

# Factorial hypersurfaces in $\mathbb{P}^4$ with nodes

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**Abstract** We prove that for  $n = 5, 6, 7$  a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$  is factorial if it has at most  $(n - 1)^2 - 1$  nodes.

**Keywords** Nodal hypersurface · Factoriality · Integral homology · Base-point-freeness

**Mathematics Subject Classifications (2000)** 14C20 · 14J17 · 14J30 · 14J70

## 1 Introduction

Unless otherwise mentioned, every variety is always assumed to be projective, normal and defined over  $\mathbb{C}$ . We also consider every divisor in the linear system  $|\mathcal{O}_{\mathbb{P}^n}(k)|$  as a hypersurface in  $\mathbb{P}^n$  for simplicity.

A variety  $X$  is called factorial ( $\mathbb{Q}$ -factorial, resp.) if each Weil divisor of  $X$  (a multiple of each Weil divisor of  $X$ , resp.) is Cartier. The factoriality and the  $\mathbb{Q}$ -factoriality are very subtle properties. They depend on both the local types of singularities and their global position. Also, they depend on the field of definition of the variety. In the present paper, we study the factoriality of hypersurfaces in  $\mathbb{P}^4$ . However, we confine our consideration to the case when they have only simple double points, i.e., nodes. Then the local class group at a node in a three fold has no torsion ([12]), and hence each Weil divisor that is  $\mathbb{Q}$ -Cartier must be a Cartier divisor on a nodal hypersurface in  $\mathbb{P}^4$ . Therefore, the  $\mathbb{Q}$ -factoriality is equivalent to the factoriality on a nodal hypersurface in  $\mathbb{P}^4$ .

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Let  $V_n$  be a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$ . We then have a small resolution  $W$  of singularities of the hypersurface  $V_n$ . The Hodge number  $h^1(W, \Omega_W^1)$  of  $W$  is the same as the rank of the Picard group  $\text{Pic}(W)$  of  $W$  and the Picard group  $\text{Pic}(W)$  is isomorphic to  $H^2(W, \mathbb{Z})$  because  $H^1(W, \mathcal{O}_W) = H^2(W, \mathcal{O}_W) = 0$ . It follows from Poincaré duality that  $\text{rank}(H^2(W, \mathbb{Z})) = h^1(W, \Omega_W^1) = \dim(H_4(W, \mathbb{Q}))$ . However, we also have  $H_4(W, \mathbb{Z}) \cong H_4(V_n, \mathbb{Z})$  because the smooth three fold  $W$  is a small resolution of  $V_n$ . It gives us a rough explanation for the fact that the variety  $V_n$  is  $\mathbb{Q}$ -factorial and hence factorial if and only if the global topological property

$$\text{rank}(H^2(V_n, \mathbb{Z})) = \text{rank}(H_4(V_n, \mathbb{Z})),$$

holds. Note that the duality mentioned above fails on singular varieties in general.

The nodes on  $V_n$  may have an effect on the integral (co)homology groups of  $V_n$ . However, the rank of the second integral cohomology group of  $V_n$  is 1 by Lefschetz theorem ([1]). Therefore, to determine whether the three fold hypersurface  $V_n$  is factorial or not, we have to see whether the rank of the fourth integral homology group of  $V_n$  is 1 or not. But it is not simple to compute the rank of the fourth integral homology group of  $V_n$ . Fortunately, the paper [6] gives us a great method to compute the rank of the fourth integral homology group of  $V_n$ , which reduces the topological problem to a rather simple combinatorial problem. To be precise, the rank of the fourth integral homology group of  $V_n$  can be obtained by the following way:

**Theorem 1.1** *Let  $V_n$  be a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$ . The rank of the fourth integral homology group  $H_4(V_n, \mathbb{Z})$  is equal to*

$$\#|\text{Sing}(V_n)| - I + 1,$$

where  $I$  is the number of independent conditions which vanishing on  $\text{Sing}(V_n)$  imposes on homogeneous forms of degree  $2n - 5$  on  $\mathbb{P}^4$ .

*Proof* See [6]. □

Therefore, the hypersurface  $V_n$  is factorial if and only if the set of nodes of the hypersurface  $V_n$  is  $(2n - 5)$ -normal\* in  $\mathbb{P}^4$ , in other words, the singular points of the hypersurface  $V_n$  impose linearly independent conditions on hypersurfaces of degree  $2n - 5$  in  $\mathbb{P}^4$ .

The geometry of the hypersurface  $V_n$  crucially depends on its factoriality. For example, in the case  $n = 4$  the hypersurface  $V_n$  is non-rational whenever it is factorial ([11]), which is not true without the factoriality condition.

Let us show an easy way to get a non-factorial hypersurface.

**Example 1.2** Suppose that  $V_n$  is given by the equation

$$x_0g(x_0, x_1, x_2, x_3, x_4) + x_1f(x_0, x_1, x_2, x_3, x_4) = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]),$$

where  $g$  and  $f$  are general homogeneous polynomials of degree  $n - 1$ . Then the hypersurface  $V_n$  has exactly  $(n - 1)^2$  nodes. They are located on the 2-plane defined by  $x_0 = x_1 = 0$ . The hypersurface  $V_n$  is not factorial because the hyperplane section  $x_0 = 0$  splits into two non-Cartier divisors while the Picard group is generated by a hyperplane section ([1]).

\* In general, a subscheme  $X$  of  $\mathbb{P}^N$  is called  $d$ -normal in  $\mathbb{P}^N$  if the first cohomology of the ideal sheaf of  $X$  twisted by  $\mathcal{O}_{\mathbb{P}^N}(d)$  is zero. Throughout this paper, we consider a finite set of points in  $\mathbb{P}^N$  as a zero-dimensional reduced subscheme of  $\mathbb{P}^N$ .

On the other hand, if  $|\text{Sing}(V_n)| < (n - 1)^2$ , every smooth surface in  $V_n$  is cut by a hypersurface in  $\mathbb{P}^4$  due to [5]. Therefore, it is natural that we should expect the following to be true.

**Conjecture 1.3** *Every nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$  with at most  $(n - 1)^2 - 1$  nodes is factorial.*

Conjecture 1.3 for  $n = 2$  is trivial. To show Conjecture 1.3 for  $n = 3$ , let  $V_3$  be a nodal cubic hypersurface in  $\mathbb{P}^4$  with at most three nodes. Then no three nodes of  $V_3$  lie on a single line. Therefore, for each node  $p$  of  $V_3$ , we can always find a hyperplane that contains the other nodes but not the node  $p$ , which shows that  $V_3$  is factorial. For the case  $n = 4$ , Conjecture 1.3 is proved in [4]. In this paper, we prove the following result.

**Theorem 1.4** *Conjecture 1.3 holds for  $n = 5-7$ .*

Note that the following result is proved in [3].

**Theorem 1.5** *A nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$  with at most  $\frac{(n-1)^2}{4}$  nodes is factorial.*

Therefore, at least asymptotically Conjecture 1.3 is not far from being true. To our surprise, the conjecture below implies Conjecture 1.3.

**Conjecture 1.6** *Let  $\Sigma$  be a set of at most  $(n - 1)^2 - 1$  points in  $\mathbb{P}^4$  such that at most  $k(n - 1)$  points in  $\Sigma$  can be contained in a curve of degree  $k$ . Then, for a general projection  $\phi_4: \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ , at most  $k(n - 1)$  points in  $\phi_4(\Sigma)$  can be contained in a curve of degree  $k$  in  $\mathbb{P}^2$ .*

Unfortunately, we are unable to prove Conjecture 1.3 now, but we believe that the proof of Theorem 1.4 can help us to find new approaches to a proof of Conjecture 1.3.

## 2 Preliminaries

### 2.1 Projections and linear systems with zero-dimensional base loci

Let  $X$  be a normal variety with a  $\mathbb{Q}$ -divisor  $B_X = \sum_{i=1}^k a_i B_i$ , where  $B_i$  is a prime divisor on  $X$  and  $a_i$  is a positive rational number, such that  $K_X + B_X$  is  $\mathbb{Q}$ -Cartier. Let  $\pi : Y \rightarrow X$  be a proper birational morphism of a smooth variety  $Y$  such that the union of all the proper transforms of the divisors  $B_i$  and all the  $\pi$ -exceptional divisors forms a divisor with simple normal crossing on  $Y$ .

Let  $B_Y$  be the proper transform of  $B_X$  on  $Y$ . Put  $B^Y = B_Y - \sum_{i=1}^n d_i E_i$ , where each  $E_i$  is a  $\pi$ -exceptional divisor and  $d_i$  is the rational number such that the equivalence

$$K_Y + B^Y \sim_{\mathbb{Q}} \pi^*(K_X + B_X)$$

holds. Then the log pair  $(Y, B^Y)$  is called the log pull back of the log pair  $(X, B_X)$  with respect to the proper birational morphism  $\pi$ , while the number  $d_i$  is called the discrepancy of the log pair  $(X, B_X)$  with respect to the  $\pi$ -exceptional divisor  $E_i$ .

By  $\mathcal{L}(X, B_X)$  we denote the subscheme of the variety  $X$  associated to the ideal sheaf

$$\mathcal{I}(X, B_X) = \pi_*\left(\mathcal{O}_Y(\lceil -B^Y \rceil)\right),$$

which is called the multiplier ideal sheaf of the log pair  $(X, B_X)$ .

We then obtain the following result due to [10] and [13].

**Theorem 2.1** *Suppose that the divisor  $K_X + B_X + H$  is numerically equivalent to a Cartier divisor for some nef and big  $\mathbb{Q}$ -divisor  $H$  on the variety  $X$ . Then for every  $i > 0$  we have*

$$H^i\left(X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(K_X + B_X + H)\right) = 0.$$

Theorem 2.1 gives us a useful tool to study the normality of a finite set in  $\mathbb{P}^N$ .

**Lemma 2.2** *Let  $\mathcal{M}$  be a linear system consisting of hypersurfaces of degree  $k$  on  $\mathbb{P}^N$ . If the base locus  $\Lambda$  of the linear system  $\mathcal{M}$  is zero-dimensional, then the finite set  $\Lambda$  is  $N(k - 1)$ -normal in  $\mathbb{P}^N$ .*

*Proof* Let  $H_1, \dots, H_r$  be general divisors in the linear system  $\mathcal{M}$ , where  $r$  is sufficiently big. We put

$$B = \frac{N}{r} \sum_{i=1}^r H_i.$$

Then the log pair  $(\mathbb{P}^N, B)$  is Kawamata log terminal in the outside of the base locus  $\Lambda$ . For each point  $p \in \Lambda$ , we have  $\text{mult}_p B \geq N$ . Therefore,  $\text{Supp}(\mathcal{L}(\mathbb{P}^N, B)) = \Lambda$ .

Since the divisor  $K_{\mathbb{P}^N} + B + H$  is  $\mathbb{Q}$ -linearly equivalent to  $N(k - 1)H$ , where  $H$  is a hyperplane, we obtain  $H^1(\mathbb{P}^N, \mathcal{I}(\mathbb{P}^N, B) \otimes \mathcal{O}_{\mathbb{P}^N}(N(k - 1))) = 0$  from Theorem 2.1. Because  $\text{Supp}(\mathcal{L}(\mathbb{P}^N, B)) = \Lambda$  and the scheme  $\mathcal{L}(\mathbb{P}^N, B)$  is zero-dimensional, the set  $\Lambda$  that is the reduced scheme of  $\mathcal{L}(\mathbb{P}^N, B)$  must be  $N(k - 1)$ -normal in  $\mathbb{P}^N$ .  $\square$

Let  $\Sigma$  be a finite set of points in  $\mathbb{P}^N$ ,  $N \geq 3$ , such that no  $k(d - 1) + 1$  points of  $\Sigma$  lie on a curve of degree  $k$  in  $\mathbb{P}^N$  for each  $k \geq 1$ , where  $d \geq 3$  is a fixed integer. Fix a 2-plane  $\Pi$  in  $\mathbb{P}^N$ . We consider the projection

$$\phi_N : \mathbb{P}^N \dashrightarrow \Pi \cong \mathbb{P}^2$$

from a general  $(N - 3)$ -dimensional linear space  $L$  onto the 2-plane  $\Pi$ .

**Lemma 2.3** *Let  $\Lambda$  be a subset of  $\Sigma$  with  $|\Lambda| > k(d - 1)$  and let  $\mathcal{M}$  be the linear system of hypersurfaces of degree  $k$  in  $\mathbb{P}^N$  that contain  $\Lambda$ . If the set  $\phi_N(\Lambda)$  is contained in an irreducible curve of degree  $k$  on  $\Pi$ , then the base locus of the linear system  $\mathcal{M}$  is zero-dimensional.*

*Proof* Suppose that the base locus of  $\mathcal{M}$  contains an irreducible curve  $Z$ . Let  $C$  be an irreducible curve of degree  $k$  on  $\Pi$  that contains  $\phi_N(\Lambda)$  and let  $W$  be the cone in  $\mathbb{P}^N$  over the curve  $C$  with vertex  $L$ . Since  $W$  is a hypersurface of degree  $k$  in  $\mathbb{P}^N$  containing the set  $\Lambda$ , it belongs to the linear system  $\mathcal{M}$ . In particular, the curve  $Z$  is contained in the hypersurface  $W$ . Therefore, the curve  $Z$  is mapped onto the curve  $C$  because the linear space  $L$  is general and the curve  $C$  is irreducible. The curve  $Z$  has degree  $k$  because the restriction  $\phi_N|_Z$  is a birational morphism to  $C$ .

If there is a point  $p$  in  $\Lambda \setminus Z$ , then the projection  $\phi_N$  maps the point  $p$  to the outside of  $C$  because of the generality of the projection  $\phi_N$ . Therefore, the set  $\Lambda$  must be contained in  $Z$  because  $\phi_N(\Lambda)$  is contained in  $C$ . However, the curve  $Z$  cannot contain more than  $k(d - 1)$  points of  $\Sigma$ .  $\square$

**Corollary 2.4** *A line on  $\Pi$  contains at most  $d - 1$  points of  $\phi_N(\Sigma)$ .*

*Proof* It immediately follows from Lemma 2.3.  $\square$

**Corollary 2.5** *For  $N = 3$ , a curve of degree  $k$  on  $\Pi$  contains at most  $k(d - 1)$  points of  $\phi_3(\Sigma)$  if  $d \geq k^2 + 1$ .*

*Proof* For  $k = 1$ , it is true because of Corollary 2.4. Assume that the claim is true for  $k < \ell$ . We then suppose that there is a subset  $\Lambda$  of  $\Sigma$  such that  $|\Lambda| > \ell(d - 1)$  and the image  $\phi_3(\Lambda)$  lie on a curve  $C$  of degree  $\ell$  on  $\Pi$ . The curve  $C$  must be irreducible because of our assumption. Therefore, it follows from Lemma 2.3 that the base locus of the linear system  $\mathcal{M}$  of hypersurfaces of degree  $\ell$  in  $\mathbb{P}^3$  containing the set  $\Lambda$  is zero-dimensional. Let  $Q_1, Q_2$  and  $Q_3$  be general members in  $\mathcal{M}$ . Then we obtain a contradictory inequality

$$\ell^3 = Q_1 \cdot Q_2 \cdot Q_3 > \ell(d - 1) \geq \ell^3.$$

Therefore, for  $d \geq k^2 + 1$ , a curve of degree  $k$  on  $\Pi$  contains at most  $k(d - 1)$  points of  $\phi_3(\Sigma)$ .  $\square$

**Corollary 2.6** *For  $N = 4$ , a curve of degree  $k$  on  $\Pi$  contains at most  $k(d - 1)$  points of  $\phi_4(\Sigma)$  if  $d \geq k^2 + 1$ .*

*Proof* Let  $\alpha: \mathbb{P}^4 \dashrightarrow H$  be the projection from a general point  $o_1 \in \mathbb{P}^4 \setminus H$ , where  $H$  is a general hyperplane containing  $\Pi$  in  $\mathbb{P}^4$ . And let  $\beta: H \dashrightarrow \Pi$  be the projection from a general point  $o_2 \in H \setminus \Pi$ . Then the projection  $\phi_4$  is obtained by the composite of the projections  $\alpha$  and  $\beta$ . We first claim that for  $d \geq k^2 + 1$  no  $k(d - 1) + 1$  points of  $\alpha(\Sigma)$  lie on a curve of degree  $k$  in  $\mathbb{P}^3$ . It is obviously true for  $k = 1$ . Assume that the claim is true for  $k < \ell$ . And then we suppose that there is a subset  $\Lambda$  of  $\ell(d - 1) + 1$  points in  $\Sigma$  such that  $\alpha(\Lambda)$  lie on a curve  $C$  of degree  $\ell$  in  $H$ . The curve  $C$  must be irreducible. Let  $\mathcal{M}$  be the linear system of hypersurfaces of degree  $\ell$  in  $\mathbb{P}^4$  passing through all the points of  $\Lambda$ . Because  $\beta(\alpha(\Lambda))$  is contained in the irreducible curve  $\beta(C)$  of degree  $\ell$  on  $\Pi$ , it follows from Lemma 2.3 that the base locus of  $\mathcal{M}$  is zero-dimensional. Let  $W$  be the cone in  $\mathbb{P}^4$  over the curve  $C$  with vertex  $o_1$ . Then we get an absurd inequality

$$\ell^3 = Q_1 \cdot Q_2 \cdot W \geq |\Lambda| > \ell^3,$$

where each  $Q_i$  is a general member of  $\mathcal{M}$ . Therefore, at most  $k(d - 1)$  points of  $\alpha(\Sigma)$  can lie on a curve of degree  $k$  in  $H$  if  $d \geq k^2 + 1$ .

For the projection  $\beta: H \dashrightarrow \Pi$ , we apply the proof of Corollary 2.5 to  $\alpha(\Sigma)$ . This completes our proof.  $\square$

## 2.2 Base-point-freeness

It is a classical result that if 6 points  $p_1, \dots, p_6 \in \mathbb{P}^2$  in general position are blown up, then the complete linear system on the blow-up corresponding to  $|\mathcal{O}_{\mathbb{P}^2}(3) - p_1 - \dots - p_6|$  is very ample as well as base-point-free. This is a key observation to classify del Pezzo surfaces.

Bese’s paper [2] developed this observation to points on  $\mathbb{P}^2$  in less general position and various divisors. The result however turned out to have a considerable generalization. Davis and Geramita [7] obtained a very ampleness and a base-point-freeness theorems on blow-ups of  $\mathbb{P}^2$  via the ideal-theoretic route that are more powerful than Bese’s.

Considering reduced zero dimensional subschemes of  $\mathbb{P}^2$  in Corollary 4.3 of [7], we can immediately obtain the following theorem that provides a strong enough tool for us to study the base-point-freeness of linear systems of certain types on blow-ups of  $\mathbb{P}^2$ .

**Theorem 2.7** *Let  $\pi: Y \rightarrow \mathbb{P}^2$  be the blow up at points  $p_1, \dots, p_s$  on  $\mathbb{P}^2$ . Then the linear system  $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^s E_i|$  is base-point-free for all  $s \leq \max\{m(d+3-m) - 1, m^2\}$ , where  $E_i = \pi^{-1}(p_i)$ ,  $d \geq 3$ , and  $m = \lfloor \frac{d+3}{2} \rfloor$ , if the set  $\Gamma = \{p_1, \dots, p_s\}$  satisfies the following:*

*no  $k(d+3-k) - 1$  points of  $\Gamma$  lie on a single curve of degree  $k$ ,  $1 \leq k \leq m$ .*

In the case  $d = 3$  Theorem 2.7 is the well-known result on the base-point-freeness of the anticanonical linear system of a weak del Pezzo surface of degree  $9 - s \geq 2$ . The theorem above immediately implies the following:

**Corollary 2.8** *Let  $\Gamma = \{p_1, \dots, p_s\}$  be a finite set of points in  $\mathbb{P}^2$ . For a given positive integer  $d \geq 3$ , if  $s \leq \max\{m(d+3-m) - 1, m^2\}$  and no  $k(d+3-k) - 1$  points of the set  $\Gamma$  lie on a curve of degree  $k \leq m$  in  $\mathbb{P}^2$ , where  $m = \lfloor \frac{d+3}{2} \rfloor$ , then for each point  $p_i \in \Gamma$  there is a curve of degree  $d$  in  $\mathbb{P}^2$  that contains all the points of the set  $\Gamma$  except the point  $p_i$ .*

The corollary above is the most important tool for this paper. Also, it makes us propose Conjecture 1.6.

### 2.3 Basic properties of nodes

As explained at the beginning, the ranks of the fourth integral homology groups of nodal hypersurfaces in  $\mathbb{P}^4$  are strongly related to the number of nodes and their position. Even though the number of nodes are given in our problem, it is necessary that we should study their position.

**Lemma 2.9** *Let  $V_n$  be a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$ .*

- (1) *A curve of degree  $k$  in  $\mathbb{P}^4$  contains at most  $k(n-1)$  nodes of  $V_n$ .*
- (2) *If a 2-plane contains  $\frac{n(n-1)}{2} + 1$  nodes of  $V_n$ , then the 2-plane is contained in  $V_n$ .*

*Proof* Suppose that the hypersurface  $V_n$  is defined by an equation  $F(x_0, x_1, x_2, x_3, x_4) = 0$ . Then the singular locus of  $V_n$  is contained in a general hypersurface  $V' = (\sum \lambda_i \frac{\partial F}{\partial x_i} = 0)$  of degree  $n-1$ . Let  $C$  be a curve of degree  $k$  in  $\mathbb{P}^4$ . Since the hypersurface  $V_n$  has only isolated singularities, the curve  $C$  cannot be contained in  $V'$ . Because the intersection number of the hypersurface  $V'$  and the curve  $C$  is  $k(n-1)$ , the curve  $C$  contains at most  $k(n-1)$  singular points of  $V_n$ .

Let  $\Pi$  be a 2-plane not contained in  $V_n$ . Then  $V' \cap \Pi$  is a curve of degree  $(n-1)$  not contained in  $V_n$ . Therefore, the curve  $V' \cap \Pi$  cannot meet  $V_n$  at more than  $\frac{n(n-1)}{2}$  nodes of  $V_n$ . □

**Lemma 2.10** *If a nodal hypersurface  $V_n$  of degree  $n$  in  $\mathbb{P}^4$  contains a 2-plane, then it has at least  $(n - 1)^2$  singular points.*

*Proof* Suppose that  $V_n \subset \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4])$  contains a 2-plane  $\Pi$ . We may assume that the 2-plane is defined by the equations  $x_0 = x_1 = 0$ . Then the nodal hypersurface  $V_n$  is defined by an equation of the form

$$x_0q_0(x_0, x_1, x_2, x_3, x_4) + x_1q_1(x_0, x_1, x_2, x_3, x_4) = 0,$$

where  $q_0$  and  $q_1$  are homogeneous polynomials of degree  $n - 1$ . For each  $i$ , let  $Q_i$  be the hypersurface in  $\mathbb{P}^4$  defined by  $q_i = 0$ . The intersection points of  $\Pi, Q_0$ , and  $Q_1$  are singular points of  $V_n$ . Because  $V_n$  has only isolated singularities, the set  $S$  of the intersection points of  $\Pi, Q_0$ , and  $Q_1$  is finite. For each point  $p \in S$ ,  $\Pi, Q_0$ , and  $Q_1$  meet transversally at  $p$ ; otherwise the hypersurface  $V_n$  would have a singularity worse than a node at  $p$ . Therefore,  $|S| = (n - 1)^2$ , which implies that  $V_n$  has at least  $(n - 1)^2$  nodes. □

### 2.4 Simple tools

To prove Theorem 1.4, we need various tools to handle hypersurfaces of certain degrees and a finite number of points in  $\mathbb{P}^4$ .

**Lemma 2.11** *Let  $\Sigma = \{p_1, \dots, p_r\}$  be a set of  $r$  points in  $\mathbb{P}^N$ ,  $N \geq 2$ . Let  $p$  be a point in  $\mathbb{P}^N \setminus \Sigma$ . Suppose that no  $m + 1$  points of  $\Sigma$  lie on a single line with the point  $p$ . Then there are at least  $\min\{r - m, \lfloor \frac{r}{2} \rfloor\}$  mutually disjoint pairs of points in  $\Sigma$  such that each pair determines a line not containing the point  $p$ .*

*Proof* We may assume that the points  $p_1, \dots, p_m$ , and  $p$  are on a single line  $L$ .

First, we suppose  $m \geq r - m$ . We then obtain  $r - m$  such pairs by choosing one point from  $\Sigma \cap L$  and the other from  $\Sigma \setminus L$ . Obviously, these pairs determine lines not passing through the point  $p$ .

Next, we suppose  $m < r - m$ . We can then find  $\lfloor \frac{r-2m}{2} \rfloor$  such pairs by choosing two points of  $\Sigma \setminus L$ ; otherwise  $m + 1$  points of  $\Sigma$  would lie on a single line. By choosing one point from the remaining points in  $\Sigma \setminus L$  and the other from  $L$  we also obtain  $m$  such pairs. The number of the pairs that we obtain is  $m + \lfloor \frac{r-2m}{2} \rfloor = \lfloor \frac{r}{2} \rfloor$ . □

**Lemma 2.12** *Let  $H$  be a hyperplane in  $\mathbb{P}^N$ ,  $N \geq 3$ , and  $X$  be a hypersurface of degree  $d \geq 2$  in  $H$ . Suppose that the hypersurface  $X$  does not contain a point  $o \in H$ . For given two points  $p, q$  in  $\mathbb{P}^N \setminus H$ , there is a hypersurface of degree  $d$  in  $\mathbb{P}^N$  such that contains  $X$  and two points  $p$  and  $q$  but not the point  $o$ .*

*Proof* Take a general 2-plane  $\Pi$  passing through the points  $p$  and  $q$  in  $\mathbb{P}^N$ . Then  $\Pi$  meets  $X$  at at least two points, say  $p'$  and  $q'$ . Then the line determined by  $p$  and  $p'$  and the line determined by  $q$  and  $q'$  meet at a point  $v$ . Then the cone over  $X$  with vertex  $v$  is a hypersurface of degree  $d$  in  $\mathbb{P}^N$  containing  $X$  and two points  $p$  and  $q$  but does not contain the point  $o$ . □

**Lemma 2.13** *Let  $\Lambda$  and  $\Delta$  be disjoint finite sets of points in  $\mathbb{P}^N$ ,  $N \geq 3$ , and let  $p$  be a point in  $\mathbb{P}^N \setminus (\Lambda \cup \Delta)$ . Suppose that  $D_0$  be a hypersurface of degree  $k$  in  $\mathbb{P}^N$  containing the set  $\Lambda$  but not the point  $p$ . If for each point  $q \in \Delta$  there is a hypersurface  $D_q$  of degree  $k$  in  $\mathbb{P}^N$  containing the set  $(\Lambda \cup \Delta \cup \{p\}) \setminus \{q\}$  but not the point  $q$ , then there is a hypersurface of degree  $k$  such that passes through  $\Lambda \cup \Delta$  but not the point  $p$  in  $\mathbb{P}^N$ .*

*Proof* Suppose that the hypersurface  $D_0$  is defined by a homogeneous polynomial  $g(x_0, \dots, x_N)$ . Also, we suppose that each hypersurface  $D_q$  is defined by a homogeneous polynomial  $f_q(x_0, \dots, x_N)$ . Then  $g(p) \neq 0$  and  $f_q(p) = 0$  for each  $q \in \Delta$ . Furthermore,  $f_q(q') \neq 0$  for some  $q' \in \Delta$  if and only if  $q = q'$ . There is a complex number  $c_q$  for each  $q \in \Delta$  such that  $g(q) + c_q f_q(q) = 0$  because  $f_q(q) \neq 0$ . Then the hypersurface defined by

$$g(x_0, \dots, x_N) + \sum_{q \in \Delta} c_q f_q(x_0, \dots, x_N) = 0$$

contains the set  $\Lambda \cup \Delta$  but not the point  $p$ . □

**Corollary 2.14** *Let  $\mathcal{M}$  be a linear system consisting of hypersurfaces of degree  $k \geq 2$  on  $\mathbb{P}^N$ ,  $N \geq 3$ . If the base locus  $\Lambda$  of the linear system  $\mathcal{M}$  is zero-dimensional, then for two distinct points  $p, q$  in  $\mathbb{P}^N \setminus \Lambda$  and a point  $o$  in  $\Lambda$ , there is a hypersurface of degree  $N(k - 1)$  such that passes through  $\Lambda \cup \{p, q\} \setminus \{o\}$  but not the point  $o$  in  $\mathbb{P}^N$ .*

*Proof* By Lemma 2.2, there is a hypersurface  $D_0$  of degree  $N(k - 1)$  in  $\mathbb{P}^N$  that passes through all the points of  $\Lambda$  except the point  $o$ . Let  $D$  be a general member in  $\mathcal{M}$ . We choose a hyperplane  $H_p$  in  $\mathbb{P}^N$  that passes through the point  $p$  but not the point  $q$ . We also choose a hyperplane  $H_q$  that passes through the point  $q$  but not the point  $p$ . We then apply Lemma 2.13 to the hypersurfaces  $D_0, D + (N(k - 1) - k)H_p$ , and  $D + (N(k - 1) - k)H_q$ . □

The result below is originally due to Edmonds [8]. It can help us to make our proofs simpler.

**Theorem 2.15** *Let  $\Sigma$  be a set of points in  $\mathbb{P}^N$  and let  $d \geq 2$  be an integer. If no  $dk + 2$  points of  $\Sigma$  lie in a  $k$ -plane for all  $k \geq 1$ , then the set  $\Sigma$  is  $d$ -normal in  $\mathbb{P}^N$ .*

*Proof* See [9]. □

### 3 Conjectural proof

In this section, we prove Conjecture 1.3 under the assumption that Conjecture 1.6 is true.

Let  $V_n$  be a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$ . Suppose that  $|\text{Sing}(V_n)| < (n - 1)^2$  and  $n \geq 4$ . Fix a point  $p \in \text{Sing}(V_n)$ . And then put  $\Gamma = \text{Sing}(V_n) \setminus \{p\}$ . To prove the factoriality of  $V_n$  it is enough to construct a hypersurface of degree  $2n - 5$  that contains all the points of the set  $\Gamma$  and does not contain the point  $p$ .

We suppose that Conjecture 1.6 holds. Let  $\phi_4: \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$  be the projection from a general line  $L$  in  $\mathbb{P}^4$ , where  $\Pi$  is a 2-plane in  $\mathbb{P}^4$ . It then follows from Conjecture 1.6 and Lemma 2.9 that the set  $\phi_4(\Gamma)$  satisfies the condition

$$\text{no } k(n - 1) + 1 \text{ points of } \phi_4(\Gamma) \text{ lie on a curve of degree } k \text{ on } \Pi \text{ for each } k \geq 1. \quad (1)$$

**Remark 3.1** It follows from Corollaries 2.4 and 2.6 that the condition (1) holds for  $k \leq \sqrt{n - 1}$  without Conjecture 1.6.

**Lemma 3.2** *For each  $1 \leq k \leq n - 1$ , any curve of degree  $k$  on  $\Pi$  cannot contain  $k(2n - 2 - k) - 1$  points of  $\phi_4(\Gamma)$ .*

*Proof* Note that  $|\phi_4(\Gamma)| \leq (n - 1)^2 - 2$  and  $n \geq 4$  by our assumption. Because  $k(n - 1) \leq k(2n - 2 - k) - 2$  if  $k < n - 1$ , the statement immediately follows.  $\square$

**Lemma 3.3** *There is a curve of degree  $2n - 5$  on  $\Pi$  which passes through all the points of  $\phi_4(\Gamma)$  but not the point  $\phi_4(p)$ .*

*Proof* It immediately follows from Corollary 2.8 and Lemma 3.2.  $\square$

**Proposition 3.4** *Conjecture 1.6 implies Conjecture 1.3.*

*Proof* Lemma 3.3 implies that there is a curve  $C$  of degree  $2n - 5$  on  $\Pi$  which passes through all the points of  $\phi_4(\Gamma)$  but not the point  $\phi_4(p)$ . We take the cone over  $C$  with vertex  $L$ . The cone then contains all the points of  $\Gamma$  but not the point  $p$ . It implies that if the hypersurface  $V_n$  has  $s < (n - 1)^2$  singular points, then these  $s$  points impose  $s$  linearly independent conditions on homogeneous forms of degree  $2n - 5$  on  $\mathbb{P}^4$ . Consequently, the rank of the fourth integral homology group of  $V_n$  is 1 by Theorem 1.1, which completes the proof.  $\square$

#### 4 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let  $V_n$  be a nodal hypersurface of degree  $n$  in  $\mathbb{P}^4$  with at most  $(n - 1)^2 - 1$  nodes. However, as we will see in the proofs, we may assume that the hypersurface  $V_n$  has exactly  $(n - 1)^2 - 1$  nodes. To prove the factoriality of the hypersurface  $V_n$ , for an arbitrary point  $p \in \text{Sing}(V_n)$ , we have to construct a hypersurface of degree  $2n - 5$  in  $\mathbb{P}^4$  that contains the set  $\text{Sing}(V_n)$  except the point  $p$ .

##### 4.1 Quintic hypersurfaces

Let  $V_5$  be a nodal quintic hypersurface in  $\mathbb{P}^4$  with 15 nodes. A line can contain at most four nodes by Lemma 2.9. If a 2-plane  $\Pi$  contains 12 nodes of  $V_5$ , then  $\Pi$  is contained in  $V_5$  by Lemma 2.9. It then follows from Lemma 2.10 that  $V_5$  must have at least 16 nodes, which contradicts our assumption. Therefore, a 2-plane can contain at most 11 nodes of  $V_5$ . Therefore, the set of nodes of  $V_5$  satisfies the condition for  $d = 5$  in Theorem 2.15 and hence the nodal quintic hypersurface  $V_5$  is factorial.

##### 4.2 Sextic hypersurfaces

Let  $V_6$  be a nodal sextic hypersurface in  $\mathbb{P}^4$  with 24 nodes. We denote the set of nodes of  $V_6$  by  $\Sigma$ . If a 2-plane  $\Pi$  contains 16 nodes of  $V_6$ , then  $\Pi$  is contained in  $V_6$  by Lemma 2.9. It then follows from Lemma 2.10 that  $V_6$  must have at least 25 nodes, which contradicts our assumption. Therefore, a 2-plane contains at most 15 nodes of  $V_6$ .

**Proposition 4.1** *If no 23 points of  $\Sigma$  lie on a single 3-plane, then the hypersurface  $V_6$  is factorial.*

*Proof* Since no 6 points of  $\Sigma$  lie on a single line and no 16 points of  $\Sigma$  lie on a single 2-plane, the set  $\Sigma$  satisfies the condition for  $d = 7$  of Theorem 2.15. Therefore, the set  $\Sigma$  is 7-normal in  $\mathbb{P}^4$  and hence the hypersurface  $V_6$  is factorial.  $\square$

Pick an arbitrary point  $p$  in  $\Sigma$  and then we denote the set  $\Sigma \setminus \{p\}$  by  $\Gamma$ . To prove the factoriality of  $V_6$ , we must find a hypersurface of degree 7 in  $\mathbb{P}^4$  that contains the set  $\Gamma$  but not the point  $p$ . Due to Proposition 4.1, we may assume that at least 23 points of  $\Sigma$  lie in a single 3-plane  $H$ . Furthermore, we may assume that all the 24 points of  $\Sigma$  lie in the 3-plane  $H$  because in what follows we will show that there is a septic hypersurface in  $H$ , not in  $\mathbb{P}^4$ , that contains  $\Gamma \cap H$  but not the point  $p$ .

We consider the projection  $\phi_3 : H \dashrightarrow \Pi$  from a general point  $o$  in  $H$ , where  $\Pi$  is a general hyperplane of  $H$ . At most five points of  $\Sigma$  can lie on a single line in  $H$  and at most ten points of  $\Sigma$  can lie on a conic on  $H$ .

**Lemma 4.2** *If there is a set  $\Lambda$  of at least 20 points of  $\Gamma$  such that  $\phi_3(\Lambda)$  is contained in a cubic curve  $C$  on  $\Pi$ , then there is a septic hypersurface in  $H$  that contains the set  $\Gamma$  but not the point  $p$ .*

*Proof* We may assume that the cubic curve  $C$  contains the point  $\phi_3(p)$ . If not, then we can easily construct a septic surface in  $H$  that contains  $\Gamma$  but not the point  $p$ . The curve  $C$  must be irreducible because a line (a conic, resp.) contains at most 5 (10, resp.) points of  $\phi_3(\Lambda)$  by Corollaries 2.4 and 2.5. It then follows from Lemma 2.3 that the linear system of cubic surfaces in  $H$  passing through  $\Lambda \cup \{p\}$  has zero-dimensional base locus. Therefore, applying Corollary 2.14, we obtain a sextic surface  $F$  that passes through 22 points of  $\Sigma$  but not the point  $p$ . Note that  $|\Gamma \setminus F| \leq 1$ . By taking a general hyperplane passing through the point in  $\Gamma \setminus F$ , we can construct a septic surface in  $H$  that contains the set  $\Gamma$  but not the point  $p$ . □

From now, we suppose that no 20 points of  $\phi_3(\Lambda)$  is contained in a cubic curve on  $\Pi$ . And then let us apply the similar technique as Lemma 2.2, which has evolved from the papers [4, 14], to the following case.

**Lemma 4.3** *If the set  $\phi_3(\Lambda)$  is contained in a quartic curve  $C$  on  $\Pi$ , then there is a septic hypersurface in  $H$  that contains the set  $\Gamma$  but not the point  $p$ .*

*Proof* The curve  $C$  must be irreducible because of our assumption. Also, we may assume that it contains the point  $\phi_3(p)$  as well. Then the linear system  $\mathcal{M}$  of quartic hypersurfaces in  $H$  passing through  $\Sigma$  has zero-dimensional base locus. Meanwhile, we have the sextic surface  $Y = H \cap V_6$  contains all the nodes of  $V_6$ . It may have non-isolated singularities. However, it is irreducible and reduced; otherwise the hypersurface  $V_6$  would have more than 24 nodes. Choose a general enough surface  $S'$  in  $\mathcal{M}$ . Then it is smooth in the outside of the base locus of  $\mathcal{M}$  and hence it is normal. Also, the surface  $Y$  gives us a reduced divisor  $D_6 \in |\mathcal{O}_{S'}(6)|$  on  $S'$ . Let  $D_4$  be a divisor in  $|\mathcal{O}_{S'}(4)|$  given by a general member of  $\mathcal{M}$ . We then consider the  $\mathbb{Q}$ -divisor  $D = (1 - \epsilon)D_6 + 2\epsilon D_4$ , where  $\epsilon$  is sufficiently small enough rational number. Then it is easy to check that the support of  $\mathcal{L}(S', D)$  is zero-dimensional and contains  $\Sigma$ . Use Theorem 2.1 to obtain  $H^1(S', \mathcal{I}(S', D) \otimes \mathcal{O}_{S'}(7)) = 0$ . Therefore, there is a divisor in  $|\mathcal{O}_{S'}(7)|$  that contains  $\Gamma$  but not the point  $p$ . The exact sequence

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(7)) \rightarrow H^0(S', \mathcal{O}_{S'}(7)) \rightarrow 0$$

completes the proof. □

**Lemma 4.4** *If the set  $\phi_3(\Gamma)$  is not contained in any quartic curve  $C$  on  $\Pi$ , then there is a septic hypersurface in  $H$  that contains the set  $\Gamma$  but not the point  $p$ .*

*Proof* In this case, the set  $\phi_3(\Lambda)$  satisfies the condition for  $d = 7$  in Theorem 2.8. Therefore, there is a septic curve on  $\Pi$  that contains the set  $\phi_3(\Lambda)$  but not the point  $\phi_3(p)$ . Then the cone over the septic curve with vertex  $o$  is a septic hypersurface in  $H$  that contains  $\Gamma$  but not the point  $p$ .  $\square$

Consequently, for an arbitrary point  $p \in \Sigma$ , we can find a septic hypersurface in  $\mathbb{P}^4$  that contains  $\Gamma$  but not the point  $p$ . Therefore, the rank of the fourth integral homology group of  $V_6$  is 1 and hence  $V_6$  is factorial.

### 4.3 Septic hypersurfaces

Let  $V_7$  be a nodal septic hypersurface in  $\mathbb{P}^4$  with 35 nodes. We again denote the set of nodes of  $V_7$  by  $\Sigma$ . Fix a point  $p$  in  $\Sigma$ . We denote the set  $\Sigma \setminus \{p\}$  by  $\Gamma$ . To prove the factoriality of  $V_7$ , we have to construct a hypersurface of degree 9 in  $\mathbb{P}^4$  that contains the set  $\Gamma$  but not the point  $p$ .

First of all, it follows from Lemma 2.10 that a 2-plane contains at most 21 points of  $\Sigma$ . Suppose that there is a 2-plane  $\Pi$  containing at least 20 points of  $\Sigma$ . The 2-plane  $\Pi$  is not contained in  $V_7$ ; otherwise the hypersurface  $V_7$  would have at least 36 nodes. We then consider the projection  $\phi_4: \mathbb{P}^4 \dashrightarrow \Pi$  from a general line  $L$ . By Corollaries 2.4 and 2.6 a line on  $\Pi$  contains at most six points of  $\phi_4(\Sigma)$  and a conic on  $\Pi$  contains at most 12 points of  $\phi_4(\Sigma)$ .

**Proposition 4.5** *If there is a 2-plane  $\Pi$  containing at least 20 points of  $\Sigma$ , then for the projection  $\phi_4: \mathbb{P}^4 \dashrightarrow \Pi$  from a general line  $L$  the set  $\phi_4(\Gamma)$  satisfies the following:*

- (1) *A line on  $\Pi$  contains at most 6 points of  $\phi_4(\Gamma)$ .*
- (2) *A conic on  $\Pi$  contains at most 12 points of  $\phi_4(\Gamma)$ .*
- (3) *A cubic on  $\Pi$  contains at most 25 points of  $\phi_4(\Gamma)$ .*
- (4) *A quartic on  $\Pi$  contains at most 30 points of  $\phi_4(\Gamma)$ .*
- (5) *A quintic on  $\Pi$  contains at most 33 points of  $\phi_4(\Gamma)$ .*
- (6) *A sextic on  $\Pi$  contains at most 34 points of  $\phi_4(\Gamma)$ .*

*Proof* The first and the second statements follow from Corollaries 2.4 and 2.6. And the last statement is obvious because  $|\phi_4(\Gamma)| = 34$ .

For a cubic, we suppose that there is a cubic  $C$  on  $\Pi$  that contains 26 points  $\phi_4(p_1), \dots, \phi_4(p_{26})$  of  $\phi_4(\Gamma)$ . The cubic  $C$  must be irreducible because of the first and the second statements. It then follows from Lemma 2.3 that the base locus of the linear system  $\mathcal{M}$  of cubic hypersurfaces in  $\mathbb{P}^4$  containing the points  $p_1, \dots, p_{26}$  is zero-dimensional and hence the restricted linear system  $\mathcal{M}|_\Pi$  of the linear system  $\mathcal{M}$  to the 2-plane  $\Pi$  also has zero-dimensional base locus. Since we have at most 15 points of  $\Sigma$  in the outside of  $\Pi$ , at least 11 points of  $p_1, \dots, p_{26}$  belong to  $\Pi$ . Therefore, there is an irreducible cubic curve  $D$  in  $\mathcal{M}|_\Pi$  that is not contained in  $V_7$  but passing through 11 nodes of  $V_7$ . However, this is impossible because  $21 = D \cdot V_7 \geq 11 \cdot 2 = 22$ .

If we have a quartic in  $\Pi$  containing 31 points of  $\phi_4(\Gamma)$ , then in the same way, we can find an irreducible quartic curve not contained in  $V_7$  but passing through 16 nodes of  $V_7$ , which is absurd.

Finally, if there is a quintic in  $\Pi$  containing 34 points of  $\phi_4(\Gamma)$ , then in the same way we can find an irreducible quintic curve not contained in  $V_7$  but passing through 19 nodes of  $V_7$ , which is also impossible.  $\square$

**Corollary 4.6** *If there is a 2-plane containing at least 20 points of  $\Sigma$ , the hypersurface  $V_7$  is factorial.*

*Proof* The proposition above shows the set  $\phi_4(\Gamma)$  satisfies the condition for  $d = 9$  in Corollary 2.8 and hence there is a curve  $C$  of degree 9 on  $\Pi$  passing through all the point of  $\phi_4(\Gamma)$  but not the point  $\phi_4(p)$ . The cone over the curve  $C$  with vertex  $L$  shows that the set  $\Sigma$  is 9-normal in  $\mathbb{P}^4$ . Therefore, the hypersurface  $V_7$  is factorial.  $\square$

From now on, we suppose that a 2-plane contains at most 19 points of  $\Sigma$ . If a hyperplane in  $\mathbb{P}^4$  contains at most 28 points of  $\Sigma$ , then the set  $\Sigma$  satisfies the condition for  $d = 9$  in Theorem 2.15 and hence it is 9-normal and  $V_7$  is factorial. Therefore, we suppose that a hyperplane  $H$  in  $\mathbb{P}^4$  contains at least 29 points of  $\Sigma$ . And let  $\Sigma' = \Sigma \cap H$  and  $\Sigma'' = \Sigma \setminus H$ . We always assume that the point  $p$  is contained in  $\Sigma'$  because, if not, then we can easily construct a hypersurface of degree 9 in  $\mathbb{P}^4$  containing  $\Gamma$  but not the point  $p$ .

We consider the projection  $\alpha : H \dashrightarrow \Pi$  from a general point  $o_1 \in H$ , where  $\Pi$  is a general 2-plane in  $H$ .

**Lemma 4.7** *If there is a set  $\Lambda$  of at least 26 points of  $\Sigma' \setminus \{p\}$  such that  $\alpha(\Lambda)$  is contained in a cubic curve  $C$  on  $\Pi$ , then there is a hypersurface of degree 9 that contains the set  $\Gamma$  but not the point  $p$ .*

*Proof* Note that the curve  $C$  is irreducible and  $r = |\Gamma \setminus \Lambda| \leq 8$ .

Suppose that the curve  $C$  does not contain the point  $\alpha(p)$ . Because a line has at most six points of  $\Sigma$ , there is a hypersurface of degree  $\min\{r - 5, \lfloor \frac{r}{2} \rfloor\} \leq 4$  in  $\mathbb{P}^4$  that passes through  $\Gamma \setminus \Lambda$  but not the point  $p$  by Lemma 2.11. Therefore, we can easily construct a hypersurface of degree 9 that passes through  $\Gamma$  but not the point  $p$ .

Suppose that the curve  $C$  contains also the point  $\alpha(p)$ . Pick two points  $p_1$  and  $p_2$  from  $\Gamma \setminus \Lambda$  in such a way that  $\Gamma \setminus (\Lambda \cup \{p_1, p_2\})$  is contained in a cubic hypersurface  $F_1$  in  $\mathbb{P}^4$  not containing the point  $p$ , which is possible because of Lemma 2.11. The linear system of cubic hypersurfaces in  $H$  containing  $\Lambda \cup \{p\}$  has zero dimensional base locus. Therefore, there is a sextic hypersurface  $F_2$  in  $\mathbb{P}^4$  that passes through  $\Lambda$  and the points  $p_1$  and  $p_2$  but not the point  $p$  by Corollary 2.14. Then the nonic hypersurface  $F = F_1 + F_2$  contains all the point of  $\Sigma$  except the point  $p$ .  $\square$

We may assume that no 26 points of  $\alpha(\Sigma' \setminus \{p\})$  lie on a cubic curve on  $\Pi$ . If a cubic curve on  $\Pi$  contains more than 18 points of  $\alpha(\Sigma' \setminus \{p\})$  then it must be irreducible.

**Lemma 4.8** *If a cubic curve  $C$  on  $\Pi$  contains 22 points of  $\alpha(\Sigma' \setminus \{p\})$ , it is unique.*

*Proof* Suppose that a cubic curve  $C'$  on  $\Pi$  contains at least 22 points of  $\alpha(\Sigma' \setminus \{p\})$ , then it meets  $C$  at  $22 - (34 - 22) = 10$  points and hence  $C = C'$ .  $\square$

To prove the factoriality of  $V_7$ , we will consider the following five cases.

**Case 1**  $|\Sigma'| = 29$ .

Because no 22 points of  $\Sigma$  are contained in a 2-plane, Lemma 4.8 enables us to choose six points  $p_1, p_2, \dots, p_6$  from the set  $\Sigma' \setminus \{p\}$  in such a way that

- no 20 points of  $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$  lie on a single cubic on  $\Pi$ ;
- $p_1, p_2$ , and  $p_3$  lie on a 2-plane  $\Pi_1$  not containing the point  $p$ ;
- $p_4, p_5$ , and  $p_6$  lie on a 2-plane  $\Pi_2$  not containing the point  $p$ .

Then the set  $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$  satisfies the condition of Corollary 2.8 for  $d = 7$ . Therefore, there is a septic curve  $D$  on  $\Pi$  containing  $\alpha(\Sigma' \setminus \{p, p_1, \dots, p_6\})$  but not the point  $p$ . Then the cone over  $D$  with vertex  $o_1$  is the septic surface  $\tilde{D}$  in  $H$  containing  $\Sigma' \setminus \{p, p_1, \dots, p_6\}$  but not the point  $p$ . Choose two points  $q_1$  and  $q_2$  from  $\Sigma''$ . By Lemma 2.12, we can construct a hypersurface  $\tilde{D}$  in  $\mathbb{P}^4$  containing  $\Sigma' \setminus \{p, p_1, \dots, p_6\}$  and  $q_1$  and  $q_2$  but not the point  $p$ . Now we choose another two points  $q_3$  and  $q_4$  from  $\Sigma''$ . Let  $H_3$  be a hyperplane passing through the point  $q_3$  but not  $q_4$  and  $H_4$  a hyperplane passing through the point  $q_4$  but not  $q_3$ . Apply Lemma 2.13 to the hypersurfaces  $\tilde{D}, H + H' + 5H_4$ , and  $H + H' + 5H_3$ , where  $H'$  is a general hyperplane in  $\mathbb{P}^4$  passing through  $q_1$  and  $q_2$ , to obtain a septic hypersurface in  $\mathbb{P}^4$  passing through  $\Sigma' \setminus \{p, p_1, \dots, p_6\}$  and  $\{q_1, q_2, q_3, q_4\}$  but not the point  $p$ . By our construction, two 2-planes  $\Pi_1$  and  $\Pi_2$  and the remaining two points of  $\Sigma''$  are contained in a quadratic hypersurface in  $\mathbb{P}^4$  not passing through the point  $p$ .

**Case 2**  $|\Sigma'| = 30$ .

For the same reason as in Case 1, we can choose three points  $p_1, p_2$  and  $p_3$  from the set  $\Sigma' \setminus \{p\}$  in such a way that

- no 23 points of  $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$  lie on a single cubic on  $\Pi$ ;
- $p_1, p_2$ , and  $p_3$  lie on a 2-plane  $\Pi_1$  not containing the point  $p$ .

Then the set  $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$  satisfies the condition of Corollary 2.8 for  $d = 8$ . Therefore, there is an octic curve  $D$  on  $\Pi$  containing  $\alpha(\Sigma' \setminus \{p, p_1, p_2, p_3\})$  but not the point  $p$ . Then the cone over  $D$  with vertex  $o_1$  is an octic surface  $\tilde{D}$  in  $H$  containing  $\Sigma' \setminus \{p, p_1, p_2, p_3\}$  but not the point  $p$ . Choose two points  $q_1$  and  $q_2$  from  $\Sigma''$ . By Lemma 2.12, we can construct a hypersurface  $\tilde{D}$  in  $\mathbb{P}^4$  containing  $\Sigma' \setminus \{p, p_1, p_2, p_3\}$  and  $q_1$  and  $q_2$ . Now we choose another two points  $q_3$  and  $q_4$  from  $\Sigma''$ . As in Case 1, we use Lemma 2.13 to get an octic hypersurface in  $\mathbb{P}^4$  passing through  $\Sigma' \setminus \{p, p_1, p_2, p_3\}$  and  $\{q_1, q_2, q_3, q_4\}$  but not the point  $p$ . Also, the 2-plane  $\Pi_1$  and the remaining one point in  $\Sigma''$  are contained in a hyperplane in  $\mathbb{P}^4$  not passing through the point  $p$ .

**Case 3**  $|\Sigma'| = 31$ .

Then the set  $\alpha(\Sigma' \setminus \{p\})$  satisfies the condition of Corollary 2.8 for  $d = 9$ . Therefore, there is a nonic curve  $D$  on  $\Pi$  containing  $\alpha(\Sigma' \setminus \{p\})$  but not the point  $p$ . Then the cone over  $D$  with vertex  $o_1$  is a nonic surface  $\tilde{D}$  in  $H$  containing  $\Sigma' \setminus \{p\}$  but not the point  $p$ . Choose two points  $q_1$  and  $q_2$  from  $\Sigma''$ . By Lemma 2.12, we can construct hypersurface  $\tilde{D}$  in  $\mathbb{P}^4$  containing  $\Sigma' \setminus \{p, p_1, p_2, p_3\}$  and  $q_1$  and  $q_2$ . Note that  $|\Sigma'' \setminus \{q_1, q_2\}| = 2$ . As in the previous, we use Lemma 2.13 to construct a nonic hypersurface in  $\mathbb{P}^4$  passing through  $\Gamma$  but not the point  $p$ .

**Case 4**  $32 \leq |\Sigma'| \leq 34$ .

Suppose that no 31 points of  $\alpha(\Sigma' \setminus \{p\})$  lie on a single quartic curve on  $\Pi$ . The set  $\alpha(\Sigma' \setminus \{p\})$  then satisfies the condition of Corollary 2.8 for  $d = 9$ . Therefore, as in Case 3, we can find a nonic hypersurface that we need.

Suppose that there is a set  $\Lambda$  of at least 31 points of  $\Sigma' \setminus \{p\}$  such that a quartic curve  $C$  on  $\Pi$  contains  $\alpha(\Lambda)$ . If the curve  $C$  does not contain the point  $\alpha(p)$ , then we can easily construct a nonic hypersurface in  $\mathbb{P}^4$  that we are looking for. Therefore, we may assume that the point  $\alpha(p)$  also belongs to  $C$ . Then it follows from Lemmas 2.2

and 2.3 that there is a nonic hypersurface in  $H$  passing through  $\Lambda$  but not the point  $p$ . Then, using Lemmas 2.12 and 2.13, we can construct a nonic hypersurface in  $\mathbb{P}^4$  that passes through all the point of  $\Gamma$  but not the point  $p$ .

**Case 5**  $|\Sigma'| = 35$ .

Suppose that there is a set  $\Lambda$  of at least 31 points of  $\Gamma = \Sigma \setminus \{p\}$  such that  $\alpha(\Lambda)$  is contained in a quartic curve  $C$  on  $\Pi$ . The curve  $C$  must be irreducible. We also assume that the curve  $C$  contains the point  $\alpha(p)$ . Then the base locus of the linear system  $\mathcal{M}$  of quartic surfaces on  $H$  containing  $\Lambda \cup \{p\}$  is zero-dimensional by Lemma 2.3. Let  $B$  be the support of the base locus of the linear system  $\mathcal{M}$  and  $\bar{\Sigma} = \Sigma \setminus B$ . Note that  $\Lambda \cup \{p\} \subset B$ . It follows from Lemma 2.2 that the set  $B$  is 9-normal in  $H$ . There is a nonic hypersurface  $F$  in  $H$  that contains  $B \setminus \{p\}$  but not the point  $p$ . Because  $|\bar{\Sigma}| \leq 3$ , for each  $q \in \bar{\Sigma}$  there is a quintic hypersurface  $Q_q$  in  $H$  such that contains the set  $\bar{\Sigma} \setminus \{q\}$  but not the point  $q$ . Choose a general element  $Q$  from the linear system  $\mathcal{M}$ . We then apply Lemma 2.13 to the nonic hypersurfaces  $F$  and  $Q + Q_q$  to obtain a nonic hypersurface passing through  $\Sigma$  except the point  $p$ . Therefore, we may assume that no 31 points of  $\alpha(\Gamma)$  lie on a quartic on  $\Pi$ .

Unless the set  $\alpha(\Gamma)$  lie on a quintic curve on  $\Pi$ , we can use Corollary 2.8 to get a nonic curve on  $\Pi$  containing the set  $\alpha(\Gamma)$  but not the point  $\alpha(p)$ , which gives us a nonic hypersurface in  $\mathbb{P}^4$  that we need.

Finally, we suppose that there is a quintic curve  $C_5$  on  $\Pi$  that contains  $\alpha(\Gamma)$ .

The curve  $C_5$  is irreducible. Also, we may assume that it contains the point  $\alpha(p)$  as well. Then the linear system  $\mathcal{D}$  of quintic hypersurfaces in  $H$  passing through  $\Sigma$  has zero-dimensional base locus. Meanwhile, we have the septic surface  $Y = H \cap V_7$  contains all the nodes of  $V_7$ , which may have non-isolated singularities. However, it is irreducible and reduced; otherwise the hypersurface would have more than 35 nodes. Choose a general enough surface  $S'$  in  $\mathcal{D}$ . Then it is smooth in the outside of the base locus of  $\mathcal{D}$  and hence it is normal. Also, the surface  $Y$  gives us a reduced divisor  $D_7 \in |\mathcal{O}_{S'}(7)|$  on  $S'$ . Let  $D_5$  be a divisor in  $|\mathcal{O}_{S'}(5)|$  given by a general member of  $\mathcal{D}$ . We then consider the  $\mathbb{Q}$ -divisor  $D = (1 - \epsilon)D_7 + 2\epsilon D_5$ , where  $\epsilon$  is sufficiently small enough rational number. Then it is easy to check that the support of  $\mathcal{L}(S', D)$  is zero-dimensional and contains  $\Sigma$ . Using Theorem 2.1, we obtain  $H^1(S', \mathcal{I}(S', D) \otimes \mathcal{O}_{S'}(9)) = 0$ . Therefore, there is a divisor in  $|\mathcal{O}_{S'}(9)|$  that contains  $\Gamma$  but not the point  $p$ . Because the sequence

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(9)) \rightarrow H^0(S', \mathcal{O}_{S'}(9)) \rightarrow 0$$

exact, we are done.

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