B.

Conversely, we first show that A is symmetrisable. We have a precise condition on A_{ij} to ensure symmetrisability, namely $A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{k-1} i_k} A_{i_k i_1} = A_{i_2 i_1} A_{i_3 i_2} \cdots A_{i_k i_{k-1}} A_{i_1 i_k}$. This is obviously satisfied unless there is a cycle in the matrix. Pick a minimal cycle

$$A_J = \begin{pmatrix} 2 & -b_1 & 0 & \dots & 0 & 0 & -b_s \\ -b'_1 & 2 & ??? & & 0 & 0 & 0 \\ 0 & ??? & 2 & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & & 2 & ??? & 0 \\ 0 & 0 & 0 & & ??? & 2 & -b_{s-1} \\ -b'_s & 0 & 0 & \dots & 0 & -b'_{s-1} & 2 \end{pmatrix}, \text{ an } (s \times s)\text{-matrix with } b_i, b'_i \in \mathbb{N}.$$

Kac claims that $\det A_J < 0$ unless $b_i = b'_i = 1$ for all i.

We want to show that if either of the conditions in the lemma on principal determinants **(Which lemma?)** holds, then $b_i = b'_i = 1$ for all i:

Suppose there exists a unique b_t or $b'_t > 1$. Then change the basis to

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, e_1, \dots, \cancel{e_t}, \dots, e_s,$$

from which $\det A_J < 0$ follows by an easy calculation ($\det A_J = s.(-x)$, where $-x$ is the unique entry in column t). So there exist at least two non-zero $b_t, b'_t > 1$. Taking the shortest route from t to t' produces a submatrix, which is proper unless $s \leq 4$.

If $s > 4$, we get a tridiagonal matrix

$$\begin{pmatrix} 2 & -b_1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -b_t \\ & & & & & -b'_t & 2 \end{pmatrix}.$$

This is a determinant that can be calculated (by expanding along row 1 and then along row t), and it is always ≤ 0. So A is symmetrisable, $A = DB$, and the principal determinants of B have the same property as A. So we can use the results about quadratic forms again. **(REFERENCE – is this Proposition 13.5?)** □

Exercise 13.13. Our proof of Kac's claim was very inelegant. Find a nicer proof.

14 Dynkin diagrams

Attach a diagram $\Delta(A)$ to an $(n \times n)$-matrix A as follows: $\Delta(A)$ has vertices $1, \ldots, n$, and if i, j are distinct vertices, then

- there is no edge between i and j if $A_{ij} = 0$,
- there is one edge between i and j if $A_{ij} = -1 = A_{ji}$,
- there is a directed double edge between i and j if $A_{ij}A_{ji} = 2$, pointing towards j if $j < i$,
- there are three edges between i and j if $A_{ij}A_{ji} = 3$, two of which form a directed double edge pointing towards j if $j < i$,
- there are four (undirected) between i and j if $A_{ij}A_{ji} = 4$ and $A_{ij} = -1$,
- there is a crossed double edge between i and j if $A_{ij}A_{ji} = 4$ and if $A_{ij} = -2 = A_{ji}$, and
- for $A_{ij}A_{ji} \geq 5$, there is one edge between i and j labelled "$|A_{ij}|\ |A_{ji}|$".

Tautological observation. Generalised Cartan matrices correspond precisely to Dynkin diagrams, and A is indecomposable if and only if $\Delta(A)$ is connected.

List of finite Dynkin diagrams

- A_n: $\square - \square - \cdots - \square$
- B_n $(n \geq 2)$: $\square - \square - \cdots - \square \Rightarrow \square$
- C_n $(n \geq 3)$: $\square - \square - \cdots - \square \Leftarrow \square$
- D_n $(n \geq 4)$: $\square - \square - \square \Rightarrow \square - \cdots - \square$
- E_6:
- E_7:
- E_8:
- F_4:
- G_2:

These diagrams are called *finite* Dynkin diagrams, and they correspond precisely to finite generalised Cartan matrices. Note that the list is closed under transposition (which corresponds to reversing the arrows in the diagrams) and taking sub-diagrams.

Exercise 14.1. Show that the determinant of any of the associated matrices is positive.

List of affine Dynkin diagrams

- $\widetilde{A}_1$:
- $\widetilde{A}'_1$:
- $\widetilde{A}_n$ $(n \geq 2)$:
- $\widetilde{B}_n$ $(n \geq 3)$ and $\widetilde{B}_n^T$:
- $\widetilde{C}_n$ $(n \geq 3)$:
- $\widetilde{C}_n^T$ $(n \geq 3)$:
- $\widetilde{D}_n$ $(n \geq ?)$:
- $\widetilde{E}_6$:
- $\widetilde{E}_7$:
- $\widetilde{E}_8$:
- $\widetilde{F}_4$:
- $\widetilde{F}_4^T$:
- $\widetilde{G}_2$:
- $\widetilde{G}_2^T$:

These diagrams are called *affine* Dynkin diagrams, and they correspond precisely to affine generalised Cartan matrices. Note that the matrix determined by $\widetilde{A}_n$ is the one in (13.1).

Exercise 14.2. Show that the determinant of any of the associated matrices vanishes.

Theorem 14.3. *Finite (affine) Dynkin diagrams are in one-to-one correspondence with finite (affine) generalised Cartan matrices.*

Proof. One direction follows from Exercises 14.1 and 14.2. To go the other way, in the finite case use the fact that if a Dynkin diagram produces a finite generalised Cartan matrix, then so does any sub-diagram (where you are allowed to remove edges, e.g. is a sub-diagram of).

Now suppose we had the following parts in a given diagram:

...

Then we get back to the finite case.

In the affine case, check what happens in rank 2: $\begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$. For rank ≥ 3, we need to check that all sub-diagrams (by removing vertices only) are finite. $\square$

15 Forms, Weyl groups and roots

Let A be a generalised Cartan matrix and $\mathscr{L}(A)$ its associated Kac-Moody Lie algebra. Recall that we have a weight space decomposition

$$\mathscr{L}(A) = \bigoplus_{\alpha} \mathscr{L}(A)_\alpha \quad \text{for} \quad \alpha \in \mathbb{Z}\{\alpha_1, \ldots, \alpha_n\}\,.$$

Theorem 15.1. *Let A be a symmetrisable generalised Cartan matrix. Then $\mathscr{L}(A)$ has a non-degenerate invariant symmetric bilinear form.*

Proof. Let $A = DB$, where $D = \operatorname{diag}(d_1,\dots,d_n)$ with $d_i \in \mathbb{N}$ and $B^T = B$. Let $(\mathfrak{h},\Pi,\Pi^\vee)$ be a minimal realisation of A, where $\Pi^\vee = (h_1,\dots,h_n)$ and $\Pi = (\alpha_1,\dots,\alpha_n)$. Let $\mathfrak{h}' = \operatorname{span}(\Pi^\vee) \subseteq \mathfrak{h}$, and let $\mathfrak{h}''$ be some complement such that $\mathfrak{h}' \oplus \mathfrak{h}'' = \mathfrak{h}$. Define a bilinear form $(-,-)$ on $\mathfrak{h}$ as follows:

$$(h_i,h) := d_i\alpha_i(h) \text{ for all } h \in \mathfrak{h}, \quad (h_1'',h_2'') := 0 \text{ for all } h_1'',h_2'' \in \mathfrak{h}''. \tag{15.1}$$

We claim that this is symmetric:

$$(h_i,h_j) := d_i\alpha_i(h_j) = d_iA_{ji} = d_id_jB_{ji} = d_jA_{ij} =: (h_j,h_i)$$

It is also non-degenerate, which we leave as an **Exercise**. In particular,

$$\ker(-,-)|_{\mathfrak{h}'} = \{x \in \mathfrak{h}' : \alpha_i(x) = 0 \text{ for all } i = 1,\dots,r\}$$

is the null space of A, which (**Exercise:**) equals the centre of $\mathscr{L}(A)$.

Now consider the *principal grading* on $\mathscr{L}(A)$,

$$\mathscr{L}(A) = \bigoplus_{k\in\mathbb{Z}} \mathscr{L}(A)_k\,,$$

where $\deg(e_i) = 1 = -\deg(f_i)$ for $i = 1,\dots,n$ and $\deg(x) = 0$ for $x \in \mathfrak{h}$. Let

$$\mathscr{L}(N) = \bigoplus_{k=-N}^{N} \mathscr{L}(A)_k\,, \text{ so } \mathscr{L}(0) = \mathscr{L}(A)_0 = \mathfrak{h}\,.$$

Define a form $(-,-)$ on $\mathscr{L}(N)$ by induction: The case $N = 0$ is given by Equation (15.1). For $N = 1$, define

$$(e_i,f_j) := \delta_{ij}d_i\,, \text{ and } (\mathscr{L}_{\pm1},\mathscr{L}_0) = 0 = (\mathscr{L}_{\pm1},\mathscr{L}_{\pm1})\,.$$

Invariance holds since

$$([e_i,f_j],h) = \delta_{ij}(h_i,h) = \delta_{ij}d_i\alpha_j(h) = (e_i,f_j)\alpha_j(h) = (e_i,[f_j,h])\,.$$

Now for $N > 1$, $(\mathscr{L}_i,\mathscr{L}_j) = 0$ if $i + j \neq 0$, and we need to define $(\mathscr{L}_{\pm N},\mathscr{L}_{\mp N})$. Take $y \in \mathscr{L}_{\mp N}$ and express it as $y = \sum_i [u_i,v_i]$ for some $u_i,v_i \in \mathscr{L}_{\mp(N-1)}$. For $x \in \mathscr{L}_{\pm N}$, define

$$(x,y) := \sum_i ([x,u_i],v_i)\,,$$

and note that $[x,u_i] \in \mathscr{L}_{\pm(N-1)}$ and $v_i \in \mathscr{L}_{\mp(N-1)}$, so this is indeed an inductive definition.

This is well-defined: Suppose $x = \sum_j [u_j',v_j']$, where $u_j',v_j' \in \mathscr{L}_{\pm(N-1)}$. Then

$$\begin{aligned}
(x,y) &= \sum_i ([x,u_i],v_i) \\
&= \sum_{i,j} ([[u_j',v_j'],u_i],v_i) \\
&= \sum_{i,j} ([[u_j',u_i],v_j'],v_i) - ([[v_j',u_i],u_j'],v_i) \\
&= \sum_{i,j} ([u_j',u_i],[v_j',v_i]) + (u_j',[[v_j',u_i],v_i]) \\
&= \sum_{i,j} (u_j',[u_i,[v_j',v_i]] + [[v_j',u_i],v_i]) \\
&= \sum_{i,j} (u_j',[v_j',[u_i,v_i]]) \\
&= \sum_j (u_j',[v_j',y])\,,
\end{aligned}$$

where we used induction to conclude the second and the penultimate line. This calculation shows that the form $(-,-)$ is well-defined, and moreover that it is symmetric. The proof that it is invariant is similar and we omit it.

The form is non-degenerate: Let $I := \ker(-,-)$. By invariance, I is an ideal. But we know that $(-,-)|_{\mathfrak{h}}$ is non-degenerate. Therefore $I \cap \mathfrak{h} = 0$, and by construction of $\mathscr{L}(A)$, $I = 0$. $\square$

Corollary 15.2. *$\big(\mathscr{L}(A)_\alpha, \mathscr{L}(A)_\beta\big) = 0$ unless $\alpha+\beta = 0$. So*

$$\big(-,-\big) \colon \mathscr{L}(A)_\alpha \times \mathscr{L}(A)_{-\alpha} \to \mathbb{C}$$

is a non-degenerate pairing, and $[x,y] = (x,y)h'_\alpha$, where $x \in \mathscr{L}(A)_\alpha$, $y \in \mathscr{L}(A)_{-\alpha}$, and h'_α is defined by

$$\big(h'_\alpha, -\big)\Big|_{\mathfrak{h}} = \alpha(-) \,.$$

Proof. Let $\alpha+\beta \neq 0$. Then there exists $h \in \mathfrak{h}$ such that $(\alpha+\beta)(h) \neq 0$. So

$$-\alpha(h)\big(x,y\big) = \big([x,h],y\big) = \big(x,[h,y]\big) = \beta(h)\big(x,y\big) \,,$$

and hence $(x,y) = 0$. Now we have $[x,y] - (x,y)h'_\alpha \in \mathfrak{h}$ if $x \in \mathscr{L}(A)_\alpha$ and $y \in \mathscr{L}(A)_{-\alpha}$. It follows that $\big([x,y]-(x,y)h'_\alpha, h\big) = 0$ from the definition for all $h \in \mathfrak{h}$. $\square$

Reminder: representation theory for $\mathfrak{sl}_2$.

$$\mathfrak{sl}(2;\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & -a\end{pmatrix}\right\} \,,$$

so $\mathfrak{sl}_2$ is of type A_1. It has a basis E, H, F, where

$$[H,E] = 2E \;\;,\quad [H,F] = -2F \;\;,\quad [E,F] = H \,.$$

By induction (inside the universal enveloping algebra $U(\mathfrak{sl}_2)$),

$$[H,F^k] = -2kF^k \;\;,\quad [H,E^k] = 2kE^k \;\;,\quad [E,F^k] = -k(k-1)F^{k-1} + kF^{k-1}H \,.$$

Theorem 15.3.

1. *If V is a representation of $\mathfrak{sl}_2$, then there exists $v \in V$ such that $Hv = \lambda v$. Setting $v_j = F^j v$, we have $Hv_j = (\lambda - 2j)v_j$.*
2. *There exists a unique $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ (where $n \geq 0$). It has a basis $(v_0, \ldots, v_n)$ and satisfies*

$$Hv_j = (n-2j)v_j, Fv_j = v_{j+1}, Ev_j = j(n-j+1)v_{j-1} \,,$$

$$\bullet \overset{F}{\underset{E}{\rightleftarrows}} \bullet \overset{F}{\underset{E}{\rightleftarrows}} \bullet \to \ldots \to \bullet \overset{F}{\underset{E}{\rightleftarrows}} \bullet \,.$$

Proof. (1) follows from the relations above (REFERENCE!), (2) is a *fun* **Exercise**. $\square$

In $\mathscr{L}(A)$, we have e_i, h_i, f_i with

$$[h_i,e_i] = 2e_i \;\;,\quad [h_i,f_i] = -2f_i \;\;,\quad [e_i,f_i] = h_i \,.$$

Lemma 15.4. *In $\mathscr{L}(A)$,*

$$(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0 \qquad \textit{and} \qquad (\operatorname{ad} f_i)^{1-A_{ij}} f_j = 0 \qquad \textit{for all } i,j = 1,\ldots,n,\ i \neq j.$$

Proof. Let us show this for the f_i's, the proof for the e_i's is analogous. Let $x = (\operatorname{ad} f_i)^{1-A_{ij}} f_j$. We prove that $[e_k, x] = 0$ for all k. First, set

$$c(x) := \{y \in \mathscr{L}(A) : [y, x] = 0\} \subseteq \mathscr{L}(A) ,$$

this is a Lie subalgebra. Therefore, $[\mathfrak{n}_+, x] = 0$. **(Please check, I think I'm missing a statement.)**

If A is symmetrisable, then $(\mathfrak{n}_+, x) = 0$ implies $x = 0$. Otherwise, if $(\mathfrak{n}_+, x) = 0$, then $U(\mathfrak{n}_+)x = \mathbb{C}x$, and

$$U(\mathfrak{n}_+)x = U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+)x = U(\mathfrak{n}_-)U(\mathfrak{h})x = U(\mathfrak{n}_-) \subseteq U(\mathfrak{n}_-) .$$

So $x \in \mathfrak{n}_-$ generates an ideal in $\mathscr{L}(A)$ wholly contained in $\mathfrak{n}_-$, so $x = 0$ by construction of $\mathscr{L}(A)$. □

Another proof. Let x be as in the first proof. Then $x \in \mathfrak{n}_- \subset \mathscr{L}(A)$. We show that $[e_k, x] = 0$ for all k, which implies that $x = 0$. We distinguish three cases:

- If $k \neq i, j$, then this is clear (check!), since $[e_k, f_i] = 0 = [e_k, f_j]$.

- If $k = j$, then

  $$[e_j, (\operatorname{ad}(f_i))^{1-A_{ij}}(f_j)] = (\operatorname{ad}(f_i))^{1-A_{ij}}[e_j, f_j] = adf_i^{1-A_ij} h_j ,$$

  and if $A_{ij} \leq -1$, we get zero. If $A_{ij} = 0$, then $[f_i, h_j] = \alpha_i(h_j) f_i = A_{ji} f_i = 0$.

- If $k = i$, then we use $\mathfrak{sl}(2;\mathbb{C})$-representation theory: Let $v = f_j$. Then $[e_i, f_j] = 0$ and $[h_i, f_j] = -\alpha_j(h_i) f_j = -A_{ij} f_j$. Now (e_i, h_i, f_i) act on $\mathscr{L}(A)$ via the adjoint action, and we consider the orbit through v. Then we use a result from representation theory:

  $$e_i.(v_{1-A_{ij}}) := [e_i, (\operatorname{ad}(f_i))^{1-A_{ij}} f_j] = (1 - A_{ij})(-A_{ij} - 1 + A_{ij} + 1) v_{1-A_{ij}} = (\operatorname{ad}(f_j))^T v = 0 \quad □$$

Definition 15.5. Let V be a representation of a Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is called *locally nilpotent* if for all $v \in V$ there exists $N(v)$ such that $x^{N(v)}.v = 0$.

Lemma 15.6. *The elements* $\operatorname{ad}(e_i)$ *and* $\operatorname{ad}(f_i)$ *act locally nilpotently on* $\mathscr{L}(A)$.

Proof. We have $(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0$, $(\operatorname{ad} e_i)^2 h = 0$, $(\operatorname{ad} e_i) e_i = 0$, $(\operatorname{ad} e_i) f_j = 0$, $(\operatorname{ad} e_i)^3 f_i = 0$. So ETS that since these are a generating set, we can act locally nilpotently on $\mathscr{L}(A)$ (i.e. what is generated). This is due to Leibniz:

$$(\operatorname{ad} x)^k [y, z] = \sum_{i=0}^{k} \binom{k}{i} \left[(\operatorname{ad} x)^i (y), (\operatorname{ad} z)^{k-i} (z)\right] ,$$

which implies the statement. □

If V is again a representation of a Lie algebra $\mathfrak{g}$, then a locally nilpotent action $x \in \mathfrak{g}$ produces an automorphism

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

on V whose inverse is $\exp(-x)$, and $x \mapsto \exp(x)$ is a Lie algebra homomorphism by the Leibniz rule.

Definition 15.7. We define the elements

$$n_i := \exp\bigl(\operatorname{ad}(e_i)\bigr) \exp\bigl(\operatorname{ad}(-f_i)\bigr) \exp\bigl(\operatorname{ad}(e_i)\bigr) \in \operatorname{Aut}\bigl(\mathscr{L}(A)\bigr) .$$

Example ($\mathfrak{sl}(2;\mathbb{C})$)**.** The Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ has a basis $\{e, f, h\}$, and so we have:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \exp(-f) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Proposition 15.8. *The element $s_i = n_i|_{\mathfrak{h}} \colon \mathfrak{h} \to \mathfrak{h}$ satisfies $s_i(x) = x - \alpha_i(x)h_i$.*

Proof. This is a fascinating calculation, left as an **Exercise**. □

Definition 15.9. The map $s_i \colon \mathfrak{h} \to \mathfrak{h}$, $x \mapsto x - \alpha_i(x)h_i$ is called a *fundamental simple reflection.* The subgroup of $\mathrm{Aut}(\mathfrak{h})$ generated by the elemets s_i is called the *Weyl* group of $\mathscr{L}(A)$, denoted by W.

We have $s_i(h_i) = -h_i$, s_i^2, $s_i(x) = x$ if $(h_i, x) = 0$. An easy calculation shows:

Lemma 15.10. *The bilinear form $(-,-)|_{\mathfrak{h}}$ is invariant under W.* □

So we have an action of W on $\mathfrak{h}^*$ via

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad \text{for all } \lambda \in \mathfrak{h}^*.$$

Proposition 15.11. *The map $n_i \colon \mathscr{L}_\alpha \mapsto \mathscr{L}_{s_i(\alpha)}$ is an identification.*

Proof. We compute directly:

$$\left[h, n_i(x)\right] = \left[n_i(s_i^{-1}h), n_i(x)\right] = n_i\left[s_i^{-1}h, x\right] = n_i\big(\alpha(s_i^{-1}h)x\big) = \alpha(s_i^{-1}h)n_i(x) = s_i(\alpha)(h)n_i(x)$$

□

Theorem 15.12 (Properties of the Weyl group)**.** *If $i \neq j$, then the element $s_i s_j \in W$ has the following order:*

$$\mathrm{order}(s_i s_j) = \begin{cases} 2 & A_{ij}A_{ji} = 0\,, \\ 3 & A_{ij}A_{ji} = 1\,, \\ 4 & A_{ij}A_{ji} = 2\,, \\ 6 & A_{ij}A_{ji} = 3\,, \\ \infty & A_{ij}A_{ji} \geq 4\,. \end{cases}$$

Proof. Let

$$K := \left\{\lambda \in \mathfrak{h}^* : \lambda(h_i) = \lambda(h_j) = 0\right\}\,,$$

so $\dim K = \dim \mathfrak{h}^* - 2$. Writing $V = \mathbb{C}\alpha_i \oplus \mathbb{C}\alpha_j$, we get a decomposition $\mathfrak{h}^* = V \oplus K$, and $s_i s_j$ acts trivially on K. On the other hand, we have

$$(s_i s_j)|_V \mapsto \begin{pmatrix} -1 + A_{ij}A_{ji} & A_{ij} \\ -A_{ji} & -1 \end{pmatrix}\,.$$

The result follows from a calculation of the eigenvalues and the Jordan Normal Form of this matrix. □

16 Root spaces

Let $L := \mathscr{L}(A)$. We have a decomposition (of vector spaces)

$$L = L_0 \oplus \bigoplus_{\alpha\in R} L_\alpha\,,$$

where $R \subset Q = Q_+ \sqcup Q_-$. We have $L_\alpha \neq 0$. Here $Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n$ is the positive weight lattice. The elements of R are called *roots*, and the decomposition of Q induces a decomposition $R = R_+ \sqcup R_-$ into *positive* and *negative* roots.

We are lead to ask the following questions: What is R, and what is $\dim L_\alpha$? Writing m_α for the multiplicity of α, we know:

- $\dim L_\alpha = \dim L_{-\alpha}$.
- $\Pi = \{\alpha_1, \dots, \alpha_n : m_{\alpha_i} = 1\} \subset R_+$.
- R is W-stable, i.e. $m_{w\alpha} = m_\alpha$ for all $w \in W$.

Definition 16.1. An element $\alpha \in R$ is called *real* if there exists $\alpha_i \in \Pi$ and $w \in W$ such that $w(\alpha_i) = \alpha$. Otherwise we call α *imaginary.*

Note. $\alpha \in R_{\mathrm{re}}$ if and only if $-\alpha \in R_{\mathrm{re}}$, since $w(s_i\alpha_i) = -w(\alpha_i)$. Furthermore, $R^+ = R^+_{\mathrm{re}} \sqcup R^+_{\mathrm{im}}$.

Lemma 16.2. $w\big(R^+_{\mathrm{im}}\big) = R^+_{\mathrm{im}}$.

Proof. Positivity is the key here. We have $\alpha = \sum_{j=1}^n k_j\alpha_j \in R^+_{\mathrm{im}}$, where $k_j \geq 0$, and at least two k_j's are non-zero. So $s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ has at least one positive k_j, so all k_j must be non-negative. □

Let $C := \big\{\lambda \in \mathfrak{h}^*_{\mathbb{R}} : \lambda(h_i) > 0 \text{ for all } i = 1, \dots, n\big\} \subset \overline{C} := \{\ldots \geq 0\}$.

Proposition 16.3. $R^+_{\mathrm{im}} \subseteq \bigcup_{w\in W} w\big(-\overline{C}\big)$.

Proof. For $\alpha \in R^+_{\mathrm{im}}$, we have a set $\{\mathrm{ht}(\beta) : \beta = w(\alpha), w \in W\}$. Pick an element β achieving minimal height. Then $s_i(\beta) = \beta - \beta(h_i)\alpha_i$. Since β has minimal height, $\beta(h_i) \leq 0$ for all i, and thus $\beta \in -\overline{C}$. □

Theorem 16.4 (Kac)**.** *Let*

$$K := \big\{\alpha \in Q_+ : \alpha \neq 0,\ \mathrm{supp}(\alpha) \text{ connected},\ \alpha \in -\overline{C}\,\big\}\,.$$

Then $R^+_{\mathrm{im}} = \bigcup_{w\in W} w(K)$.

Remark 16.5. Here $\mathrm{supp}(\alpha) := \{j : k_j \neq 0\}$, where $\alpha = \sum_j k_j\alpha_j$; $\mathrm{supp}(\alpha)$ corresponds to the vertices in the Dynkin diagram.

Exercise 16.6. Show that for [DIAGRAMME], $-\overline{C} = \{k(\alpha_1 + \alpha_2 + \alpha_3) : k \in \mathbb{N}\}$ in any minimal realisation, and $w(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$.

Corollary 16.7. $\alpha \in R^+_{\mathrm{im}} \Rightarrow k\alpha \in R^+_{\mathrm{im}}$ *for all* $k \in \mathbb{N}$. □

Corollary 16.8. *If A is an indecomposable generalised Cartan matrix, then*

1. *if A is of finite type, then* $R^+_{\mathrm{im}} = \varnothing$*, i.e. there are only real roots,*

2. *if A is affine, then there exists an integral $u > 0$ such that $Au = 0$, and u has no common factors. Let $u = (a_1, \ldots, a_n)$ and $\delta = \sum_{i=1}^n a_i \alpha_i$. Then $R^+_{\mathrm{im}} = \{k\delta : k \in \mathbb{Z} \setminus \{0\}\}$.*

3. *if A is indefinite, then there exists $\alpha \in R^+_{\mathrm{im}}$ such that $\alpha = \sum_{i=1}^n k_i \alpha_i$, $k_i > 0$ and $\alpha(h_i) < 0$ for all i.*

Proof. **Exercise.** □

Corollary 16.9. *Since A is symmetrisable by (REFERENCE), if $\alpha \in R_{\mathrm{re}}$, then $(\alpha, \alpha) > 0$, and if $\alpha \in R_{\mathrm{im}}$, then $(\alpha, \alpha) \le 0$.*

Proof. $(h_i, x) := d_i \alpha_i(x)$ where $h_i \leftrightarrow d_i \alpha_i$, so

$$\left(\alpha_i, \alpha_j\right) = \left(\tfrac{h_i}{d_i}, \tfrac{h_j}{d_j}\right) = d_j^{-1} A_{ji}\ .$$

So $(\alpha_i, \alpha_j) = \frac{2}{d_i} > 0$, which proves the first statement.

If $\alpha = \sum_{i=1}^n k_i \alpha_i$ is imaginary with $k_i \ge 0$ and $\alpha(h_i) \le 0$ for all i, then

$$\left(\alpha, \alpha\right) = \sum_{i=1}^n k_i \left(\alpha_i, \alpha\right) = \sum_{i=1}^n \frac{k_i}{d_i} \alpha(h_i) \le 0\ .$$ □

We prove Theorem 16.4 in several steps.

Step 1: $\alpha \in R \Rightarrow \operatorname{supp}(\alpha)$ is connected.

Proof. Without loss of generality, let $\alpha \in R^+$. Let $\operatorname{supp}(\alpha) = J \subseteq \{1, 2, \ldots, n\}$ with $J = J_1 \cup J_2$ disconnected (i.e. $A_{j_1 j_2} = 0$ for all $j_1 \in J_1$ and $j_2 \in J_2$). Let $i \in J_1$, $j \in J_2$. Then $[e_i, e_j] = (\operatorname{ad} e_i)^{1 - A_{ij}} e_j = 0$. So $L_\alpha \subset \mathfrak{n}_+$, and we claim that $L_\alpha = 0$.

But L_α is spanned by Lie monomials in the e_k's. We show by induction on degree that any Lie monomial involving e_i and e_j (with i, j chosen as above) is zero.

... □

Step 2: $\alpha \in R$, $\alpha \ne \pm\alpha_i$ with $\alpha \pm \alpha_i \notin R \Rightarrow \alpha(h_i) = 0$. $\alpha \in R$, $\alpha \ne -\alpha_i$ with $\alpha + \alpha_i \notin R \Rightarrow \alpha(h_i) \ge 0$.

Proof. Pick $x \in L_\alpha$. Then $n_i(x) \in L_{s_i \alpha}$. Now calculate $n_i(x)$:

$$\operatorname{ad} e_i(x) = 0 \Rightarrow \exp(\operatorname{ad} e_i) x = x$$

$$\operatorname{ad} f_i(x) = 0 \Rightarrow \exp(\operatorname{ad}(-f_i)) x = x$$

Since $\operatorname{ad} e_i(x) \in L_{\alpha + \alpha_i}$ and $\operatorname{ad} f_i(x) \in L_{\alpha - \alpha_i}$, we conclude that $n_i(x) = x$. So $s_i \alpha = \alpha$, and so $\alpha(h_i) = 0$. The second claim is a variation on this theme. □

Step 3: $\alpha = \sum_i k_i \alpha_i \in K$, $\Psi := \{\beta \in R^+ : \beta = \sum_i = 1^n m_i \alpha_i,\ m_i \le k_i \text{ for all } i\}$. Let $\widehat{\beta}$ have the largest height amongst the $\beta \in \Psi$. Then $\operatorname{supp}(\widehat{\beta}) = \operatorname{supp}(\alpha)$.

Proof. First note that $\operatorname{supp}(\widehat{\beta}) \subseteq \operatorname{supp}(\alpha)$. If equality did not hold, then there would exist $j \in \operatorname{supp}(\alpha) \setminus \operatorname{supp}(\widehat{\beta})$ and $j' \in \operatorname{supp}(\widehat{\beta})$ such that $A_{jj'} \ne 0$. So in $\widehat{\beta}$, $m_j = 0$, and so $\widehat{\beta} - \alpha_j \notin R^+$ and $\widehat{\beta} + \alpha_j \notin R^+$. So $\beta(h_j) = 0$, and for a contradiction note that $\beta(h_j) = \sum_{i \in \operatorname{supp}(\widehat{\beta})} m_i \alpha_i(h_j) = \sum_{i \in \operatorname{supp}(\widehat{\beta})} m_i A_{ji} < 0$. □

Step 4: $J := \{i \in \mathrm{supp}(\alpha) : k_i = m_i\} = \mathrm{supp}(\alpha)$. (This implies $K \subseteq R^+$.)

Proof. Let $i \in \mathrm{supp}(\alpha) \setminus J$. Then $\widehat{\beta} + \alpha_i \notin R^+$. So by $\widehat{\beta}(h_i) \geq 0$ by A PREVIOUS Lemma.

Let $M \subseteq \mathrm{supp}(\alpha) \setminus J$ be the connected component of i. So $\widehat{\beta}(h_j) \geq 0$ for all $j \in M$. Let $\widehat{\beta}' := \sum_{j \in M} m_j \alpha_j$, so

$$\widehat{\beta}'(h_i) = \widehat{\beta}(h_i) - \sum_{j \in \mathrm{supp}(\alpha) \setminus M} m_j \alpha_j(h_i) .$$

Recall that $\alpha(j)(h_i) = A_{ij}$. If $j \in M$, then $\widehat{\beta}(h_j) \geq 0$, and since $A_{ij} \leq 0$, $\widehat{\beta}'(h_i) \geq 0$. By choosing i carefully (there exist $i' \in M$ and $j' \in \mathrm{supp}(\alpha) \setminus M$ such that $A_{i'j'} < 0$), we have $\widehat{\beta}'(h_i) > 0$ for some $i \in M$.

Let A_M be the principal matrix $(A_{ij})_{i,j \in M}$, and $u = (m_j)^T_{j \in M}$. So

$$\widehat{\beta}'(h_i) = \sum_{j \in M} A_{ij} m_j \text{ for all } i \in M,$$

i.e. $A_M u \geq 0$ (and $\neq 0$). So A_M has finite type BY (REFERENCE). Let $\gamma := \sum_{i \in M} (k_i - m_i)\alpha_i$, where we have $k_i - m_i > 0$ for all $i \in M$. Then

$$\alpha - \widehat{\beta} = \sum_{t \in \mathrm{supp}(\alpha) \setminus J} (k_t - m_t)\alpha_t .$$

So

$$(\alpha - \widehat{\beta})(h_i) = \sum_{t \in \mathrm{supp}(\alpha) \setminus J} (k_t - m_t)\alpha_t(h_i) = \sum_{j \in M} (k_j - m_j) A_{ij} ,$$

since M is the connected component of i in $\mathrm{supp}(\alpha) \setminus J$. But $\alpha(h_i) \leq 0$ since $\alpha \in K$, we have $\widehat{\beta}(h_i) \geq 0$ from above. Therefore, $\gamma(h_i) \leq 0$ for all $i \in M$.

Set $u = (k_i - m_i)^T_{i \in M}$, and so $A_M u \leq 0$. Since A_M has finite type, $u = 0$. (That is, $A_M(-u) \geq 0$, and since A_M has finite type, either $-u > 0$ or $u = 0$.) □

Step 5: $K \subseteq R^+_{\mathrm{im}}$.

Proof. We already know that $K \subseteq R^+$. If $\alpha \in K$, then also $2\alpha \in K$. Hence $2\alpha \in R^+$, and so α is imaginary. □

This concludes the proof of Theorem 16.4.

17 Affine Lie algebras and Kac-Moody Lie algebras

Simple Lie algebras of finite type are determined by the Dynkin diagram of their Cartan matrix, which is one of A_n, B_n, C_n, D_n, E_6, E_7 E_8, F_4 or G_2. Write $\mathfrak{g}$ for the corresponding simple Lie algebra. It acts on itself by the adjoint representation. There exists a unique highest root $\theta = \sum_i a_i \alpha_i \in \mathfrak{h}^*$.

Example. The Lie algebra associated to A_n has $a_i = 1$ for all $i = 1, \ldots, n$, and the root spaces are E_{ij}, $i \neq j$.

We had constructed for all Kac-Moody Lie algebras a "standard form" $\langle -, - \rangle$. We know that simple Lie algebras have a unique invariant inner product up to scale, the Killing form, so that the standard form must be a multiple of the Killing form. For the "long root" θ, we have $\langle \theta, \theta \rangle = 2$. (This follows from W-invariance of the form and $\langle \alpha_i, \alpha_i \rangle = 2/\alpha_i$.)

Let h_θ be the *coroot* corresponding under $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ to the element $\frac{2\theta}{\langle \theta, \theta \rangle} = \theta$.

Observation. Let A be the Cartan matrix for a Dynkin diagram of finite type. Let $\widehat{A}$ be the matrix given by

$$\begin{aligned}\widehat{A}_{ij} &= A_{ij} \text{ for } i,j \in \{1,\ldots,n\},\\ \widehat{A}_{00} &= 2\,,\\ \widehat{A}_{i0} &= -\sum_{j=1}^{n} a_j A_{ij} \text{ for } i \in \{1,\ldots,n\},\\ \widehat{A}_{0j} &= -\sum_{i=1}^{n} c_i A_{ij} \text{ for } j \in \{1,\ldots,n\}.\end{aligned}$$

This process produces the so-called *untwisted affine Dynkin diagrams.*

Exercise 17.1. Show that the matrix $\widehat{A}$ is an affine generalised Cartan matrix.

Example. Starting with G_2, we the Weyl group is $W(G_2) = \langle s_1, s_2 : s_1^2 = s_2^2 = e, (s_1 s_2)^6 = e\rangle \cong D_6$ (the dihedral group). The Cartan matrix is $\left(\begin{smallmatrix} 2 & -1 \\ -3 & 2\end{smallmatrix}\right)$, and $d = \operatorname{diag}(1,3)$. The positive roots are

$$R^+ = R^+_{\mathrm{re}} = \left\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_1+3\alpha_2, 2\alpha_1+3\alpha_2\right\}.$$

To find the coroot h_θ, we write $h_\theta = c_1 h_1 + c_2 h_2$ and compute:

$$\begin{aligned}\langle\theta,\alpha_1\rangle &= 2\langle\alpha_1,\alpha_1\rangle + 3\langle\alpha_2,\alpha_1\rangle = 4-3 = 1 = \alpha_1(h_\theta) = 2c_1 - 3c_2\\ \langle\theta,\alpha_2\rangle &= 0 = \alpha_2(h_\theta) = -c_1 + 2c_2\end{aligned}$$

This implies that $h_\theta = 2h_1 + h_2$. So the affine Cartan matrix $\widehat{A}$ is

$$\widehat{A} = \begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -3 & 2\end{pmatrix}.$$

Recall that we had the affine Kac-Moody Lie algebra

$$\widehat{\mathscr{L}\mathfrak{g}} = \mathfrak{g}\left[t^{\pm1}\right] \oplus \mathbb{C}d \oplus \mathbb{C}c$$

with structure

$$\left[X\otimes f + \lambda d + \mu c, Y\otimes g\lambda' + \mu' c\right] = [X,Y]\otimes fg + \lambda Y \otimes t\frac{dg}{dt} - \lambda' X\otimes t\frac{df}{dt} - \left(\mathrm{Res}_{t=0} f\, dg\right)\langle X,Y\rangle c\,.$$

Theorem 17.2. *If $\mathfrak{g}$ is simple with Dynkin diagram of finite type, then $\widehat{\mathscr{L}\mathfrak{g}}$ is isomorphic to the Kac-Moody Lie algebra of the associated untwisted affine Dynkin diagram.*

Proof. We have generators and relations for the Kac-Moody Lie algebra: $e_0,\ldots,e_n$, $f_0,\ldots,f_n$, $h_0,\ldots,h_n$. Since $\mathfrak{g}$ is simple, we already have generators $E_1,\ldots,E_n$, $F_1,\ldots,F_n$ and $H_1,\ldots,H_n$. Set

$$e_i := E_i \otimes 1\ ,\quad f_i := F_i\otimes 1\ ,\quad h_i := H_1 \otimes 1 \quad \in \mathfrak{g}\left[t^{\pm1}\right].$$

We need to define the generators e_0, f_0, h_0. We know that $\dim\mathfrak{g}_\theta = \dim\mathfrak{g}_{-\theta} = 1$, and we have an involution $\omega\colon \mathfrak{g}\to\mathfrak{g}$ such that $\omega(E_i) = -F_i$. There exists a non-degenerate pairing

$$\langle -,-\rangle\colon \mathfrak{g}_\theta \times \mathfrak{g}_{-\theta} \to \mathbb{C}\,.$$

Choose $E_\theta, F_\theta \in \mathfrak{g}_{\pm\theta}$ with $\langle E_\theta, F_\theta\rangle = 1$ and $\omega(E_\theta) = -F_\theta$. (For example, pick $\widetilde{F}_\theta \in \mathfrak{g}_{-\theta}$, set $\widetilde{E}_\theta := -\omega(F_\theta) \in \mathfrak{g}_\theta$, then $\langle\widetilde{E}_\theta, \widetilde{F}_\theta\rangle = \xi \neq 0$, so set $F_\theta := \widetilde{F}_\theta/\sqrt{\xi}$ etc.) Then let

$$e_0 := F_\theta \otimes t\ ,\quad f_0 := E_\theta\otimes t^{-1}\,.$$

So we have
$$H := (\mathfrak{h}\otimes 1)\oplus \mathbb{C}d \oplus \mathbb{C}c \quad \ni \quad h_0 =: -h_\theta \otimes 1 + c\,.$$
It follows that
$$[e_0, f_0] = [F_\theta, E_\theta]\otimes 1 + \langle F_\theta, E_\theta\rangle c = h_0\,,$$
where we used $[F_\theta, E_\theta] = \langle F_\theta, E_\theta\rangle h'_{-\theta} = -h_\theta$.

We need a realisation of $\widehat{A}$ given by $\alpha_0, \alpha_1, \dots, \alpha_n \in H^*$, where $\alpha_1, \dots, \alpha_n$ are extended from $\mathfrak{h}^*$ by $\alpha_i(d) = \alpha_i(c) = 0$ for all $i = 1, \dots, n$. Do the same for $\theta \in \mathfrak{h}^*$, which gives $\theta \in H^*$. Let δ be the dual of d. Then $\alpha_0 := -\theta + \delta \in H^*$, and
$$\big(H, \Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}, \Pi^\vee = \{h_0, h_1, \dots, h_n\}\big)$$
is a minimal realisation of $\widehat{A}$. (Clearly the α_i and h_i are all linearly independent.) So we get
$$\begin{aligned}
\alpha_j(h_i) &= A_{ij} \text{ for all } i, j = 1, \dots, n,\\
\alpha_0(h_i) &= -\theta(h_i) + \delta(h_i) = -\sum_{j=1}^n a_j \alpha_j(h_i) = -\sum_{j=1}^n a_j A_{ij} = \widehat{A}_{i0} \text{ for } i \neq 0,\\
\alpha_j(h_0) &= \text{similarly for } j \neq 0.
\end{aligned}$$
The relations are as follows:
$$\begin{aligned}
&[e_i, f_i] = h_i \text{ for all } i, && [e_i, f_j] = 0 \text{ for } i \neq j,\\
&[x,\ e_i] = \alpha_i(x) e_i \text{ for } x \in H, \text{ all } i, && [x,\ f_i] = -\alpha_i(x) f_i \text{ for } x \in H, \text{ all } i.
\end{aligned}$$
To see the relation $[e_i, f_j] = 0$ when $j \neq i = 0$, note that $[e_0, f_j] = [F_\theta, F_j]\otimes t = 0$. To see the relation $[x, e_i] = 0$ when $i = 0$, note that (with $x_0 \in \mathfrak{h}$)
$$[x_0 + \lambda d + \mu c, e_0] = [x_0, F_\theta]\otimes t + \lambda F_\theta \otimes t = -\theta(x_0) F_\theta \otimes t + \lambda F_\theta \otimes t = (\delta - \theta)(x) e_0 = \alpha_0(x) e_0\,.$$

What about the map $\widetilde{L}(\widehat{A}) \to \widehat{\mathscr{L}\mathfrak{g}}$ – is it surjective, what is its kernel? It is surjective, i.e. the Lie algebra M generated by $e_0, \dots, e_n$, $f_0, \dots, f_n$ and H is $\widehat{\mathscr{L}\mathfrak{g}}$. Consider the set $S := \{x \in \mathfrak{g} : x \otimes t \in M\}$:

- $S \neq 0$ because S contains F_θ.
- $\mathfrak{g}$ acts on S, i.e. S is an ideal in $\mathfrak{g}$, $[y\otimes 1, x \otimes t] = [y, x]\otimes t$.
- By simplicity, $S = \mathfrak{g}$.

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $[\mathfrak{g}\otimes t, \mathfrak{g}\otimes t^{k-1}] = \mathfrak{g}\otimes t^k$ for $k \geq 2$, and this $\mathfrak{g}\big[t^{\pm 1}\big] \subseteq M$, where $c, d \in H$. Hence $\widehat{\mathscr{L}\mathfrak{g}} = M$. For $x \in H$,
$$\big[x, v_\alpha \otimes t^i\big] = \big[x_0 + \mu d + \lambda c, v_\alpha \otimes t^i\big] = \alpha(x_0) v_\alpha \otimes t^i + \mu i v_\alpha \otimes t^i = (\alpha + i\delta)(x) v_\alpha \otimes t^i\,,$$
i.e. $\widehat{\mathscr{L}\mathfrak{g}} = L_0 \oplus \sum_{(i,\alpha)\neq(0,0)} \big(\widehat{\mathscr{L}\mathfrak{g}}\big)_{\alpha + i\delta}$. Next,
$$J = (L_0 \cap J) \oplus \sum_{(i,\alpha)\neq(0,0)} \big(L \cap (\widehat{\mathscr{L}\mathfrak{g}})_{\alpha+i\delta}\big)\,,$$
so $J \cap L_{\alpha+i\delta} \neq 0$ for some (i, α) if $J \neq 0$. But $x \otimes t^i \in J$ for $0 \neq x \in \mathfrak{g}_\alpha$. Pick $y \in \mathfrak{g}_{-\alpha}$ such that $\langle x, y\rangle \neq 0$. But
$$0 = \big[x \otimes t^i, y \otimes t^{-i}\big] = [x, y]\otimes 1 - i\langle x, y\rangle c \in H\,,$$
so $i = 0$, so $[x, y] = 0 = \langle x, y\rangle h'_\alpha \neq 0$, contradiction. □

Remark 17.3. Observe that the real roots are $\{\alpha + i\delta : \alpha \neq 0\}$, the imaginary roots are $\{i\delta : i \neq 0\}$, and the multiplicity of the set of imaginary roots is the rank of $\mathfrak{g}$.

The Kac-Moody Lie algebras associated to affine Dynkin diagrams $\mathscr{L}(\tilde{\mathfrak{g}})$ correspond precisely to affine Lie algebras $\mathfrak{g}[t^{\pm}] \oplus \mathbb{C}d \oplus \mathbb{C}c =: \widehat{\mathscr{L}\mathfrak{g}}$.

Dynkin diagrams have automorphisms: [Explain the procedure that gets from the orbit space to the affine Lie algebra.] [something] produces the so-called *twisted affine Dynkin diagrams* from the classical Dynkin diagrams.

$$\begin{array}{rcl}
A_{2n-1} & \to & \widetilde{B}_n^T = {}^2\widetilde{A}_{2n-1} \\
A_{2n} \ \ (n \geq 2) & \to & \widetilde{C}_n' = {}^2\widetilde{A}_{2n} \\
D_{n+1} & \to & \widetilde{C}_n' = {}^2\widetilde{D}_{n+1} \\
A_2 & \to & ? \\
E_6 & \to & \widetilde{F}_4^T = {}^2\widetilde{E}_6 \\
D_4 & \to & \widetilde{G}_2^T = {}^3\widetilde{D}_4
\end{array}$$

The diagram automorphism σ extends to an automorphism, also denoted by σ, of $\mathfrak{g}$ and H via

$$\sigma(c) = c\,, \quad \sigma(d) = d\,, \quad e_i \mapsto e_{\sigma(i)}\,, \quad f_i \mapsto f_{\sigma(i)}\,, \quad h_i \mapsto h_{\sigma(i)}\,.$$

All that remains is to define the action of σ on t. If σ has order m, let $\xi = e^{2\pi i/m}$, and let τ act on $\mathrm{Aut}(\widehat{\mathscr{L}\mathfrak{g}})$ by

$$\tau(x \otimes t^i) = \xi^{-i}\sigma(x) \otimes t^i\,, \quad \tau(c) = c\,, \quad \tau(d) = d\,.$$

Theorem 17.4. *If Δ is the (classical) Dynkin diagram of the simple Lie algebra $\mathfrak{g}$, then the* twisted affine Kac-Moody Lie algebra

$$\left(\widehat{\mathscr{L}\mathfrak{g}}\right)^T = \left\{x \in \widehat{\mathscr{L}\mathfrak{g}} : \tau(x) = x\right\}$$

is isomorphic to the Kac-Moody Lie algebra corresponding to the twisted affine Dynkin diagram $\widetilde{\Delta}$.

18 The Weyl-Kac formula

Throughout this section, let $\mathscr{L} := \mathscr{L}(A)$ denote a symmetrizable Kac-Moody Lie algebra. We want to study $\mathscr{L}$ through its representations.

18.1 Category $\mathscr{O}$

The idea is to study representations of $\mathscr{L}$ by their combinatorics.

Definition 18.1. Let V be a representation of $\mathscr{L}$. We say that V *belongs to category* $\mathscr{O}$, or $V \in \mathscr{O}$, if the following hold:

- $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where $V_\lambda = \left\{v \in V : h.v = \lambda(h)v \text{ for all } h \in H\right\}$.
- $\dim_{\mathbb{C}} V_\lambda < \infty$ for all $\lambda \in H^*$.
- There exist $\lambda_1, \ldots, \lambda_s \in H^*$ such that if $V_\lambda \neq 0$, then $\lambda < \lambda_i$ for some i. (This is to say that $\lambda_i - \lambda \in Q_+$, i.e. $\lambda_i - \lambda = \sum_{j=1}^n n_j \alpha_j$ for $n_j \in \mathbb{Z}_{\geq 0}$.)

Definition 18.2. There is a *character function*

$$\mathrm{ch}\colon \mathscr{O} \to \mathrm{Fun}(H^*)\,, \quad V \mapsto \mathrm{ch}(V) := \sum_{\lambda\in H^*} \dim V_\lambda e_\lambda\,,$$

where e_λ is the character function for $\lambda \in H^*$, and $e_\lambda e_\mu = e_{\lambda+\mu}$. **Explain what e_λ is. Is it the same as $e(\lambda)$ below?** We define

$$\mathscr{R} := \left\{ f \in \mathrm{Fun}(H^*) : \exists \lambda_1,\ldots,\lambda_s \text{ s.t. } \mathrm{supp}(f) \subseteq S(\lambda_1)\cup\cdots\cup S(\lambda_s)\right\}\,,$$

where

$$S(\lambda) := \lambda - Q_+ = \left\{\mu \in H^* : \mu \le \lambda\right\}\,.$$

Observe that the set $\mathscr{R}$ is a ring and that the character function is in fact $\mathrm{ch}\colon \mathscr{O} \to \mathscr{R}$.

Remark 18.3. Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence of $\mathscr{L}$-representations. Then if $B \in \mathscr{O}$, it follows that $A, C \in \mathscr{O}$, but the converse does not hold. However, if $D, E \in \mathscr{O}$, then $D \otimes E \in \mathscr{O}$. (Here $x.(d\otimes e) = x.d\otimes e + = d \otimes x.e$.)

The category $\mathscr{O}$ has universal objects (cf. the Virasoro algebra, in particular Equation (7.1)): Recall that $\mathscr{L} = \mathfrak{n}_- \oplus H \oplus \mathfrak{n}_+$. For $\lambda \in H^*$, we define the *Verma module*

$$M(\lambda) := U(\mathscr{L}) \otimes_{U(\mathfrak{n}_+\oplus H)} \mathbb{C}(\lambda)\,, \tag{18.1}$$

where the action of $U(\mathfrak{n}_+ \oplus H)$ is as follows:

$$1_\lambda \in \mathbb{C}(\lambda)\,, \quad \mathfrak{n}_+.\mathbb{C}(\lambda) = 0\,, \text{ and } h.1_\lambda = \lambda(h)1_\lambda \text{for all } h \in H.$$

There is a special element $v_\lambda := 1 \otimes 1_\lambda$ that satisfies

$$e_i.v_\lambda = 0 \qquad \text{and} \qquad h.v_\lambda = \lambda(h)v_\lambda\,.$$

By the Poincaré-Birkhoff-Witt Theorem (Theorem 6.5), $M(\lambda) \cong_H U(\mathfrak{n}_-)\otimes \mathbb{C}_\lambda$. **explain this notation; is it "isomorphism as H-algbras"?** We have further

$$M(\lambda)_\mu = \begin{cases} 0 & \text{if } \mu \not\le \lambda, \\ \mathscr{P}(\lambda-\mu) & \text{if } \mu \le \lambda, \end{cases}$$

where $\mathscr{P}$ is the Kostant partition function: $\mathscr{P}(\nu) := \dim U(\mathfrak{n}_-)_{-\nu}$, which is by the Poincaré-Birkhoff-Witt Theorem the number of ways to write $\nu \in Q_+$ as a sum of positive roots.

Lemma 18.4.

$$\mathrm{ch}\, M(\lambda) = \frac{e_\lambda}{\prod_{\alpha\in\Phi_+} \left(1-e_{-\alpha}\right)^{m_\alpha}}\,,$$

where Φ_+ is the set of positive roots and m_α the multiplicity of the root α.

Proof. We need

$$\sum_{\nu\in Q_+} \mathscr{P}(\nu)e_{-\nu} := \sum_{\nu\in Q_+} \dim U(\mathfrak{n}_-)_{-\nu} e_{-\nu} = \Big(\prod_{\alpha\in\Phi_+}\left(1-e_{-\alpha}\right)^{m_\alpha}\Big)\,.$$

By Poincaré-Birkhoff-Witt, a basis for $U(\mathfrak{n}_-)_{-\nu}$ is

$$\prod_{\alpha\in\Phi_+} \left(f_\alpha^{(i)}\right)^{a_{\alpha,i}}\,, \text{ where } \sum_{\alpha,i} a_{\alpha,i}\alpha = \nu\,. \qquad \square$$

A very similar argument shows that the representation $M(\lambda)$ has a unique maximal submodule $J(\lambda)$, and hence a unique maximal (irreducible) quotient:

Definition 18.5. If $M(\lambda)$ is the Verma module (as in (18.1)) for the Kac-Moody Lie algebra $\mathscr{L}$ and $J(\lambda) \subset M(\lambda)$ is the unique maximal submodule, let

$$L(\lambda) := M(\lambda)\big/J(\lambda)$$

be the unique maximal quotient of $M(\lambda)$.

Observe that every irreducible object in $\mathscr{O}$ is of the form $L(\lambda)$ for some λ. (We did something similar for the Virasoro algebra.) The goal is to compute $\operatorname{ch} L(\lambda)$, which is unkonwn in general.

Proposition 18.6. *Let $V \in \mathscr{O}$ and $\lambda \in H^*$. Then there exists a filtration*

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_t = 0$$

such that either $V_i\big/V_{i-1} \cong L(\mu)$ for some $\mu \geq \lambda$ or $(V_i\big/V_{i-1})_\mu = 0$ for all $\mu \geq \lambda$.

If $\mu \geq \lambda$, then the number of factors $L(\mu)$ is independent of the choice of the filtration and of the choice of λ.

Definition 18.7. We define $\left[V : L(\mu)\right]$ to be the multiplicity of $L(\mu)$ in a filtration V from Proposition 18.6.

Proof of Proposition 18.6. Existence: V has spectrum bounded above, $a(V,\lambda) = \sum_{\mu\geq\lambda} \dim V_\mu$ is finite. We use induction on $a(V,\lambda)$. If $a(V,\lambda) = 0$, then we have a filtration $V = V_0 \supset V_1 = 0$.

If $a(V,\lambda) > 0$, then there exists a weight μ with $\mu \geq \lambda$ in V. Choose a maximal weight and let $0 \neq v \in V_\mu$. By maximality, $\mathfrak{n}_+.v = 0$, so $N := U(\mathscr{L}).v \subset V$, so N is a quotient $M(\mu) \twoheadrightarrow N$, $v_\mu \mapsto v$. Let N' be the image of $J(\mu) \subset M(\mu)$ under the quotient map. Then $V \supseteq N \supset N' \supseteq 0$, and

$$a(N',\lambda) < a(V,\lambda)\ , \quad a(V/N,\lambda) < a(V,\lambda)$$

because $(N\big/N')_\mu \neq 0$. So induction works.

The proof of uniqueness is tricker, but not very illuminating. □

Corollary 18.8. *If $V \in \mathscr{O}$, then* $\operatorname{ch} V = \sum_{\lambda\in H^*} \left[V : L(\lambda)\right] \operatorname{ch} L(\lambda)$. □

18.2 Kac's Casimir

If $\mathscr{L}$ is a *finite-dimensional* simple Lie algebra, then there is a special element, the *Casimir operator*

$$\Omega^{\mathrm{fd}} := \sum_i x_i y_i \in U(\mathscr{L})\ ,$$

where $\{x_i\}$ and $\{y_i\}$ are dual bases with respect to the non-degenerate invariant form on $\mathfrak{g}$. It is easy to see that in fact Ω^{fd} lies in the centre of $U(\mathscr{L})$.

There exists an explicit version (via the Killing form). If $\{h'_1,\dots,h'_n\}$ is a basis for $\mathfrak{h}^*$ and $\{h''_1,\dots,h''_n\}$ is the dual basis, and if $e_\alpha \in \mathscr{L}_\alpha$, $f_\alpha \in \mathscr{L}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h'_\alpha$ for all $\alpha \in \Phi_+$, then

$$\begin{aligned}
\Omega^{\mathrm{fd}} &= \sum_{i=1}^n h'_i h''_i + \sum_{\alpha\in\Phi_+} \left(e_\alpha f_\alpha + f_\alpha e_\alpha\right) \\
&= \sum_{i=1}^n h'_i h''_i + \sum_{\alpha\in\Phi_+} \left(2 f_\alpha e_\alpha + h'_\alpha\right) \\
&= \sum_{i=1}^n h'_i h''_i + 2\sum_{\alpha\in\Phi_+} \left(f_\alpha e_\alpha\right) + 2h'_\rho\ ,
\end{aligned}$$

where $2\rho = \sum_{\alpha\in\Phi_+}\alpha$ satisfies $\rho(h_i') = 1$, $\langle h_\rho', h\rangle = \rho(h)$.

For infinite-dimensional Lie algebras, we define

$$\Omega := \sum_i h_i' h_i'' + 2h_\rho' + 2\sum_{\alpha\in\Phi_+}\sum_i f_\alpha^{(i)} e_\alpha^{(i)} ,$$

where the $f_\alpha^{(i)}$ form a basis for $\mathscr{L}_{-\alpha}$ and the $e_\alpha^{(i)}$ form a basis for $\mathscr{L}_\alpha$ such that $\left\langle f_\alpha^{(i)}, e_\alpha^{(j)}\right\rangle = \delta_{ij}$; the h_i' and h_i'' are mutually dual bases of H, and $\langle h_\rho', h\rangle = \rho(h)$ defined by $\rho(h_i) = 1$ for all $i = 1,\dots,n$. (Note that ρ is not unique, as the h_i do not necessarily form a basis!)

Then Ω acts on V for $V\in\mathscr{O}$, denoted by Ω_V.

Lemma 18.9 (Technical manipulation of elements)**.**

$$\Omega \in \mathrm{End}_{\mathscr{O}}\big(\mathrm{id}_{\mathscr{O}}\big)\ , \quad u\colon V\to V\ , \Omega_V.u = u.\Omega_V \textit{ for } u\in U(\mathscr{L})$$

explain this more verbosely

Proposition 18.10. *If $M(\lambda)$ is the Verma module for the Kac-Moody Lie algebra $\mathscr{L}$ as in (18.1), then*

$$\Omega_{M(\lambda)} = \big(\langle\lambda+\rho,\lambda+\rho\rangle - \langle\rho,\rho\rangle\big)\,\mathrm{id}_{M(\lambda)}\ ,$$

where the inner product on H^* is induced from inner product on H.

Proof. With $v_\lambda\in M(\lambda)$, we have

$$\begin{aligned}\Omega.v_\lambda &= \big(\sum_i h_i' h_i'' + 2h_\rho' + 2\sum_{\alpha\in\Phi_+}\sum_i \cancel{f_\alpha^{(i)} e_\alpha^{(i)}}\big).v_\lambda\\ &= \big(\sum_i \lambda(h_i')\lambda(h_i'') + 2\lambda(h_\rho')\big)v_\lambda\\ &= \big(\langle\lambda,\lambda\rangle - 2\langle\lambda,\rho\rangle\big)v_\lambda\ .\end{aligned}$$

Now use Lemma 18.9 an the fact that $M(\lambda) = U(\mathscr{L}).v_\lambda$. □

Corollary 18.11.

$$\Omega_{L(\lambda)} = \big(\langle\lambda+\rho,\lambda+\rho\rangle - \langle\rho,\rho\rangle\big)\,\mathrm{id}_{L(\lambda)}\ .$$

18.3 The Weyl-Kac formula

Definition 18.12. A representation V of $\mathscr{L}$ is *integrable* if $V = \bigoplus_{\lambda\in H^*} V_\lambda$ and $e_i, f_i\colon V\to V$ act locally nilpotently for $i = 1,\dots,n$.

Example. The adjoint representation $\mathscr{L}$ is integrable. Any finite-dimensional representation is integrable.

Lemma 18.13. *If V is integrable, then* $\dim V_\lambda = \dim V_{w(\lambda)}$ *for all $w\in W$.*

Sketch of proof. It is easy to show that $\dim V_\lambda = \dim V_{s_i(\lambda)}$ for each simple reflection s_i. An $\mathfrak{sl}_2$-calculation for $\langle e_i, h_i, f_i\rangle$ acting on V shows that integrability implies that V is locally finite. Finally, $\mathfrak{sl}_2$-representation theory shows that $\dim V_\lambda = \dim V_{s_i(\lambda)}$. □

Proposition 18.14. *$L(\lambda)$ is integrable if and only if $\lambda(h_i)\in\mathbb{Z}_{\ge 0}$ for all $i = 1,\dots,n$. In this case we say that λ is* dominant *and* integral.

Remark 18.15. Since integrability says that elements of the Lie algebra act locally nilpotently, the exponential is well defined and we may pass from the Lie algebra to a Lie group. In some sense integrability is an analogue of finite-dimensionality (of the representation).

Proof. We know the $\mathfrak{sl}_2$-representation theory. Let v_λ be the highest weight of $L(\lambda)$. Then $\{f_i^r v_\lambda\}_{r\geq 0}$ span a $\mathfrak{sl}_2$-representation (where $\mathfrak{sl}_2 = \langle e_i, f_i, h_i\rangle$). This representation is finite-dimensional if and only if $f_i^r v_\lambda = 0$ for $r \gg 0$, and only if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$

Conversely, if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all i, then take $v \in L(\lambda)$ such that $v = u.v_\lambda$ for some $u \in U(\mathscr{L})$ and apply the Leibniz rule:

$$f_i^N(u.v_\lambda) = \sum_{k=0}^{N} \binom{N}{k} \big((\operatorname{ad} f_i)^k(u)\big).(f_i^{N-k}.v_\lambda) = 0\,,$$

where $(\operatorname{ad} f_i)^k = 0$ for $k \gg 0$ and $f_i^{N-k} = 0$ for $N-k \gg 0$. So for $N \gg 0$, $f_i^N(u.v_\lambda) = 0$. □

Theorem 18.16 (Weyl-Kac character formula)**.** *Let $\lambda \in X^+$, where*

$$X^+ := \{\lambda \in H^* : \lambda(h_i) \in \mathbb{Z},\ \lambda(h_i) \geq 0 \text{ for all } i = 1,\dots,n\}\,.$$

Then

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W} \epsilon(w) e\big(w(\lambda+\rho)-\rho\big)}{\prod_{\alpha\in\Phi_+} \big(1-e(-\alpha)\big)^{m_\alpha}}\,,$$

where $\epsilon\colon W \to \{\pm 1\}$ is defined as a group homomorphism with $\epsilon(s_i) = -1$.

Clarify whether $e(-\alpha)$ is the same as $e_{-\alpha}$ before, and maybe recall what it is.

Proof. We have

$$\operatorname{ch} M(\lambda) = \sum_{\mu\in H^*} \big[M(\lambda) : L(\mu)\big] \operatorname{ch} L(\mu)\,,$$

and $\mu \leq \lambda$ if $\big[M(\lambda) : L(\mu)\big] \neq 0$. The Casimir operator Ω acts on

$$\begin{aligned} M(\lambda) \quad &\text{via} \quad \|\lambda+\rho\|^2 - \|\rho\|^2\,, \\ \text{and on } L(\mu) \quad &\text{via} \quad \|\mu+\rho\|^2 - \|\rho\|^2\,. \end{aligned}$$

Therefore $\|\lambda+\rho\|^2 = \|\mu+\rho\|^2$ [**check; condition missing?**], and

$$\operatorname{ch} M(\lambda) = \sum_{\substack{\mu\leq\lambda \\ \|\lambda+\rho\|^2 = \|\mu+\rho\|^2}} \big[M(\lambda) : L(\mu)\big] \operatorname{ch} L(\mu)\,. \tag{18.2}$$

Now define

$$C_\lambda := \Big\{\mu \leq \lambda : \|\lambda+\rho\|^2 = \|\mu+\rho\|^2\Big\}\,.$$

Then $\mu \in C_\lambda \Rightarrow C_\mu \leq C_\lambda$. Also note that $\big[M(\lambda) : L(\lambda)\big] = 1$. Running through the entire set C_λ and looking at the formula for $\operatorname{ch} M(\mu)$ in Equation (18.2) we find that for all $\mu \in C_\lambda$

$$\operatorname{ch} M(\mu) = \sum_{\tau\in C_\lambda} a_{\tau\mu} \operatorname{ch} L(\tau)\,,$$

where $a_{\mu\mu} = 1$ and $a_{\tau\mu} = 0$ unless $\tau \leq \mu$ (i.e. with a total ordering on C_λ that refines the ordering $\leq$). The matrix $(a_{\tau\mu})_{\tau,\mu\in C_\lambda}$ is triangular with 1 on the diagonal. Hence

$$\operatorname{ch} L(\lambda) = \sum_{\mu\in C_\lambda} b_{\lambda\mu} \operatorname{ch} M(\mu)$$

with $b_{\lambda\mu} \in \mathbb{Z}$. For example,

$$e(\rho) \prod_{\alpha\in\Phi_+} \big(1-e(-\alpha)\big)^{m_\alpha} \operatorname{ch} L(\lambda) = \sum_{\mu\in C_\lambda} b_{\lambda\mu} e(\mu+\rho)\,.$$

Since $L(\lambda)$ is integrable, $\operatorname{ch} L(\lambda)$ is W-invariant. So

$$\begin{aligned}
& s_i\Big(e(\rho) \prod_{\alpha\in\Phi_+\setminus\{\alpha_i\}} \big(1-e(-\alpha)\big)^{m_\alpha} \big(1-e(-\alpha_i)\big) \operatorname{ch} L(\lambda)\Big) \\
= \;& e(\rho-\alpha_i) \prod_{\alpha\in\Phi_+\setminus\{\alpha_i\}} \big(1-e(-\alpha)\big)^{m_\alpha} \big(1-e(-\alpha_i)\big) \operatorname{ch} L(\lambda) \\
= \;& -e(\rho) \underbrace{\prod_{\alpha\in\Phi_+} \big(1-e(-\alpha)\big)^{m_\alpha}}_{\Delta} \operatorname{ch} L(\lambda)\,,
\end{aligned}$$

whence we deduce

$$\begin{aligned}
\Rightarrow \quad & w\big(e(\rho)\Delta \operatorname{ch} L(\lambda)\big) = \epsilon(w) e(\rho)\Delta \operatorname{ch} L(\lambda) \\
\Rightarrow \quad & w\big(\sum_\mu b_{\lambda\mu} e(\mu+\rho)\big) = \epsilon(w) \sum_\mu b_{\lambda\mu} e(\mu+\rho) \\
\Rightarrow \quad & b_{\lambda\mu} = \epsilon(w) b_{\lambda\nu} \text{ if } w(\mu+\rho) = \nu+\rho\,.
\end{aligned}$$

Suppose that $b_{\lambda\mu} \neq 0$. Then $b_{\lambda\mu} \neq 0$ for all ν such that $w(\mu+\rho) = \nu+\rho$ and $\nu \leq \lambda$. Pick ν such that $\lambda-\nu$ has minimal height. Then $\nu+\rho \in X^+$ since

$$s_i w(\mu+\rho) = s_i(\nu+\rho) = \nu+\rho-(\nu+\rho)(h_i)\alpha_i\,.$$

We have $\langle \nu+\rho, \nu+\rho\rangle = \langle \lambda+\rho, \lambda+\rho\rangle$ since $\nu \in C_\lambda$, and so $(\lambda+\rho)-(\nu+\rho) = \sum_i k_i\alpha_i$ with $k_i \geq 0$ for all i. So

$$\begin{aligned}
0 = \|\lambda+\rho\|^2 - \|\nu+\rho\|^2 &= \langle \lambda+\nu+2\rho, \lambda-\nu\rangle \\
&= \sum_i k_i \langle \lambda+\nu+2\rho, \alpha_i\rangle \\
&= \sum_i k_i \frac{\langle \alpha_i,\alpha_i\rangle}{2} \big(\lambda+\nu+2\rho\big)(h_i)
\end{aligned}$$

But note that each summand is non-negative: $k_i \geq 0$, $\langle \alpha_i, \alpha_i\rangle \geq 0$ and $(\lambda+\rho)(h_i), (\mu+\rho)(h_i) \geq 0$. Therefore $\lambda = \nu$. So

$$\begin{aligned}
b_{\lambda\mu} \neq 0 \quad \Rightarrow \quad & \mu+\rho = w(\lambda+\rho) \text{ for some } w \in W \\
\Rightarrow \quad & b_{\lambda\mu} = \epsilon(w) b_{\lambda\lambda} = \epsilon(w) \\
\Rightarrow \quad & e(\rho)\Delta \operatorname{ch} L(\lambda) = \sum_{w\in W} \epsilon(w) e\big(w(\lambda+\rho)\big) \\
\Rightarrow \quad & \operatorname{ch} L(\lambda) = \frac{\sum_{w\in W} \epsilon(w) e\big(w(\lambda+\rho)-\rho\big)}{\Delta}\,.
\end{aligned}$$

□

Corollary 18.17 (Kac denominator formula)**.**

$$e(\rho) \prod_{\alpha\in\Phi_+} \big(1-e(-\alpha)\big)^{m_\alpha} = \sum_{w\in W} \epsilon(w) e\big(w(\rho)\big)$$

Proof. Take $\lambda = 0$, i.e. the trivial representation $L(0) = \mathbb{C}$. □

There exists a formula for $\dim L(\lambda)_\mu$ for $\lambda \in X^+$ (the *Kostant partition function*).

Exercise 18.18. Using this style of representation theory (and more work), one can show the presentation for a symmetrisable Kac-Moody Lie algebra is what we used plus

$$(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0\,, \quad (\operatorname{ad} f_i)^{1-A_{ij}} f_j = 0$$

(see Carter §19.4, Kac Theorem 9.11).

19 W − K + A = M

Let $\mathscr{L} = \widehat{\mathfrak{g}}$ be an untwisted Lie algebra (i.e. $\widehat{\mathfrak{g}} = \mathscr{L}(\widehat{A})$ for some untwisted, affine generalised Cartan matrix $\widehat{A}$), where $\mathfrak{g} = \mathscr{L}^0$ is a simple, finite-dimensional Lie algebra with root system Φ^0 and Weyl group W^0. We know the roots of $\mathscr{L}$:

$$\Phi = \{\alpha + n\delta^{\mathrm{real}} : \alpha \in \Phi^0, n \in \mathbb{Z}\} \cup \{n\delta^{\mathrm{im}} : n \neq 0\} ,$$

where the imaginary roots have multiplicity $\mathrm{rk}\,\mathscr{L}^0 =: l$. The root $\delta = a_0\alpha_0 + \cdots + a_l\alpha_l$ is dual to $a_0^{-1}d$ (but $a_0 = 1$ in the untwisted case). The positive roots are

$$\Phi_+ = \{\alpha \in \Phi^0_+\} \cup \{\alpha + n\delta : n \geq 0, \alpha \in \Phi^0\} \cup \{n\delta : n > 0\} .$$

19.1 Affine Weyl groups

The Weyl group of $\mathscr{L}$ is $W = \langle s_0, s_1, \ldots, s_l\rangle$, and $W^0 = \langle s_1, \ldots, s_l\rangle$. Let $\theta := a_1\alpha_1 + \cdots + a_l\alpha_l \in \Phi^0$, so $\langle\theta,\theta\rangle = 2$. We can write

$$\theta = \sum_{i=1}^{l} a_i \frac{\langle\alpha_i,\alpha_i\rangle}{2}\frac{2\alpha_i}{\langle\alpha_i,\alpha_i\rangle} ,$$

and the associated reflection is $s_\theta \in W^0$. Writing $s_\theta(h) = h - \theta(h)h_\theta$, we get

$$h_\theta = \sum_{i=1}^{l} a_i \frac{\langle\alpha_i,\alpha_i\rangle}{2} h_i = \sum_{i=1}^{l} \frac{2c_i}{2} h_i = c - c_0 h_h = c - h_0 .$$

The elements $\{s_0 s_\theta, s_1, \ldots, s_l\}$ generate W. We have

$$\begin{aligned} s_0 s_\theta(h) &= s_0\big(h - \theta(h)h_\theta\big) \\ &= h - \alpha_0(h)h_0 - \theta(h)\big(h_\theta - \alpha_0(h_\theta)h_0\big) \\ &= h + \delta(h)h_\theta - \big(\theta(h) + \delta(h)\big)c \\ &= h + \delta(h)h_\theta - \big(\langle h_\theta, h\rangle + \tfrac{1}{2}\langle h_\theta, h_\theta\rangle\delta(h)\big)c . \end{aligned}$$

For $x \in H^0$, there is a map

$$t_x \colon H \to H , \quad h \mapsto h + \delta(h)x - \big(\langle x, h\rangle + \tfrac{1}{2}\langle x, x\rangle\delta(h)\big)c .$$

Lemma 19.1. $t_x t_y = t_{x+y}$ *and* $w t_x w^{-1} = t_{w(x)}$ *for* $w \in W^0$.

Proof. Easy calculation. □

Proposition 19.2. *$W = t(M) \rtimes W^0$, where $t(M) = \{t_m : m \in M\}$ and $M \subset H^0$ is the space generated as an abelian group by $w(h_\theta)$ for all $w \in W^0$.*

Remark 19.3. In the untwisted case, an elementary computation shows that $M = \bigoplus_{i=1}^{l} \mathbb{Z}h_i$.

Proof. We have $H = H^0 \oplus \mathbb{C}c \oplus \mathbb{C}d$, and $\delta\big(w(h)\big) = (w^{-1}\delta)(h) = \delta(h)$. The set

$$H_1 := \{h \in H : \delta(h) = 1\} = \{h^0 + \lambda c + d\}$$

has an action of W. Since $w(c) = c$ for all $w \in W$, this action descends to an action of W on $H_1/\mathbb{C}c \cong H^0$ (and W^0 acts on H^0 as usual, and $t_m(h^0) = h^0 + m$); $h^0 + \lambda c + d \mapsto h^0 + d \mapsto h^0$.

Since $H \xrightarrow{\cong} H^*$, we need the action of W on H^*. We have $W = t(M^*) \rtimes W^0$, where M^* is the lattice spanned by long roots and

$$t_\alpha(\lambda) = \lambda + \lambda(c)\alpha - \big((\lambda,\alpha) + \frac{1}{2}(\alpha,\alpha)\lambda(c)\big)\delta .$$

□

19.2 Formulae

Recall that $\rho(h_i) = 1$ and $\rho(d) = 0$. The number $\rho(c) = h^\vee = c_0 + \cdots + c_l$ is the *dual Coxeter number*. With $\rho^0 \in (H^0)^*$ such that $\rho^0(h_i) = 1$ for $i = 1, \dots, l$ we have

$$\begin{aligned} w(\rho) - \rho &= w^0 t_\alpha(\rho) - \rho \\ &= w^0\Big(\rho + \rho(c)\alpha - \big(\langle \rho, \alpha\rangle + \tfrac{1}{2}\langle \alpha, \alpha\rangle \rho(c)\big)\delta\Big) - \rho \\ &= w^0(h^\vee\alpha + \rho^0) - \rho^0 - \frac{\big(\langle \rho^0 + h^\vee\alpha, \rho^0 + h^\vee\alpha\rangle - \langle \rho^0, \rho^0\rangle\big)\delta}{2h^\vee} . \end{aligned}$$

We define

$$c(\lambda) := \left\|\lambda + \rho^0\right\|^2 - \left\|\rho^0\right\|^2 ,$$

so that the finite-dimensional Casimir operator on $L^0(\lambda)$ is $\Omega_{L^0(\lambda)} = c(\lambda)\,\mathrm{id}$ (cf. Corollary 18.11). Then we have a character formula

$$\chi^0(\lambda) := \mathrm{ch}\, L^0(\lambda) = \sum_{w\in W^0} \epsilon(w) e\big(w(\lambda + \rho^0) - \rho^0\big) \Big/ \sum_{w \in W^0} \epsilon(w) e\big(w(\rho^0) - \rho^0\big) .$$

So

$$\begin{aligned} \sum_{w\in W} \epsilon(w) e\big(w(\rho) - \rho\big) &= \sum_{\alpha\in M^*}\sum_{w\in W^0} \epsilon(w^0) e\big(w^0(h^\vee\alpha + \rho^0) - \rho^0\big) e\Big(\frac{-c(h^\vee\alpha)}{2h^\vee}\delta\Big) \\ &= \sum_{w\in W^0} \epsilon(w^0) e(w^0\rho^0 - \rho^0) \sum_{\alpha\in M^*} \chi^0(h^\vee\alpha) e\Big(\frac{-c(h^\vee\alpha)}{2h^\vee}\delta\Big) \\ &= \prod_{\alpha\in\Phi^0_+}\big(1 - e(-\alpha)\big) \sum_{\alpha\in M^*} \chi^0(h^\vee\alpha) e\Big(\frac{-c(h^\vee\alpha)}{2h^\vee}\delta\Big) . \end{aligned}$$

Theorem 19.4 (MacDonald)**.** *Set $q := e(-\delta)$. Then*

$$\prod_{n>0}(1 - q^n)^l \prod_{\alpha\in\Phi^0}(1 - q^n e_{-\alpha}) = \sum_{\alpha\in M^*} \chi^0(h^\vee\alpha)\; q^{c(h^\vee\alpha)/2h^\vee} , \tag{19.1}$$

where $q = e(-\delta)$.

Example. Consider $\widetilde{A}_1$. We have $l = 1$, $\Phi^0 = \{\pm\alpha_1\}$, $h^\vee = 2$, $M^* = \mathbb{Z}\alpha_1$ and $\rho^0 = \frac{1}{2}\alpha_1$. Set $z := e_{-\alpha_1}$; then

$$\chi^0(n\alpha_1) = \frac{e(n\alpha_1) - e(-(n+1)\alpha_1)}{1 - e(-\alpha_1)} = \frac{z^{-n} - z^{n+1}}{1 - z} .$$

The Casimir operator is multiplication by

$$c(2n\alpha_1) = \big\langle \big(2n + \tfrac{1}{2}\big)\alpha_1, \big(2n + \tfrac{1}{2}\big)\alpha_1\big\rangle - \big\langle \tfrac{1}{2}\alpha_1, \tfrac{1}{2}\alpha_1\big\rangle = 4n(2n+1) .$$

Hence the right-hand side of Equation (19.1) is

$$\sum_{n\in\mathbb{Z}} \frac{z^{-2n} - z^{2n+1}}{1 - z} q^{n(2n+1)} = \frac{1}{1 - z} \sum_{m\in\mathbb{Z}} (-1)^m z^m q^{m(m-1)/2} .$$

Plugging in the left-hand side, we get

$$\prod_{n>0}(1 - q^n)(1 - q^{n-1}z)(1 - q^n z^{-1}) = \sum_{m\in\mathbb{Z}} (-1)^m z^m q^{m(m-1)/2} ,$$

which is the *Jacobi triple product identity*.

Exercise 19.5. Do this for $\widetilde{A}_1'$:

$$\prod_{n>0}(1-q^n)(1-q^nz^{-1})(1-q^{n-1}z)(1-q^{2n-1}z^{-2})(1-q^{2n-1}z^2) = \sum_{n\in\mathbb{Z}}\left(z^{3n}-z^{-3n+1}\right)q^{n(3n-1)/2}$$

This is the *quintuple product identity*, known already in antiquity.

If we set $e_\alpha = 1$ for all $\alpha\in\Phi^0$, this produces other identities. (For instance, $\dim L^0(\lambda) = d^0(\lambda)$.)

Theorem 19.6. *Let* $\phi(q) := (1-q)(1-q^2)(1-q^3)\cdots$. ***What about convergence?*** *Then*

$$\phi(q)^{\dim\mathscr{L}^0} = \sum_{\alpha\in M^*} d^0(h^\vee\alpha)\, q^{c(h^\vee\alpha)/2h^\vee}\ .$$

Example. For $\widetilde{A}^1$, $\phi(q)^3 = \sum_{n\in\mathbb{Z}}(4n+1)q^{n(2n+1)}$.

Exercise 19.7. Repeat the entire section for the twisted version.

20 KP hierarchy and Lie theory

In 1834 John Scott Russel created the first *soliton* in history in historic Edinburgh. The basic governing equation of a soliton (a solitary wave) is the *Korteveg-de Vries (KdV) equation*:

$$u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u\,u_x = 0 \tag{20.1}$$

It is a non-linear partial differential equation in $\mathbb{R}^{1+1}$ that describes the behaviour of a (one-dimensional) wave in shallow water. There is a two-dimensional version, called the KP equation:

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u\,u_x\right)_x = \frac{3}{4}u_{yy} \tag{20.2}$$

Greek mythology. An *integrable* system is a Procrustean bed. Integrable systems have

- exact solutions (in this case solitons),
- remarkable symmetries,
- Lax-pair formalism.

The KP equation fints into an infinite family of PDEs, all of which are governed simultaneously. We will see (a) solutions of Equation (20.2), and (b) a family of solutions, which is an inifinite-dimensional homogeneous manifold. ("If you have one solution, you have every solution.")

Recall from Section 9.1 that from a space $V = \bigoplus_{n\in\mathbb{Z}}\mathbb{C}v_n$ we can form the space $F^{(0)}$, on which we have an action of $\mathfrak{gl}_\infty \subset \mathrm{End}(V)$, which is spanned by elements E_{ij} of "infinite matrices with only finitely many non-zero entries". The Lie group attached to this algebra is

$$GL_\infty := \left\{A = (a_{ij})_{i,j\in\mathbb{Z}} : A \text{ is invertible and almost all } a_{ij} - \delta_{ij} = 0\right\}$$

We define

$$\Omega := GL_\infty.v_{0,-1,-2,\dots} = \left\{u_0\wedge u_{-1}\wedge u_{-2}\wedge\cdots : u_{-m} = v_{-m} \text{ for } m \gg 0\right\},$$

where $v_{0,-1,-2,\ldots}$ is the *vacuum vector* in degree 0. There is a natural scaling action on Ω, so if we define a Graßmannian-like object

$$\mathrm{Gr} := \left\{ U < V : \text{ there is } k \text{ such that } U \geq V^{\leq k} = \bigoplus_{i \leq k} \mathbb{C} v_i \text{ and } \dim\left(U/V^{\leq k}\right) = k \right\} ,$$

then we have a bijection (**Exercise:** Check!)

$$\mathbb{P}\Omega \xrightarrow{\cong} \mathrm{Gr} \ , \quad u_0 \wedge u_{-1} \wedge u_{-2} \wedge \cdots \mapsto \bigoplus_{i \leq 0} \mathbb{C} u_i \ .$$

There are two operators, $F^{(0)} \to F^{(1)}$ and $F^{(0)} \to F^{(-1)}$, respectively via V and V^* (the restricted dual, cf. Theorem 7.3), defined as follows. For $v \in V$ and $f \in V^*$, we have operations

$$\begin{aligned} \widehat{v}.v_{i_0} \wedge v_{i_1} \wedge \cdots &= v \wedge v_{i_0} \wedge v_{i_1} \text{ , and} \\ \widehat{f}.v_{i_0} \wedge v_{i_1} \wedge \cdots &= \sum_{j \geq 0} (-1)^j v_{i_0} \wedge v_{i_1} \wedge \cdots \wedge v_{i_{j-1}} f(v_{i_j}) \wedge v_{i_{j+1}} \wedge \cdots \ . \end{aligned}$$

The element E_{ij} acts on $F^{(0)}$ via $\widehat{v}_i \widehat{v}_j^*$.

Lemma 20.1. *If $\tau \in \Omega$, then*

$$\sum_{j \in \mathbb{Z}} \widehat{v}_j(\tau) \otimes \widehat{v}_j^*(\tau) = 0 \ , \tag{20.3}$$

and if $\tau \neq 0$ and τ satisfies Equation (20.3), then $\tau \in \Omega$.

Proof. For $\tau = v_{0,-1,-2,\ldots}$ Equation (20.3) is obviously satisfied. We prove that the left-hand side of Equation (20.3) is GL_∞-invariant:

$$A\left(\sum_{j \in \mathbb{Z}} \widehat{v}_j(\tau) \otimes \widehat{v}_j^*(\tau)\right) A^{-1} = \sum_{j \in \mathbb{Z}} \widehat{v}_j(\tau) \otimes \widehat{v}_j^*(\tau) \ .$$

Then it follows that Equation (20.3) holds in general, since $\tau = A v_{0,-1,-2,\ldots}$, and so

$$A\left(\sum_{j \in \mathbb{Z}} \widehat{v}_j(\tau) \otimes \widehat{v}_j^*(\tau)\right) A^{-1} A\, v_{0,-1,-2,\ldots} = 0 \ .$$

Conversely, assume that τ satisfies Equation (20.3) and that $\tau \neq 0$. Write $\tau = \sum_{k=1}^N c_k \tau_k$, where τ_k are semi-infinite monomials and all $c_k \neq 0$. Without loss of generality, τ_1 has maximal degree among the τ_k. If τ_2 is of the form $\tau_2 = E_{ij}.\tau_1$ with $i < j$, then $E_{ij}^2.\tau_1 = 0$, and so $\exp\left(-c_2 E_{ij}\right).\tau$ removes the τ_2-term.

(Note that the degree is $\deg(v_{\mathbf{i}}) = \sum_{m \geq 0} (i_m + m)$ and that $\exp\left(-c_2 E_{ij}\right) \in GL_\infty$.)

Now by GL_∞-invariance, the new expression also satisfies Equation (20.3), and by iterating we obtain $\tau = \tau_1 + \phi$, where ϕ has no terms of the form $E_{ij}.\tau_1$. Then Equation (20.3) implies

$$\sum_{j \in \mathbb{Z}} \left(\widehat{v}_j.\tau_1 \otimes \widehat{v}_j^*.\phi + \widehat{v}_j.\phi \otimes \widehat{v}_j^*.\tau_1 + \widehat{v}_j.\phi \otimes \widehat{v}_j^*.\phi \right) = 0 \ .$$

So for each j we must have, up tp sign, that

$$\widehat{v}_j.\tau_1 = \widehat{v}_k.(\text{semi-infinite wedge in } \phi) \ ,$$

which forces $E_{kj}.\tau_1$, contradiction. **Clarify this sentence.** We conclude that $\phi = 0$. □

Recall further that we had shown that $F^{(0)} \cong \mathbb{C}[x_1, x_2, \ldots] = B(0,1)$, the bosonic representation. Since $\mathfrak{gl}_\infty$ acts on $F^{(0)}$, it also acts on $\mathbb{C}[x_1, x_2, \ldots]$. How does it act, and where do the v_λ $(= v_{\mathbf{i}})$ go? We introduce the *vertex operators*

$$X\colon F^{(0)} \to \widehat{F^{(1)}}\,,\ X(u) := \sum_{j\in\mathbb{Z}} u^j\,\widehat{v}_j\,, \quad \text{and}\quad X^*\colon F^{(0)} \to \widehat{F^{(1)}}\,,\ X^*(u) := \sum_{j\in\mathbb{Z}} u^{-j}\,\widehat{v}_j^*\,.$$

These become operators on $B(0,1)$, too, by the following fact:

$$X(u) \mapsto u\exp\Big(\sum_{j\ge 1} u^j x_j\Big)\exp\Big(-\sum_{j\ge1}\frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\Big)\,,\ X^*(u) \mapsto \exp\Big(-\sum_{j\ge 1} u^j x_j\Big)\exp\Big(\sum_{j\ge1}\frac{u^{-j}}{j}\frac{\partial}{\partial x_j}\Big)$$

This answers the question how $\mathfrak{gl}_\infty$ acts. For the image of v_λ, note that $v_\lambda \mapsto S_\lambda(x)$, the Schur polynomial, given by $S_\lambda(x) = \det\big(S_{\lambda_i+j-1}(x)\big)$, where the *elementary Schur polynomials* S_k are such that

$$\sum_{k\in\mathbb{Z}} S_k(x)\,z^k = \exp\Big(\sum_{k=1}^{\infty} x_k\,z^k\Big)\,.$$

We translate Ω to $B(0,1) = \mathbb{C}[x_1, x_2, \ldots]$ by identifying

$$F^{(0)} \otimes F^{(0)} \ \text{ with }\ \mathbb{C}[x_1', x_2', \ldots] \otimes \mathbb{C}[x_1'', x_2'', \ldots]\,.$$

Then $X(u).\tau \otimes X^*(u).\tau$ has vanishing u^0-term, i.e.

$$u\exp\Big(\sum_{j\ge1} u^j\big(x_j' - x_j''\big)\Big)\exp\Big(\sum_{j\ge1}\frac{u^{-j}}{j}\big(\partial_{x_j''} - \partial_{x_j'}\big)\Big).\tau(x')\tau(x'')$$

has vanishing u^0-term. Rewrite $x' = X - Y$, $x'' = X + Y$. So $(-).\tau \in \mathbb{C}[x_1, x_2, \ldots]$ belongs to Ω if and only if the u^0-term vanishes in the expression

$$u\exp\Big(-\sum_{j\ge1} 2u^j Y_j\Big)\exp\Big(\sum_{j\ge1}\frac{u^{-j}}{j}\frac{\partial}{\partial Y_j}\Big).\tau(X-Y)\tau(X+Y)\,. \tag{20.4}$$

The Hirota form. Let $P \in \mathbb{C}[x_1, x_2, \ldots]$ with a finite number of x_i's, and let f and g be functions in those variables. Define an operation

$$(Pf.g)(x) := P\Big(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \ldots\Big)\big(f(x_1-u_1, x_2-u_2, \ldots)\,g(x_1+u_1, x_2+u_2, \ldots)\big)\Big|_{u=0}\,.$$

So Equation (20.4) becomes

$$u\Big(\sum_{j\ge\mathbb{Z}} u^j S_j(-2Y)\Big)\Big(\sum_{j\ge\mathbb{Z}} u^{-j} S_j(\widetilde{\partial}_Y)\Big).\tau(X-Y)\tau(X+Y)\,,$$

where $Y = (Y_1, Y_2, \ldots)$ and $\widetilde{\partial}_Y = \big(\frac{\partial}{\partial Y_1}, \frac12\frac{\partial}{\partial Y_2}, \frac13\frac{\partial}{\partial Y_3}, \ldots\big)$. It follows that Equation (20.3) is equivalent to

$$\sum_{j\ge0} S_j(-2Y)\,S_{j+1}(\widetilde{\partial}_Y)\,\tau(X-Y)\,\tau(X+Y) = 0\,. \tag{20.5}$$

We compute:

$$\begin{aligned}
S_{j+1}(\widetilde{\partial}_Y)\,\tau(X-Y)\,\tau(X+Y) &= S_{j+1}(\widetilde{\partial}_u)\,\tau(X-Y-u)\,\tau(X+Y+u)\Big|_{u=0}\\
&= S_{j+1}(\widetilde{\partial}_u)\exp\Big(\sum_{s\ge1} Y_s\frac{\partial}{\partial u_s}\Big)\tau(X-u)\,\tau(X+u)\Big|_{u=0}\\
&= S_{j+1}(\widetilde{X})\exp\Big(\sum_{s\ge1} Y_s X_s\Big)\tau(X)\,\tau(X)\Big|_{u=0}\,,
\end{aligned}$$

where $\widetilde{X} = \big(X_1, \frac12 X_2, \frac13 X_3, \ldots\big)$.

Exercise 20.2. Show that $Pf.f = 0$ if $P(x) = -P(-x)$.

Theorem 20.3. *$\tau \in \Omega$ if and only if τ is a solution of*

$$\Big(\sum_{j=0}^{\infty} S_j(-2Y)\, S_{j+1}(\widetilde{X})\exp\big(\textstyle\sum_{s\geq 1} Y_s X_s\big)\Big)\tau(X)\,\tau(X) = 0\,, \tag{20.6}$$

where $Y_1, Y_2, \dots$ are parameters, i.e. we have an infinite family of differential equations.

Now expand Equation (20.6) in the Y_i. The variable Y_r appears in $\exp\big(\sum_{s\geq 1} Y_s X_s\big)$ once with coefficient X_r. In the expression $S_j(-2Y)$ the variable Y_r appears when $j = r$, with coefficient -2. So Equation (20.6) has the Y_r-term

$$\big(S_1(\widetilde{X})X_r - 2S_{r+1}(\widetilde{X})\big)\tau\,\tau = 0 \quad \Rightarrow \quad \big(X_1 X_r - 2S_{r+1}(\widetilde{X})\big)\tau\,\tau = 0\,.$$

If $r = 1$ or $r = 2$, this is odd and there are no conditions. If $r = 3$, the term in parentheses is $x_1^4 + 3x_2^2 - 4x_1 x_3$. Setting $x := x_i$, $y := x_2$ and $t := x_3$, we obtain

$$\Big(\frac{\partial^4}{\partial u_1^4} + 3\frac{\partial^2}{\partial u_2^2} - 4\frac{\partial^2}{\partial u_1\,\partial u_3}\Big)\tau(X+u)\,\tau(X-u)\Big|_{u=0} = 0\,.$$

A long exercise shows that $u(x, y, t) = 2\frac{\partial^2}{\partial x^2}\big(\log \tau\big)$ is a solution of the KP equation.

21 The Kazhdan-Lusztig Conjecture

Let A be a symmetrisable generalised Cartan matrix.

21.1 The Shapovalov form and Kac-Kazhdan formulas

The character forlumals for integrable irreducible reprepresentations $L(\mu)$ involves the Weyl group. Recall that we had the Verma modules $M(\lambda)$, and for $M(\lambda+\mu)$ the character formula has the form $L\big(w(\lambda+\rho)\big)$. The Kac-Kazhdan formula is interesting because it shows that the Weyl group W does play a rôle in general.

Definition 21.1. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra. We have a formula for the universal enveloping algebra of $\mathfrak{g}$:

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(H) \otimes U(\mathfrak{n}_+) = U(H) \oplus \big(\mathfrak{n}_-.U(\mathfrak{g}) + U(\mathfrak{g}.\mathfrak{n}_+\big)\,.$$

Denote by $\pi\colon U(\mathfrak{g} \to U(H)$ the projection onto the first direct summand.

Recall that $U(H)$ is a polynomial ring. There exists an anti-linear involution $\sigma\colon \mathfrak{g} \to \mathfrak{g}$, $\sigma(e_i) = f_i$, and $\sigma\colon \mathfrak{h} \to \mathfrak{h}$. Let

$$F\colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(H)\,, \quad F(x, y) = \pi\big(\sigma(x)y\big)\,.$$

This is a family of symmetric, bilinear forms over H^*.

Definition 21.2. The *Shapovalov form* $F(\lambda)\colon M(\lambda) \otimes M(\lambda) \to \mathbb{C}$ is defined by

$$F(\lambda)(x v_\lambda, y v_\lambda) = F(x, y)(\lambda)\,.$$

Exercise 21.3. Check that the Shapovalov form is well defined.

Lemma 21.4. *The Shapovalov form has the following properties:*

- $F(\lambda)\big(xv, w\big) = F(\lambda)\big(v, \sigma(v)w\big)$ *for* $x \in U(\mathfrak{g})$, $v, w \in M(\lambda)$.
- $F(\lambda)\big(M(\lambda)_{\mu_1}, M(\lambda)_{\mu_2}\big) = 0$ *if* $\mu_1 \neq \mu_2$.
- $\operatorname{rad} F(\lambda)$ *is the maximal submodule of* $M(\lambda)$.

We would like to find out about $F(\lambda)$, or at least about $\det F(\lambda)_\eta$, the restriction of $F(\lambda)$ to $M(\lambda)_{\lambda-\eta}$, to help us understand simplicity or composition factors of the $M(\lambda)$'s.

Definition 21.5. Let F_η be the restriction of F to $U(\mathfrak{n}_-)_{-\eta}$.

Theorem 21.6 (Theorem A)**.**

$$\det F_\eta = \prod_{\alpha\in\Delta_+}\prod_{n=1}^{\infty}\Big(h_\alpha + \big\langle \rho - \tfrac{n\alpha}{2}, \alpha\big\rangle\Big)^{\operatorname{mult}(\alpha)\mathscr{P}(\eta-n\alpha)},$$

where $\mathscr{P}$ *is the Kostant partition function, and* $\det F_\eta \in U(H)$.

Theorem 21.7 (Theorem B)**.** $L(\mu)$ *is an irreducible subquotient of* $M(\lambda)$ *if and only if there exists* $\beta_1, \dots, \beta_n \in \Delta_+$ *and* $n_1, \dots, n_k \in \mathbb{N}$ *such that*

$$\big\langle \lambda + \rho - n_1\beta_1 - n_2\beta_2 - \cdots - n_{i-1}\beta_{i-1}, \beta_i\big\rangle = \frac{n_i}{2}\big\langle \beta_i, \beta_i\big\rangle \quad \text{for all } \ i = 1, \dots, k \tag{21.1}$$

and $\lambda - \mu = \sum_{i=1}^{k} n_i\beta_i$.

Observation. Pick $\lambda \in H^*$ such that $\langle\lambda, \delta\rangle = -h^\vee$, the dual Coxeter number, and $\langle \lambda + \rho, \alpha\rangle \notin \mathbb{N}$ for each positive *real* root α. (Such weights exist, e.g. $\lambda = -\rho$.) This weight is called a *KK weight* (for "Kac-Kazhdan"). Then $\big\langle \lambda + \rho, \delta, =\big\rangle\, \frac{n}{2}\langle\delta, \delta\rangle = 0$ for all $n \in \mathbb{N}$, and so $L(\lambda - n\delta)$ is a factor of $M(\lambda)$ for all $n \in \mathbb{N}$. But for $\alpha \in \Delta_+^{\mathrm{re}}$,

$$\big\langle \lambda + \rho - n\delta, \alpha\big\rangle = \big\langle \lambda + \rho, \alpha\big\rangle \notin \mathbb{N}$$

and

$$\frac{n}{2}\big\langle \alpha, \alpha\big\rangle = nd_i \ \text{, where } \ \alpha = w(\alpha_i)\,,$$

and so there are no subquotients of the form $L(\lambda - n\delta - m\alpha)$. In particular, the Weyl group plays no rôle.

Conjecture 21.8 (Kac-Kazhdan, Theorem: Hayashi; Goodman, Wallach)**.** *Let* λ *be a KK weight on an affine Lie algebra. Then*

$$\operatorname{ch} L(\lambda) = e^\lambda \prod_{\alpha\in\Delta_+^{\mathrm{re}}}\big(1 - e^{-\alpha}\big)^{-1}.$$

What are te ingredients of the proof?

- $\det F_\eta$ is a polynomial in $U(H)$, so it can be factored. First calculate the "leading term" and get
$$\prod_{\alpha\in\Delta_+^{\mathrm{re}}}\prod_{n=1}^{\infty} h_\alpha^{\operatorname{mult}(\alpha)\mathscr{P}(\eta-n\alpha)}\,. \tag{21.2}$$
- Use Kac's Casimir to observe that if $v \in M(\lambda)_{\lambda-\beta}$ is a highest-weight vector, then
$$\big\|\lambda + \rho\big\|^2 - \big\|\rho\big\|^2 = \big\|\lambda - \beta + \rho\big\|^2 - \big\|\rho\big\|^2\,,$$
i.e. $\big\langle \lambda + \rho, \beta\big\rangle = \frac{1}{2}\langle\beta, \beta\rangle$.

So if this cannot be satisfied, then $M(\lambda)$ is irreducible and so $F(\lambda)$ is non-degenerate; hence $\det F_\eta$ is a product of linear factors $\big(h_\beta + \langle\rho - \frac{1}{2}\beta, \beta\rangle\big)$, and the formula for leading terms shows that $\beta = n\alpha$ for some $\alpha \in \Delta_+$, which we call a *quasi-root*. So we get products of $\big(h_\alpha + \langle\rho - \frac{1}{2}\alpha, \alpha\rangle\big)$.

The trick. We had already shown that the leading term of $\det F_\eta$ (given by Equation (21.6)) is the expression (21.2), and F_η is a product of finitely many linear terms of the form $h_\alpha + \langle \rho - \frac{n\alpha}{2}, \alpha\rangle$. (Note that the Kostant partition function $\mathscr{P}(\eta - n\alpha)$ vanishes for large n or large α.) Now extend everything to $\mathbb{C}[t]$. For example, if we have the universal enveloping algebra $\widetilde{U}(\mathfrak{g})$ extended over $\mathbb{C}[t]$, then

$$\widetilde{M}(\lambda) = \widetilde{U}(\mathfrak{g}) \otimes_{\widetilde{U}(\mathfrak{n}_+ \oplus H)} \mathbb{C}[t]_{\widetilde{\lambda}} ,$$

where $\widetilde{\lambda} \in H^* \otimes \mathbb{C}[t]$ is given by

$$\widetilde{\lambda}(\mathfrak{n}_+) = 0 \qquad \text{and} \qquad \widetilde{\lambda}(h) = \lambda(h) + t\, z(h) ,$$

where $z \in H^*$ is some generic element in the sense that that $z(h_\alpha) \neq 0$ for all $\alpha \in Q_+ \setminus \{0\}$. This produces a form $F^t_\eta(\lambda)$ taking values in $\mathbb{C}[t]$, and when we specialise $t \to 0$, we recover $F_\eta(\lambda)$, and $\widetilde{M}(\lambda) \to M(\lambda)$. Filter

$$\widetilde{M}(\lambda) = \widetilde{M}^0 \supseteq \widetilde{M}^1 \supseteq \widetilde{M}^2 \supseteq \cdots ,$$

where

$$\widetilde{M}^i = \left\{ v \in \widetilde{M}(\lambda) : F^t(\lambda)(v,w) \in (t^i) \text{ for all } v, w \in \widetilde{M}(\lambda) \right\} ,$$

and specialise $t \to 0$:

$$M(\lambda) = M^0 \supset M^1 \supseteq M^2 \supseteq \cdots$$

This is called *Jantzen's filtration*; here M^1 is a maximal submodule.

Pick λ such that $\lambda(h_\beta) + \langle \rho - \frac{1}{2}\beta, \beta\rangle$ vanishes for some quasi-root but does not vanish for all other quasi-roots. Then we see that

1. $M(\lambda - \beta)$ is irreducible, and
2. $M(\lambda)$ has a submodule U which is a direct sum of modules of the form $M(\lambda - \beta)$.

Now by construction of $\widetilde{\lambda}$ we see that the highest power of t dividing $\det F^t_\eta(\lambda)$ equals the oerder of the linear factor $h_\alpha + \langle \rho - \frac{n\alpha}{2}, \alpha\rangle$ in $\det F_\eta(\lambda)$. But $M^1 = \bigoplus M(\lambda - \beta)$, so if $M(\lambda-\beta)$ appears in M^i with multiplicity $s_i(\beta)$, we see that the order of t dividing $\det F^t_\eta(\lambda)$ is $\sum_i s_i(\beta)\mathscr{P}(\eta - \beta)$.

Example. Suppose $M(\lambda - \beta)$ contains three highest-weight modules of highest weight β, with multiplicities respectively 1, 2 and 5. Then $M^1 = M(\lambda-\beta)^{\oplus 3}$, $M^2 = M(\lambda-\beta)^{\oplus 2}$, and $M^3 = M^4 = M^5 = M(\lambda - \beta)$.

The functions $\phi_\beta \colon Q_+ \to \mathbb{R}$, $\eta \mapsto \mathscr{P}(\eta - \beta)$ are all linearly independent. But comparing the leading term of $F_\eta(\lambda)$ (cf. (21.2)) with the multiplicities in Jantzen's filtration for the quasi-root $\beta = n\alpha$, we get

$$\prod_{\alpha \in \Delta_+^{\mathrm{re}}} \prod_{n=1}^{\infty} h_\alpha^{\mathrm{mult}(\alpha)\mathscr{P}(\eta - n\alpha)} = \prod_{\Delta_+} \prod_{n=1}^{\infty} h_\alpha^{\mathscr{P}(\eta - n\alpha)\sum_i s_i(n\alpha)} .$$

This proves Theorem A.

The proof of Theorem B is an inductive variation on the proof of Theorem A, producing the *Jantzen Sum Formula*

$$\sum_{i \geq 1} \operatorname{ch} M(\lambda)^i = \sum_{(\alpha,n) \in D_\lambda} \operatorname{ch} M(\lambda - n\alpha) , \tag{21.3}$$

where the right-hand-side sum is over the set

$$D_\lambda := \left\{ (\alpha, n) \in \widetilde{\Delta}_+ \times \mathbb{N} : \langle \lambda + \rho - \tfrac{1}{2}\alpha, \alpha\rangle = 0 \right\} .$$

(Here $\widetilde{\Delta}_+$ is the set of positive roots with multiplicities.)

21.2 Bringing in the Weyl group

Definition 21.9. We define an equivalence relation $\sim$ on H^* by $\lambda \sim \mu$ if there exists a sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_n = \mu$ such that for each $0 \le i < k$, either $\{\lambda_i, \lambda_i + 1\}$ or $\{\lambda_{i+1}, \lambda_i\}$ satisfy (21.1) for some $\beta_1, \dots, \beta_n \in \Delta_+$ and $n_1, \dots, n_k \in \mathbb{N}$.

Definition 21.10. Given equivalence class $[\lambda] \equiv \mathscr{C}(\lambda) =: \Lambda$ in $H^* / \sim$, we say that a module $M \in \mathscr{O}$ has *type* Λ if all its "composition factors" belong to Λ. (Cf. Section 18.1 for the category $\mathscr{O}$.) We write $\mathscr{O}_\Lambda$ for the subcategory of modules of type Λ.

Example. It follows from Theorem B of Kac-Kazhdan (Theorem 21.7) that $M(\lambda)$ has type Λ.

Theorem 21.11 (Deodhar, Gabher, Kac)**.** *Let $M \in \mathscr{O}$. Then there exists a unique set $\{M_\Lambda\}_\Lambda$ such that $M = \bigoplus_\Lambda M_\Lambda$. Furthermore,* $\mathrm{Ext}^1_{\mathscr{O}}(F, E) = 0$ *if $E \in \mathscr{O}_\Lambda$, $F \in \mathscr{O}_{\Lambda'}$ and $\Lambda \neq \Lambda'$.*

Theorem 21.11 says that to understand category $\mathscr{O}$, we only need to understand $\mathscr{O}_\Lambda$. The moral will be that some choices of Λ are better than others.

For $w \in W$, we define an action w_* on H^* by $w_* \lambda = w(\lambda + \rho) - \rho$. We set

$$C := \left\{\lambda \in H^* : \left\langle \lambda + \rho, \alpha_i \right\rangle \ge 0 \text{ and } \left\langle \lambda + \rho, \alpha \right\rangle \neq 0 \text{ if } \alpha \in \Delta_+^{\text{isotropic}} \right\},$$

and define *Tits' cone* of "good" elements

$$K^{\mathrm{g}} := \bigcup_{w \in W} w_* C .$$

Define

$$\begin{aligned} S_\alpha &:= \left\{\lambda \in H^* : \langle \lambda - \rho, \alpha \rangle = \tfrac{1}{2} \left\langle \alpha, \alpha \right\rangle \right\}, \\ S &:= \bigcup_{\alpha \in \Delta_+^{\mathrm{im}}} S_\alpha \text{ , and} \\ K^{\mathrm{w.g.}} &:= H^* \setminus S . \end{aligned}$$

Elements of S are "critical hyperplanes", and W acts on the set $K^{\mathrm{w.g.}}$ of "weakly good" elements.

Exercise 21.12. Compare H^*, K^{g} and $K^{\mathrm{w.g.}}$ in the cases

1. finite generalised Cartan matrices,
2. affine generalised Cartan matrices, and
3. $\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ for $a \ge 3$.

(It is clear that $K^{\mathrm{g}} \subseteq K^{\mathrm{w.g.}}$.)

Definition 21.13. Define an equivalence relation $\approx$ as for $\sim$, using K^{g} instead of H^*, and an equivalence relation $\approx^\circ$ as for $\sim$, using $K^{\mathrm{w.g.}}$ instead of H^*.

Proposition 21.14. *If $\lambda, \mu \in K^{w.g.}$, then $\lambda \approx^\circ \mu$ if and only if there exists $w \in W(\lambda)$, the integral Weyl group generated by*

$$\left\{ s_\beta : \beta \in \Delta_+^{\mathrm{re}} \text{ such that } \frac{2\langle \lambda + \rho, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \right\},$$

such that $w_* \lambda = \mu$.

Proof. We prove a better result. Let $\lambda \in K^{\text{w.g.}}$. Then $L(\mu)$ is a subquotient of $M(\lambda)$ if and only if there exist $\beta_1, \dots, \beta_t \in \Delta_+^{\text{re}}$ with $\lambda > (s_{\beta_1})_* \lambda > \cdots > (s_{\beta_t} \cdots s_{\beta_1})_* \lambda = \mu$.

We pick $\lambda - \mu \in Q_+$ and do induction on $\text{ht}(\lambda - \mu)$: If $\text{ht}(\lambda - \mu) = 0$, then $\lambda = \mu$ and there is nothing to do. Now assume $\text{ht}(\lambda - \mu) > 0$. Then $L(\lambda) = M(\lambda)/M^1$ from Jantzen's filtration. Since $\lambda \neq \mu$, $L(\mu)$ is a subquitient of M^1, and hence by Jantzen's Sum Formula (21.3) also a subquotient of $M(\lambda - n\alpha)$ for some $(\alpha, n) \in D_\lambda$.

Now if we assume that $\alpha \in \Delta_+^{\text{im}}$, we have $\langle \lambda + \rho, \alpha \rangle = \frac{n}{2} \langle \alpha, \alpha \rangle$. Set $\beta := n\alpha \in \Delta_+^{\text{im}}$ and $\langle \lambda + \rho, \beta \rangle = \frac{1}{2} \langle \beta, \beta \rangle$; and so $\lambda \in S_\beta$, contradiction. So $\alpha \in \Delta_+^{\text{re}}$, and then $\langle \lambda + \rho, \alpha \rangle = \frac{n}{2} \langle \alpha, \alpha \rangle$ with $\langle \alpha, \alpha \rangle \neq 0$, and so $n \equiv n(\alpha)$ is uniquely determined.

Finally we note that $\lambda - n(\alpha)\alpha = (s_\alpha)_* \lambda \in K^{\text{w.g.}}$, and by induction we conclude the "if" direction. The converse is just Jantzen's Sum Formula (21.3). □

Remark 21.15. The two equivalence relations from Definiton 21.13 are related by $\approx = \approx^\circ|_{K^{\text{g}}}$, since both have the same description in terms of W. In the affine (and finite) case, $\approx^\circ = \sim|_{K^{\text{w.g.}}}$, but this is false in general.

Definition 21.16. Let $\mathcal{O}^{\text{g}}$ be the full subcategory of $\mathcal{O}$ whose objects have composition factors belonging to K^{g}. We say that $M \in \mathcal{O}^{\text{g}}$ has *type* Λ if each composition factor of M belongs to Λ.

Example. If $\lambda \in K^{\text{g}}$, then $M(\lambda) \in \mathcal{O}^{\text{g}}$, and $M(\lambda)$ has type Λ.

There is a decomposition

$$\mathcal{O}^{\text{g}} = \bigoplus_{\Lambda} \mathcal{O}^{\text{g}}_{\Lambda} .$$

Moreover, the equivalence classes are labelled by $\bigcup_{\lambda \in C} W/W(\lambda)$ thanks to Proposition 21.14.

Exercise 21.17. Check that if $\lambda \approx \lambda'$, then $W(\lambda) = W(\lambda')$.

21.3 The Kazhdan-Lusztig Conjecture

A References

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- Kumar: J. Algebra 108 (1987)

B Notational reference

Witt	the Witt algebra over $\mathbb{C}$
Vir	the Virasoro algebra over $\mathbb{C}$, a central extension of Witt
$\mathrm{Lie}(G)$	the Lie algebra of the Lie group G, $\mathfrak{g} = T_e G$
$\mathfrak{g}$	either a general Lie algebra or a finite-dimensional, simple one
$\widehat{\mathfrak{g}}$	a central extension of a Lie algebra $\mathfrak{g}$; $0 \to \mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0$
$\mathscr{L}\mathfrak{g}$	the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$
$\overline{\mathscr{L}\mathfrak{g}}$	$= \mathscr{L}\mathfrak{g} \rtimes \mathbb{C}d$, where $d = t\frac{d}{dt} \in \mathrm{Der}\big(\mathbb{C}[t, t^{-1}]\big)$
	Note: $H^2\big(\mathscr{L}\mathfrak{g}; \mathbb{C}\big) \cong \mathbb{C} \cong H^2\big(\overline{\mathscr{L}\mathfrak{g}}; \mathbb{C}\big)$
$\widehat{\mathscr{L}\mathfrak{g}}$	$= \mathscr{L}\mathfrak{g} \oplus \mathbb{C}d \oplus \mathbb{C}c$, the central extension of $\overline{\mathscr{L}\mathfrak{g}}$, affine Kac-Moody Lie algebra
$\mathscr{L}(A)$	the Kac-Moody Lie algebra corresponding to the generalised Cartan matrix A
$U(\mathfrak{g})$	the universal enveloping algebra of the Lie algebra $\mathfrak{g}$; $U(\mathfrak{g}) = T(\mathfrak{g})\big/(x \otimes y - y \otimes x - [x, y])$
$\langle -, - \rangle$	a positive-definite sesquilinear form on a representation V, non-degenerate if V is unitary
$\widehat{A}$	the affine, untwisted gen. Cartan matrix derived from the Cartan matrix A
$\widetilde{A}$	the affine, twisted generalised Cartan matrix derived from A

C Generation of Lie algebras

Lie algebra Name	 Notation	Dynkin diagram	Generalised Cartan Matrix
simple	$\mathfrak{g}$	Δ, classical	A, finite type
loop	$\mathfrak{g}[t^{\pm 1}]$	n/a	n/a
affine	$\mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}c$	n/a	n/a
?	$\mathfrak{g}[t^{\pm 1}] \rtimes \mathbb{C}(\frac{d}{dt})$	n/a	n/a
affine Kac-Moody	$\mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}d \oplus \mathbb{C}c$	$\widehat{\Delta}$, untwisted affine	$\widehat{A}$, untwisted affine
twisted affine Kac-Moody	$\left(\mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}d \oplus \mathbb{C}c\right)^T$	$\widetilde{\Delta}$, twisted affine	$\widetilde{A}$, twisted affine