Augmented base loci and restricted volumes in normal varieties

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Frontiers of rationality
Longyearbyen (Spitsbergen), July 16, 2014

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and Salvatore Cacciola (Roma Tre University)

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As we all know, knowledge of the behavior of these maps often says a lot about the geometry of X itself.
In particular, there are some closed subsets, associated to L, that govern, asymptotically, this behavior.

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(introduced in 2000 by Nakamaye, and in 2006 by Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa)

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Just to mention a few instances, we recall the fundamental papers of Takayama, Hacon and McKernan on the birationality of pluricanonical maps

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For every subvariety $Z \subseteq X$ not contained in $\mathrm{B}_+(L)$, it is easily seen that the restriction of L to Z is big. But there is more: the space of sections of $|mL|_Z|$ that extend to X has maximal growth. In fact if we denote by $H^0(X|Z, mL)$ the image of the restriction map $H^0(X, mL) \to H^0(Z, mL_{|Z})$ and set $d = \dim Z$, then we claim that

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$$\begin{array}{ccc} H^0(mA) & \twoheadrightarrow & H^0(mA_{|Z}) \\ \big\downarrow & & \big\downarrow \\ H^0(msL) & \longrightarrow & H^0(msL_{|Z}) \end{array}$$

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$\lim_{m \to +\infty} \mathrm{vol}_{X|Z}(L + \frac{1}{m}A) = 0$
for every ample A and for every irreducible component Z of $\mathrm{B}_+(L)$.

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We can assume that L is big. Let Z be an irreducible component of $\mathrm{B}_+(L)$. Recall that $\dim Z \geq 1$ by a well known result of Zariski.

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$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle \mu_m}\downarrow & & \downarrow{\scriptstyle \nu_m} \\ X & \underset{\varphi_{mL}}{\dashrightarrow} & \varphi_{mL}(X) \end{array}$$

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where μ_m is the normalized blow-up of X along the base ideal of $|mL|$,

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with A_m ample on Y_m, F_m effective and such that $\mathrm{Supp}(F_m) = \mu_m^{-1}\left(\mathbf{B}(L)\right) \subseteq \mu_m^{-1}\left(\mathbf{B}_+(L)\right).$

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If $x \in \mathrm{Exc}(\mu_m)$, then $\mu_m(x) \in \mathbf{B}(L) \subseteq \mathbf{B}_+(L)$, whence $x \in \mu_m^{-1}\left(\mathbf{B}_+(L)\right)$.

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If $x \notin \mathrm{Exc}(\mu_m)$ and $\mu_m(x) \notin \mathbf{B}_+(L)$, then μ_m, φ_{mL} and ν_m are isomorphisms in a neighborhood of x, contradicting the fact that f_m is not an isomorphism in a neighborhood of x.

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$$\mu_m^{-1}\left(\mathbf{B}_+(L)\right) \subseteq \mathbf{B}_+(\mu_m^* L) \subseteq \mathrm{Exc}(f_m) \cup \mathrm{Supp}(F_m).$$

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From the previous inclusions

$$\mu_m^{-1}(\mathbf{B}_+(L)) \subseteq \mathbf{B}_+(\mu_m^* L) \subseteq \mathrm{Exc}(f_m) \cup \mathrm{Supp}(F_m).$$

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Recall that we have an irreducible component Z of $\mathbf{B}_+(L)$ such that $Z \not\subseteq \mathbf{B}(L)$ and that we want to prove that $\mathrm{vol}_{X|Z}(L) = 0$.

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we conclude that

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Let $\mathcal{L}$ be a line bundle on a variety Z.
A graded linear series W is a sequence of subspaces

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This lemma follows from a deep result of Kaveh-Kovanskii (using Okounkov bodies), but we will give a simple proof ispired by a paper of Di Biagio-Pacienza.

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We can assume that the base filed is uncountable, since the estimate is invariant under base field extension.

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(here we use that the base field is uncountable, as we impose countably many conditions).
We now claim that the restriction map

$$W_m \to H^0(T, m\mathcal{L}_{|T}) \text{ is injective for } m \gg 0$$

(and this will prove the Lemma).

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the maps φ_{mL} are an isomorphism on $X - \mathbf{B}_+(L)$ for $m \gg 0$.
Using the same method of proof of the previous theorem, we can prove

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So the two theorems work for $\mathbb{Q}$-divisors. What about real divisors? (I must say that ELMNP's theorem, in its "continuity version", also holds for real divisors)

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Nevertheless we are able to generalize to $\mathbb{R}$-divisors using a recent idea of Birkar (used to prove Nakamaye's theorem on any scheme).

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Now the two theorems above go through.