Symplectic Reflection Algebras

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- G finite group
- $\mathfrak{h}$ a finite dimensional $\mathbb{C}$–vector space with

$$G \hookrightarrow GL(\mathfrak{h}).$$

- $V = \mathfrak{h} \oplus \mathfrak{h}^*$
  - G acts on V diagonally
  - V is symplectic with

$$\omega((x, f), (x', f')) = f(x') - f'(x).$$

  - so G preserves ω

• We want to study such a situation, or more generally the siutation of a finite group G acting symplectically on any finite dimensional symplectic $\mathbb{C}$–vector space.

• Why?

– Arises in algebraic geometry: V/G satisfies Beauville's notion of a symplectic singularity.

– Arises in representation theory: quiver varieties; symplectic reflection algebras.

– It's a basic object: $V = T^*\mathfrak{h}$.

Goal: Understand the G–equivariant geometry of V.

This is not well posed. For example what do we mean for $G = \{1\}$?

– subschemes of V?

– subspaces of V?

– $\underline{V \text{ itself}}$.

Now suppose $|G| > 1$. Algebraic geometers and algebraists might come up with the same answer.

– stack theoretic G–quotient

– G–equivariant coherent sheaves

– $\mathbb{C}[V] * G$

Problem: these are all tautological statements

We can see everything, but usually it's hard to extract any information.

A (slightly) more specific first goal might be:

– How complicated is the orbit space V/G?

Geometry: How complicated is a resolution? Does a nice = crepant = symplectic resolution of V/G exist?

$$\tau : \widetilde{V/G} \longrightarrow V/G.$$

Such resolutions are of interest to algebraic geometers. They need not exist; when they do exist they need not be unique.

Algebra: How complicated is the orbit map?

$$V \quad \xrightarrow{\pi} \; V/G \qquad\qquad\qquad (v \mapsto G.v)$$

$$\mathbb{C}[V] \; \xleftarrow{\pi^*} \; \mathbb{C}[V]^G = \{p \in \mathbb{C}[V] : g \cdot p = p \text{ for all } g\}$$

• What is the structure of $\pi^*(0)$? i.e. of the ring of coinvariants

$$\mathbb{C}[V]^{\mathrm{co}G} = \frac{\mathbb{C}[V]}{\langle \mathbb{C}[V]^G_+ \rangle}.$$

(e.g used for invariant theory in characteristic p, Knop.)

A beautiful theorem unites the algebraic and geometric approaches.

Theorem (Bezrukavnikov–Kaledin, 2004) *Suppose* V/G *has a symplectic resolution. Then there is an equivalence of triangulated categories*

$$D^b(\mathsf{Coh}\widetilde{V/G}) \xrightarrow{\sim} D^b(\mathbb{C}[V] * G\text{–}\mathsf{mod}).$$

Taking Grothendieck groups relates G–modules to the cohomology of $\widetilde{V/G}$: a generalised McKay correspondence.

Example 1: $G = \frac{\mathbb{Z}}{2\mathbb{Z}}, \mathfrak{h} = \mathbb{C}$, so $V = \mathbb{C}^2$ with G acting by multiplication by -1.

- $\mathbb{C}[V]^G = \mathbb{C}[x,y]_{\text{even}} = \mathbb{C}[x^2, xy, y^2] = \frac{\mathbb{C}[a,b,c]}{\langle ac - b^2 \rangle}$

- Here $D^b(\text{Coh}\ \widetilde{V/G}) \xrightarrow{\sim} D^b(\mathbb{C}[V] * G\text{–mod})$ is a theorem of Gonzalez-Sprinberg–Verdier (1983), and Kapranov–Vasserot (2000): a special case of the original McKay correspondence.

- $\mathbb{C}[V]^{\text{co}G} = \frac{\mathbb{C}[x,y]}{\langle x^2, xy, y^2 \rangle}$

The coinvariants have a basis $\{\overline{1}, \overline{x}, \overline{y}\}$.

Example 2: $G = S_n, \mathfrak{h} = \mathbb{C}^n, V = \mathbb{C}^{2n}$, with G acting by permutation of coordinates. Then

$$\mathbb{C}^{2n}/S_n = (\mathbb{C}^2)^n/S_n = S^n\mathbb{C}^2,$$

the variety of n unordered points in the plane.

There exists a symplectic resolution

$$\tau : \mathrm{Hilb}^{\mathrm{n}}\mathbb{C}^2 \longrightarrow S^n\mathbb{C}^2$$

Theorem (Bridgeland–King–Reid, Haiman, 2001)

$$D^b(\text{Coh } \mathrm{Hilb}^{\mathrm{n}}\mathbb{C}^2) \xrightarrow{\sim} D^b(\mathbb{C}[V] * G\text{–mod}).$$

Theorem (Haiman, 2002)

$$\dim \mathbb{C}[V]^{\mathrm{co}G} = (n+1)^{n-1}.$$

Problem: How can we extract information from G–equivariant geometry? For instance how do we see whether a symplectic resolution exists.

Principal idea: Rigidify/deform G–equivariant geometry.

Theorem (Ginzburg–Kaledin, 2004) *Suppose there exists $Y = \widetilde{V/G}$, a symplectic resolution of V/G. Then there exists a family of resolutions over B*

$$\tau_B : Y_B \longrightarrow (V/G)_B$$

such that for generic $b \in B$ τ_b is an isomorphism.

We deform $\mathbb{C}[V] * G$ – the centre $\mathbb{C}[V]^G$ will deform too.

$$H_\kappa = \frac{TV * G}{\langle y \otimes x - x \otimes y - \kappa(x,y) : x, y \in V \rangle}$$

where $\kappa : \bigwedge^2 V \longrightarrow \mathbb{C}G$.

For example:

– if $\kappa = 0$ then $H_\kappa = \mathbb{C}[V] * G$

– if $\kappa = \omega$ then $H_\kappa = A_n(\mathbb{C}) * G$, an extension of the Weyl algebra.

Generally, H_κ does not vary continuously with κ.

H_κ is a filtered ring:

$$F^0 = \mathbb{C}G, F^1 = V + \mathbb{C}G, F^i = (F^1)^i.$$

Let $\operatorname{gr} H_\kappa = \bigoplus_{i \geq 0} \frac{F^i}{F^{i-1}}$, the associated graded ring.

We say H_κ has the **PBW property** if $\operatorname{gr} H_\kappa \cong \mathbb{C}[V] * G$.

Theorem (Etingof–Ginzburg, 2002) *H_κ has the PBW property if and only if for all $x, y \in V$*

$$\kappa(y,x) = t\omega(y,x) + \sum_{s \in \mathcal{S}} c(s)\omega_s(y,x)s.$$

- $t \in \mathbb{C}$
- $\mathcal{S} = \{s \in G : \mathrm{rank}(1_V - s_V) = 2\}$ = set of symplectic reflections.
- $c : \mathcal{S} \longrightarrow \mathbb{C}$, invariant under conjugation
- $\omega_s = \omega|_{\mathrm{im}(1_V - s_V)}$

We write $H_{t,c}$ instead of H_κ and call such an algebra a **symplectic reflection algebra**.

This highlights groups generated by symplectic reflections. There are *essentially* two families (classified by Huffman–Wales, Cohen, Guralnik–Saxl):

(1) $G = S_n \wr \Gamma (= \Gamma^n \rtimes S_n)$ acting on $(\mathbb{C}^2)^n$ where $\Gamma \leq SL_2(\mathbb{C})$

(2) $G =$ complex reflection group acting on $\mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is the reflection representation.

These groups had appeared before in a familiar context.

Theorem (Verbitsky, 2000) *If V/G has a symplectic resolution then G is generated by symplectic reflections.*

Why does $H_{t,c}$ really help us?

Let $e = \frac{1}{|G|}\sum_{g\in G} g \in \mathbb{C}G$ be the trivial idempotent. We call $eH_{t,c}e$ the **spherical subalgebra** of $H_{t,c}$.

It can be shown that

- $\mathrm{gr}(eH_{t,c}e) \cong \mathbb{C}[V]^G$
- $Z(H_{0,c}) \cong eH_{0,c}e$
- $Z(H_{t,c}) = \mathbb{C}$ if $t \neq 0$.

In particular, for $t = 0$, we have a family of deformations of V/G,

$$X_c = \mathrm{Spec}\, Z(H_{0,c})$$

Taking central characters yields a mapping

$$\chi_c : \{\text{Simple } H_{0,c}\text{–modules}\} \longrightarrow X_c$$

There is a general theorem of Artin–Procesi, LeBruyn, Brown–Goodearl which applies to show

$$(X_c)_{\mathsf{sm}} = \{\chi(M) : \quad M \text{ a simple } H_{0,c}\text{–module of maximal dimension}\}.$$

The set on the right hand side is called the **Azumaya locus** of $H_{0,c}$.

In our first example

$$H_{0,0} = \mathbb{C}[x,y] * \frac{\mathbb{Z}}{2\mathbb{Z}}, \quad Z(H_{0,0}) = \mathbb{C}[x,y]_{\text{even}}.$$

Thus X_0 has a singularity at 0.

Generically, the simple $H_{0,0}$–modules are two–dimensional.

But we find that there are 2 killed by x^2, xy, y^2, both of which are one–dimensional:

$$x \cdot v = 0 = y \cdot v; \quad g \cdot v = \pm v.$$

Theorem (Verbitsky, Etingof–Ginzburg, G, Ginzburg–Kaledin, 2004) *V/G has a symplectic resolution if and only if (G, V) belongs to family (1).*

Proof: $\Leftarrow$ Take the appropriate Hilbert scheme of the minimal resolution a Kleinian singularity.

$\Rightarrow$ By the Ginzburg–Kaledin theorem, if V/G has a resolution then a generic Poisson deformation of it should be smooth. Another result of Ginzburg–Kaledin shows that the deformations X_c are "generic enough", i.e. X_c must be smooth for generic c.

By the Azumaya locus theorem, X_c is smooth if and only if all representations of $H_{0,c}$ have the same dimension. A little representation theory shows that for all G not belonging to family (1) there are always small representations. Contradiction.

Let's concentrate on family (2) now and assume that $t = 1$. In particular G is a finite (complex) reflection group.

Usually $H_{1,c}$ is a simple ring behaving quite like a Weyl algebra.

For some **singular** values of c, however, this is not so. For instance, there can be finite dimensional modules.

e.g. if $\mathbb{C}v$ is the trivial G–module, solving

$$0 = (yx - xy)v = (\omega(y, x) + \sum_{s \in \mathcal{S}} c(s)\omega_s(x, y)s)v$$

for c shows that $H_{1,c}$ has a one–dimensional representation when c is the constant function $1/$Coxeter(G).

Applying a (Heckman–Opdam) shift functor produces an interesting finite dimensional $H_{1,c+1}$–module, L. In particular, on degeneration (i.e. passage to associated graded module) we get

Theorem (G, 2003) *For any finite Coxeter group, $\mathbb{C}[V]^{coG}$ has a quotient of dimension* $(\text{Coxeter}(G) + 1)^{\operatorname{rank} G}$ *with good combinatorial properties.*

This has recently been generalised by Vale to some other complex reflection groups.

In the $G = S_n$ case intriguing analogues with semisimple Lie algebras appear.

The previous theorem is interpreted as a Borel–Weil type theorem

$$L \otimes \mathsf{sign} \cong H^0(\mathsf{Hilb}_0^n \mathbb{C}^2, \mathcal{P}).$$

Moreover, there is a diagram

$$\begin{array}{ccccc} & & & & \mathrm{Hilb^n}\mathbb{C}^2 \\ & & & & \big\downarrow \tau \\ & & H_{1,c}\text{–}\mathsf{mod} & \xrightarrow{\mathrm{gr}} & S^n\mathbb{C}^2 \\ & & \big\uparrow \mathsf{inc} & & \\ \mathcal{H}_q(n) & \xleftarrow{\mathsf{KZ}} & \mathcal{O}_{1,c} & & \end{array}$$

$$
\begin{array}{ccccc}
 & & \text{Coh } T_{1,c} & \xrightarrow{\mathrm{gr}} & \mathrm{Hilb}^{\mathrm{n}}\mathbb{C}^2 \\
 & & \cong \big\downarrow & & \big\downarrow \tau \\
 & & H_{1,c}\text{–mod} & \xrightarrow{\mathrm{gr}} & S^n\mathbb{C}^2 \\
 & & \big\uparrow \text{inc} & & \\
\mathcal{H}_q(n) & \xleftarrow{\text{KZ}} & \mathcal{O}_{1,c} & &
\end{array}
$$

• $H_{1,c}$ should be a deformation of $\mathrm{Hilb^n}\mathbb{C}^2$, a Beilinson–Bernstein type result. We expect finitely generated modules give rise to coherent sheaves on $\mathrm{Hilb^n}\mathbb{C}^2$ (G–Stafford) with cohomology vanishing conditions.

Then much combinatorics of $\mathrm{Hilb^n}\mathbb{C}^2$ should be constructed in the category of $H_{1,c}$–modules.

• Category $\mathcal{O}_{1,c}$ should be a subcategory of $H_{1,c}$–mod equivalent to modules over the q–Schur algebra, a Soergel type result (Rouquier).

Then its combinatorics is governed by parabolic KL polynomials.

To prove either hope (properly) we need to localise.

In fact, much of $\mathrm{Hilb}^{\mathrm{n}}\mathbb{C}^2$ and the q–Schur algebra is best understood by considering all n together (Grojnowski, Nakajima, Vasserot–Varagnolo).

We don't even know how to restrict from $H_{1,c}(S_n)$ to $H_{1,c}(S_{n-1})$!