Plan

1) Definitions

- stably conjugate
- stably linearizable
- strongly stably linearizable.

Example $\mathbb{Z}/2 \times \mathbb{Z}/2$ acts on $\mathbb{P}^1$

— " — $\mathbb{A}^2$

not stably conjugate.

Remark faithful representations are stably conjugate

$$\begin{tikzcd} V_1 \oplus V_2' \arrow[d] & V_1 \oplus V_2 \arrow[l, dashed] \arrow[r, dashed] \arrow[dl] \arrow[dr] & V_1' \oplus V_2 \arrow[d] \\ V_1 & & V_2 \end{tikzcd}$$

Lemma K/k finite Galois ext. $G = Gal(K/k)$

W/K vector space $G \curvearrowright W$ semi-linear

$$W = W^G \otimes_k K$$

Pf. w_i basis K/k $\{\sigma_i\} = G$ $\sigma_1 = 1$

$v \in W$ $u_i = \sum_j \sigma_j(w_i \cdot v) \in W^G$

$$\| \qquad \qquad (\sigma_j(w_i)) \text{ invertible}$$

$$\sum \sigma_j(w_i) \cdot \sigma_j(v)$$

$W^G \otimes_k K \to W$
есть отобр.

$v = \sigma_1(v)$ можно выразить как лин. комбинацию u_i

$\sum \lambda_i v_i = 0$ K-lin. comb $v_i \in W^G$ $\Rightarrow$ contr.

$\exists\, i_0, \alpha \in K$ $Tr_{K/k}(\alpha \lambda_{i_0}) \neq 0$ $\sum Tr(\alpha \lambda_i) v_i = 0$ $/k$

Invariants

$$H^1(G, \mathrm{Pic}(X))$$

$H^1(G, \mathrm{Pic}(X)) = 0$ if stably linearizable

Moreover, $H^1(G, \mathrm{Pic}(X)) = 0$ if $G \subset$ connected group $\subset \mathrm{Aut}(X)$

Th $G \subset Cr_2$ $\quad$ ($C \subset$ Fix curve of $g > 0$ (only one such a curve).
$G = \langle \delta \rangle_p$

$$\Rightarrow H^1(G, \mathrm{Pic}(X)) = (\mathbb{Z}/p)^{2g},$$

Corollary. **Equiv.** $G = \langle \delta \rangle_p$

(i) $G \subset Cr_2$ stably linearizable

(ii) $G \subset Cr_2$ linearizable

(iii) does not fix a curve of positive genus.

Invariant For $A \subset G$ $\exists$ a fixed point.

Strongly stably linearizable $\Rightarrow$ $\exists$ a fixed point $\forall A \subset G$.

Remark **Arithmetic situation**

(a) de Jonquières

(b) Geizer

(c) Bertini

(d) $p = 3$ $K^2 = 3$ $g = 1$

(e) $p = 3$ $K^2 = 1$ $g = 2$

(f) $p = 5$ $K^2 = 1$ $g = 1$

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Example Exc. conic bundle

$\longrightarrow X_4 \subset \mathbb{P}^1$

$\mathbb{Z}/4 \times \mathbb{Z}/2$ no fixed points
not strongly stably lin.

$H^1(G', \mathrm{Pic}(X)) = 0$

$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$+ \quad - \quad -$

$\{x_1 x_2 = y_1^4 - y_2^4\} \subset \mathbb{P}(2,2,1,1)$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ -1 $-$

$\downarrow$

$\mathbb{P}(1,1)$

Châtelet surface

Trivial Example. $\boxed{\mathbb{Z}/2 \times \mathbb{Z}/2} \curvearrowright \mathbb{P}^1 \times \mathbb{P}^1$

not linearizable

stably linearizable

not strongly stably linearizable

$$\begin{tikzcd} \underset{\text{trivial}}{\mathbb{P}^1 \times \mathbb{A}^1} \arrow[dr] & \mathbb{P}^1 \times \mathbb{A}^1 \arrow[l, dashed] \arrow[d] \\ & \mathbb{P}^1 \end{tikzcd}$$

Conic bundle $G_F = \{1\}$

$$\begin{tikzcd} X/G \arrow[d] & X \arrow[l] \arrow[d] \\ B/G & B = \mathbb{P}^1 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} k(u,v) = k(X/G) \arrow[r, phantom, "\overset{n}{\subset}"] & k(X) \supset k(t,v) \\ k(B/G) \arrow[u] \arrow[r, phantom, "\overset{n}{\subset}"] & k(B) = k(t) \arrow[u] \end{tikzcd}$$

$k(B/G) = k(u)$

$k(t,v) \cong \mathbb{C}^1$

$X \underset{G\text{-equiv}}{\sim} B \times \mathbb{A}^1$ stably linearizable

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Th 1 $G \subset \mathrm{Aut}(X)$, X del Pezzo
minimal

$\Rightarrow$ Equiv.

(1) $H^1(G', \mathrm{Pic}(X)) = 0 \quad \forall G' \subset G$

(2) $\forall\, g \in G$ does not fix a curve of positive genus

(3) either $K_X^2 \geq 5$ or

$K_X^2 = 4 \qquad \varepsilon^3 = 1$

$x_1^2 + \varepsilon x_2^2 + \varepsilon^2 x_3^2 + x_4^2 = x_1^2 + \varepsilon^2 x_2^2 + \varepsilon x_3^2 + x_5^2 = 0$

$G \simeq \mathbb{Z}/3 \rtimes \mathbb{Z}/4$

$\gamma : (x_1, \ldots x_5) \longmapsto (x_2, x_3, x_1, \varepsilon x_4, \varepsilon^2 x_5)$

$\beta : (\quad) \longmapsto (x_1, x_3, x_2, -x_5, x_4)$

$\beta^2 : (x_1, \ldots, x_5) \longmapsto (x_1, x_2, x_3, -x_4, -x_5)$

$(1, 1, 1, 0, 0)$ fixed pt for $\mathbb{Z}/3 \times \mathbb{Z}/2$

$K^2 = 4 \qquad W = (\mathbb{Z}/2)^4 \rtimes S_5$

$\sum x_i^2 = \sum a_i x_i^2 = 0 \qquad \tau_i$ reflections $i = 0, \ldots, 4$

$A \simeq (\mathbb{Z}/2)^4$

$G \to PGL_2$

$A_G \quad \{1\},$ $\qquad G \not\ni \tau_i \qquad G \to S_5$

$\{1, \tau_i \tau_j\}$

$\{1, \tau_i \tau_j, \tau_j \tau_k, \tau_k \tau_i\}$ $\quad G/A = 1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5, S_3, D_5$

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Conic bundles

\# degen fibers ≥ 4

$\rho: G \to \mathrm{Aut}(\mathrm{Pic}\, X)$ $\qquad K_X^2 \leq 4$

Case $\quad \mathrm{Ker}\, \rho = 0$

$G \simeq \tilde{D}_n \qquad n = 6 - K_X \quad$ odd

$G_F = \mathbb{Z}/2 \qquad G_B \simeq D_n$

Construction

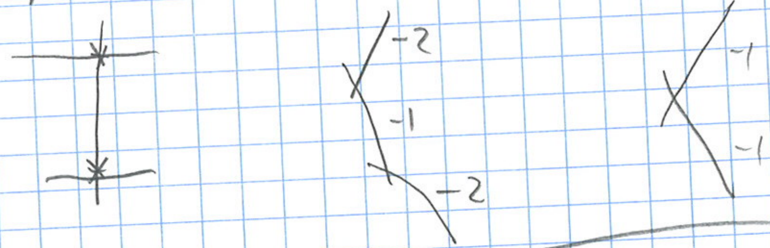
$\sigma_1 : D_n \to \mathrm{Aut}(\mathbb{P}^1) \qquad \lambda : x \mapsto x^2$

$\sigma_2 : D_n \to \mathrm{Aut}(\mathbb{P}^1) \qquad \lambda \cdot \sigma_1 = \sigma_2 \cdot \lambda$

$\sigma = \sigma_1 \times \sigma_2 : D_n \longrightarrow \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$

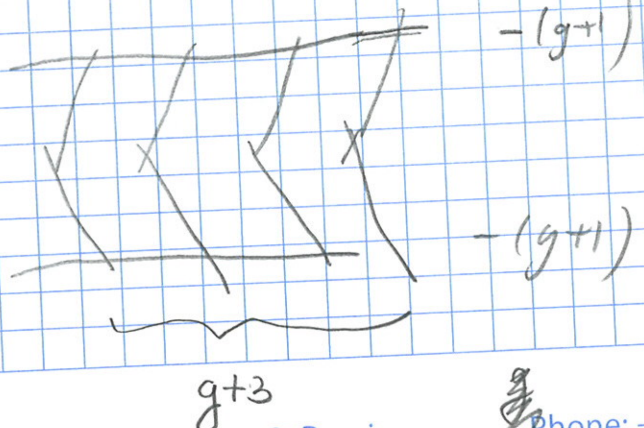
$\Gamma = \{x^2 = y\} \qquad L = \{y^n = 1\}$

$Y \xrightarrow{2:1} \mathbb{P}^1 \times \mathbb{P}^1$ branched over $L + \Gamma$



Exceptional conic bundles $\qquad g$

$\{x_1 x_2 = f(y_1, y_2)\}$
$\subset \mathbb{P}(1, g+1, g+1,$



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Case $\ker \rho \neq 0$. $\ker \rho \subset G_F$

$\forall\, \delta \in \ker \rho$ $\quad \mathrm{Fix}(\delta) \xrightarrow{2:1} B$

C (-1) section $\quad F_i = F_i' + F_i''$ degen. fiber

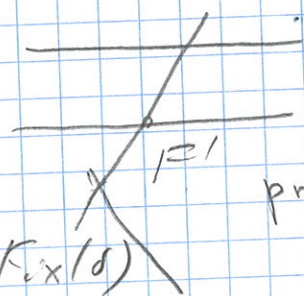
$\# C'$ — " — $\quad C \cap F'$ fixed point

$\delta: C \to C$

$C' \to C'$

$C \cap F_i$ fixed point $\Rightarrow$ $C \cup C' \subset \mathrm{Fix}(\delta)$

$\Rightarrow C \cap C' = \emptyset$



prohibited

otherwise $F' \subset \mathrm{Fix}(\delta)$

Conclusion: $\pi: X \to B$ exceptional.

$$\{x_1 x_2 = \psi(y_1, y_2)\} \subset \mathbb{P}(g+1, g+1, 1, 1)$$

K_X^2 even

$G_F = \ker \rho \subset Z(G)$ cyclic

$G_B = D_4$ or S_4

Example. $g = 2$ $\quad S_4 = G_B$

$\psi = \psi_6$

$$\{(h, \delta) \in GL_2 \times S_4 \mid h(x_1, x_2) = \mathrm{sgn}(\delta) \cdot x_1 x_2\}$$

$\cup$

$G \xrightarrow{\text{surj}}$