Some questions and results in birational geometry related to cubic hypersurfaces

Plan: " Use cubics to tie together some loosely related structural problems in birational geometry "

§1 : Gromov's question

§2 : Birational isotriviality of fiber spaces

§3 : Dynamical spectra as a possible venue to irrationality of general (smooth) cubic fourfolds

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(work over the ground field $\mathbb{C}$ for definiteness throughout)

§1. Gromov's question

<u>Def.</u> : Call a variety X <u>uniformly rational</u> if every point $x \in X$ has a Zariski open neigh. $U \ni x$ isomorphic to a Zariski open set $V \subseteq \mathbb{P}^n$ ($n = \dim X$).

<u>Question</u> : Is every <u>smooth</u> rational X uniformly rational?

For curves, surfaces it's obviously yes, otherwise unknown.

<u>Prop. (Bogomolov)</u> : If X is uniformly rat., $Y \subseteq X$ smooth, then $Bl_Y X$ is uniformly rational.

Proof (Sketch):

Can suppose $y \in Y \subset \mathbb{A}^n$, $\dim Y = m \leq n-2$.

- $\exists \ \varphi : \mathbb{A}^n \dashrightarrow \mathbb{A}^n$ birational, def. at y, and isom. around y

  mapping Y bir. onto a hypersurface in some $L \simeq \mathbb{A}^{m+1} \subseteq \mathbb{A}^n$

  (choose splitting $\mathbb{A}^n = L \oplus M$ generically, $\pi_M : \mathbb{A}^n \to L$ will map a neighborhood of y in Y isom. onto $\pi_M(Y) \cap U$, some $U \subset L$ open affine. i.e. the trivial $M \simeq \mathbb{A}^{n-m-1}$ bundle over U, has a section over $\pi_M(Y) \cap U$ and this extends (since U is affine) to a section $\sigma : U \to U \oplus \mathbb{A}^{n-m-1}$. Then

  $$\varphi(u,m) = (u, m - \sigma(u)) \text{ does the job.})$$

- Can assume $L \simeq \mathbb{A}^{m+1}$ with coord. $x_1, \dots, x_{m+1}$, M coord. $x_{m+2}, \dots, x_n$, $Y \subset L$ def. by

  $$f(x_1, \dots, x_{m+1}) = 0.$$

  $Bl_Y \mathbb{A}^n$ is then the closure of the graph of

  $$\mathbb{A}^n \dashrightarrow \mathbb{P}^{n-m-1}$$

  $$(x_1, \dots, x_n) \longmapsto (Y_0, \dots, Y_\ell) = (f, x_{m+2}, \dots, x_n).$$

The open $V \subseteq Bl_Y \mathbb{A}^n$ where $Y_0 \neq 0$ can be described as

$$V = \{ (x_1, \dots, x_n), (y_1, \dots, y_k) \subseteq \mathbb{A}^n \times \mathbb{A}^k \mid$$
$$x_{m+2} = y_1 f, \dots, x_n = y_k f \}, \quad y_i = \frac{Y_i}{Y_0}$$

so $V \simeq \mathbb{A}^n$ via proj. to $(x_1, \dots, x_{m+1}, y_1, \dots, y_k)$.

- For given $p \in \{Y_0 = 0\} \subseteq Bl_Y \mathbb{A}^n$ $\exists$ an autom.
  $\psi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ mapping $Y \to Y'$ and
  $Bl_Y \mathbb{A}^n \longrightarrow Bl_{Y'} \mathbb{A}^n$ s.t. in the image coord.
  $x_1', \dots, x_n', Y_0', \dots, Y_k'$, $\psi(p) \notin \{Y_0' = 0\}$. $\square$

Rem.: Thus it would suffice to prove stability of uniform rat. under blow-downs, i.e.

let X' be (w.l.o.g. projective), $E \subseteq X'$ divisor with str. of proj. bundle

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \wr & & \\ \mathbb{P}(\mathcal{E}) & & \end{array} \quad \text{s.t.}$$

conormal bundle of E in X' cuts out $\mathcal{O}(1)$ on every fiber of $\mathbb{P}(\mathcal{E})$ and $\exists$ line bunde $\mathcal{L}'$ on X'

gen. by global sections whose restr. to E is is inverse image of an ample line bundle on Y

$\Longrightarrow$ E can be blown down to Y

Ishii

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \cup & & \cup \\ E & \longrightarrow & Y \end{array}$$

X projective. Is X unif. rational?

As a nontrivial class of examples, one can look at nodal cubic hypersurfaces $X_3^3 \subseteq \mathbb{P}^4$ and their small and big resolutions:

Thm. (– , Bogomolov) :

(a) The analogue of Gromov's question has a negative answer in the category of Moishezon manifolds, i.e. if e.g. X is a Lefschetz cubic (just one node) and $\tilde{X}$ a small resolution of the node, then $\tilde{X}$ is a non-alg. Moishezon manifold and for no point x on the exceptional $\mathbb{P}^1$ can one find a bimerom. map $\tilde{X} \dashrightarrow \mathbb{P}^3$ defined and locally an isom. around x.

(b) For small alg. resolutions of nodal cubics X Gromov's question has a pos. answer.

(c) For big resolutions as well.

Ideas of proof :

(a) The node x is resolved zero homologously, i.e. the exc. $C \cong \mathbb{P}^1$ is homologous to zero. This excludes the existence of bimerom. maps of the given/envisaged type by intersection theory.

(b) Let $p \in X$ be a node. Projection away from p gives an isom.

$$Bl_p X \simeq Bl_C \mathbb{P}^3$$

where $C \subset Q \subset \mathbb{P}^3$ ← hyperplane in $\mathbb{P}^4$ at ∞

↑ proj. tangent cone of X at p

is the <u>associated curve</u> as seen from p, (consisting of directions in the tang. cone corresp. to lines contained in X passing through p.

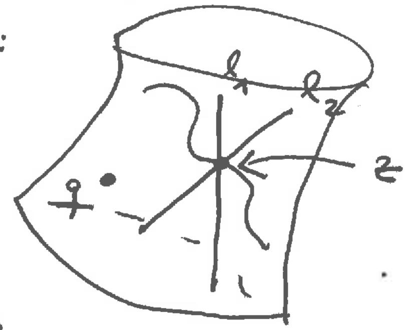
The nodes away from X are in 1-1 correspondence with nodes of C. If a small resol. <u>is algebraic</u>, all nodes of C are such that two smooth <u>components</u> of C cross transversally at them; say $q \in C$ node, C_1, C_2 cross transv. at q. corresponds to $p' \in X$ node

Then a neigh. of a small resolut. of p' in $Bl_p X$ is obtained by blowing up first C_1 in $\mathbb{P}^3$ and then C_2, or vice versa, hence by Bogomolov's proposition, the result follows.

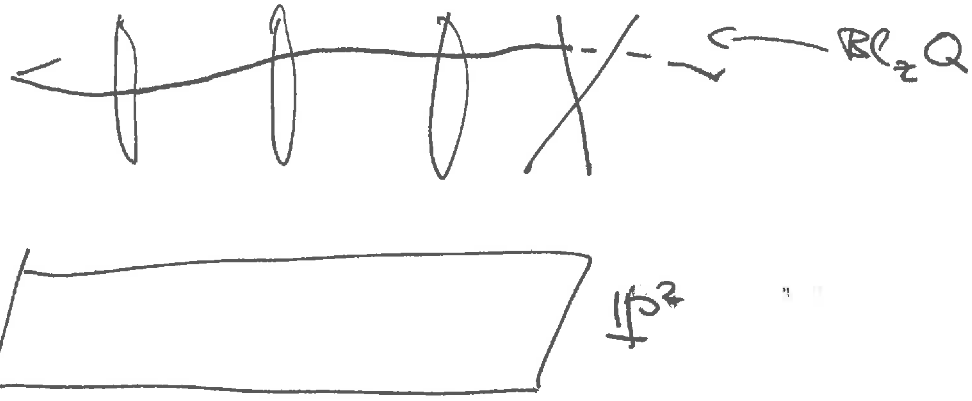
(c) We take a point $p \in X$ (node) and replace it by $Q = \mathbb{P}^1 \times \mathbb{P}^1$ in $\tilde{X} = Bl_p X$.

We have to show that every point $q \in Q$ has a Zariski neigh. in $\tilde{X}$ isom. to an open in $\mathbb{P}^3$.

We can find a line $\ell \subset X$ through p corresp. to a point on the assoc. curve $C \subset Q$ s.t. ~~the~~ none of the two lines on Q through that point pass through q:



Projection from ℓ gives $\tilde{\tilde{X}} = Bl_\ell\, Bl_p X$ the structure of a conic bundle over $\mathbb{P}^2$, and $Bl_z Q \subseteq \tilde{\tilde{X}}$ is a section of this conic bundle away from the strict trafos of the two lines ℓ_1, ℓ_2 which are blown down:



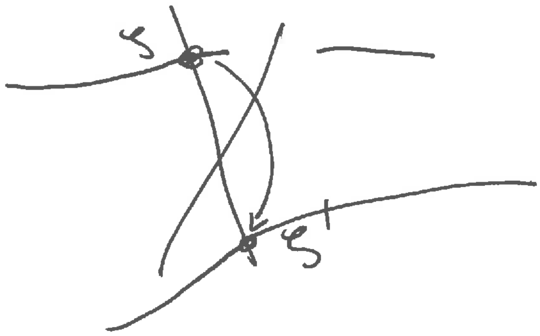
Moreover, we know that X has the Gromov property away from the nodes (use reflections in general smooth points). $Bl_z Q$ intersects each fiber in a smooth point of the fiber. Writing the conic bundle

$$a(\underline{t})\,x^2 + b(\underline{t})\,y^2 + c(\underline{t})\,z^2$$

$c(\underline{0}) = 0$, $a(\underline{0}) \neq 0$, $b(\underline{0}) \neq 0$, locally around a singular fiber, we can map

$$x \longmapsto -x \quad \text{or} \quad y \longmapsto -y \quad \text{or} \quad z \longmapsto -z,$$

and thus move the intersection point ξ away from $Bl_Z Q$:



But around ξ' we already have good charts. $\square$

§2. Birational isotriviality

The following Thm. is useful sometimes in dealing with the family of all smooth cubic 4-folds, say, assum. they were rational:

Thm.: (v. Bothmer, Bogomolov) $\boxed{k = \mathbb{C}}$

Let $\begin{array}{c} X \\ \downarrow f \\ B \end{array}$ be a flat family of varieties s.t.

all fibers X_b, $b \in B$ a closed point, a birational to each other. Then it is birationally isotrivial in the sense that:

$\exists \quad B' \longrightarrow B$ Zar. étale neigh. of the gen. pt. of B

and

$$\begin{tikzcd} X \times_B B' = X_{B'} \arrow[rr, dashed, "\Phi"] \arrow[dr] & & B' \times X_0 \arrow[dl] \\ & B' & \end{tikzcd}$$

with Φ birational.

However, there are open questions:

- Can we always choose the trivialization s.t. Φ is defined in the gen. point of a given fiber? (Gromov type question)
- If we assume k only alg. closed, not $\mathbb{C}$, then does the Thm. remain true? E.g. over $\overline{\mathbb{Q}}$? If we have a family $X \longrightarrow B$ def. over $\overline{\mathbb{Q}}$ s.t. all fibers over $\overline{\mathbb{Q}}$-val. points are bir. to each other, is then the same true for all points in a Zariski open subset of $B(\mathbb{C})$?

Ideas of the proof:

let $\quad S \subset \mathbb{P}^s = \mathbb{P}(\mathbb{C}^{s+1})$ projective

$$\mathcal{P}_d(\underline{x}) = \mathbb{P}\left((\mathrm{Sym}^d (\mathbb{C}^{s+1})^\vee)^{t+1} \right).$$

These $(t+1)$-tuples of hom. polynomials

$$\underline{p} = (p_0, \dots, p_t) \text{ in } \mathcal{P}_d(\underline{x}) \quad \text{s.t.}$$

– the rat. map given by f is def. on S:

$$S \dashrightarrow \mathbb{P}^t$$

and maps S onto an image with Hilbert polynomial h and is birational, with inverse definable by an $(s+1)$-tuple of hom. polyn. of degree d' form a quasi-proj. subvar. $\mathcal{B}^h_{d,d'}(S,\mathbb{P}^t)$ of $\mathcal{P}_d(\underline{x})$.

– the subset of $\mathrm{Hilb}_{h,\mathbb{P}^t}$ consisting of ~~smooth~~ subvar. bir. to a fixed model $X_0 \subseteq \mathbb{P}^s$ is equal to (contained in) countably many locally closed subvarieties (into which the images of the $\mathcal{B}^h_{d,d'}(S,\mathbb{P}^t)$ in Hilb decompose).

The image of $\alpha : \mathcal{B} \to \mathrm{Hilb}_{h,\mathbb{P}^t}$ intersects one of these subvarieties in a dense open subset[4] of both:

pull-back of f to $\mathcal{B}'$ bir. trivial, since over $\tilde{\mathcal{B}}$ family bir. trivial.

loc. closed

$$\begin{tikzcd}
\mathcal{B}' \arrow[r] \arrow[d, "\text{étale}"'] \arrow[dr, phantom, "\#"] & \tilde{\mathcal{B}} \subset \mathcal{B}^h_{d,d'}(X_0,\mathbb{P}^t) \arrow[d, "\text{étale}"] \\
\mathcal{B} \supset \mathcal{B}^0 \arrow[r] & \mathcal{U}
\end{tikzcd}$$

$\square$

§3. Dynamical degrees & a view toward irrationality of very general cubic hypersurfaces (?)

An intuition: for $X^n_3 \subseteq \mathbb{P}^{n+1}$ $(n \geq 3)$ general,

$\mathrm{Bir}(X^n_3)$ should be "smaller" than $\mathrm{Bir}(\mathbb{P}^n)$

in a suitable sense.

Given any variety X, dim. n, one can associate to it a subset $\Lambda(X) \subseteq \mathbb{R}^{n+1}$ with interesting point set,

$$\Lambda(X) = \{(\lambda_0, \dots, \lambda_n) \mid \lambda_i \in \mathbb{R} \text{ satisf. certain conditions}\}$$

metric and topological (?) properties, ~~...~~ depending only on the birational equivalence class of X, the dynamical spectrum of $\mathrm{Bir}(X)$:

to a Cremona transformation, one can associate a multidegree:

$f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, d_i = degree of transform of general codim. i linear subspace.

The dynamical degrees are the exponential growth rates of the d_i under taking iterates of f, more generally:

Given

$$f : X \dashrightarrow X \underset{\text{(any var.)}}{} , \quad H \text{ ample on } X,$$

$$\lambda_i(f) = \liminf_{n \to \infty} \left((f^n)^* H^i \cdot H^{d-i} \right)^{\frac{1}{n}},$$

$d = \dim(X)$ (one can also define $\lambda_i(f)$ as the exp. growth rate of the spectral radii of the maps $(f^n)^*$ induced by f on $A^i(X)$, Chow gp. of codim. i cycles mod. numerical equivalence).

$\Lambda(X)$ = totality of dynamical degrees $(\lambda_0(f), \dots, \lambda_n(f))$, f ranging over $\mathrm{Bir}(X)$.

For surfaces, $\Lambda(\mathbb{P}^n)$ is very different from $\Lambda(S)$, S irrational (Blanc-Cantat) (latter is discrete, former has accumulation points, but also gaps to the right of every dyn. degree, and infinite Cantor-Bendixson rank).

<u>Question</u>: Can point-set or metric properties of $\Lambda(X_3^n)$ be used to distinguish it from $\Lambda(\mathbb{P}^n)$? (some refinement of Noether-Iskovskikh-Manin method?)