Halphen pencils on quartic threefolds

Ivan Cheltsov *, Ilya Karzhemanov

University of Edinburgh, Edinburgh, United Kingdom

Received 22 February 2008; accepted 26 August 2009

Communicated by Ludmil Katzarkov

Abstract

For any smooth quartic threefold in \( \mathbb{P}^4 \) we classify pencils on it whose general element is an irreducible surface birational to a surface of Kodaira dimension zero.

© 2009 Elsevier Inc. All rights reserved.

Keywords: Quartic threefold; Birational automorphisms; \( K_3 \) surfaces; Fano varieties

1. Introduction

Let \( X \) be a smooth quartic threefold in \( \mathbb{P}^4 \). The following result is proved in [4].

**Theorem 1.1.** The threefold \( X \) does not contain pencils whose general element is an irreducible surface that is birational to a smooth surface of Kodaira dimension \(-\infty\).

On the other hand, one can easily see that \( X \) contains infinitely many pencils whose general elements are irreducible surfaces of Kodaira dimension zero (cf. [1–3]).

**Definition 1.2.** A Halphen pencil is a one-dimensional linear system whose general element is an irreducible subvariety birational to a smooth variety of Kodaira dimension zero.

The following result is proved in [2].
Theorem 1.3. Suppose that $X$ is general. Then every Halphen pencil on $X$ is cut out by
\[ \lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4, \]
where $l_1$ and $l_2$ are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$.

The assertion of Theorem 1.3 is erroneously proved in [1] without the assumption that the threefold $X$ is general. On the other hand, the following example is constructed in [3].

Example 1.4. Suppose that $X$ is given by the equation
\[ w^3 x + w^2 q_2(x, y, z, t) + w x p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4, \]
where $q_i$ and $p_i$ are forms of degree $i$. Let $\mathcal{P}$ be the pencil on $X$ that is cut out by
\[ \lambda x^2 + \mu (w x + q_2(x, y, z, t)) = 0, \]
where $(\lambda : \mu) \in \mathbb{P}^1$. Then $\mathcal{P}$ is a Halphen pencil if $q_2(0, y, z, t) \neq 0$ by [2, Theorem 2.3].

The purpose of this paper is to prove the following result.

Theorem 1.5. Let $\mathcal{M}$ be a Halphen pencil on the threefold $X$. Then

- either $\mathcal{M}$ is cut out on $X$ by the pencil
  \[ \lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4, \]
  where $l_1$ and $l_2$ are linearly independent linear forms, and $(\lambda : \mu) \in \mathbb{P}^1$,
- or the threefold $X$ can be given by the equation
  \[ w^3 x + w^2 q_2(x, y, z, t) + w x p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4 \]
  such that $q_2(0, y, z, t) \neq 0$, and $\mathcal{M}$ is cut out on the threefold $X$ by the pencil
  \[ \lambda x^2 + \mu (w x + q_2(x, y, z, t)) = 0, \]
  where $q_i$ and $p_i$ are forms of degree $i$, and $(\lambda : \mu) \in \mathbb{P}^1$.

Let $P$ be an arbitrary point of the quartic hypersurface $X \subset \mathbb{P}^4$.

Definition 1.6. The mobility threshold of the threefold $X$ at the point $P$ is the number
\[ \iota(P) = \sup\{ \lambda \in \mathbb{Q} \text{ such that } |n(\pi^*(-K_X) - \lambda E)| \text{ has no fixed components for } n \gg 0 \}, \]
where $\pi : Y \to X$ is the ordinary blow up of $P$, and $E$ is the exceptional divisor of $\pi$.

Arguing as in the proof of Theorem 1.5, we obtain the following result.
Theorem 1.7. The following conditions are equivalent:

- the equality $\iota(P) = 2$ holds,
- the threefold $X$ can be given by the equation
  \[
  w^3 x + w^2 q_2(x, y, z, t) + w x p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,
  \]
  where $q_i$ and $p_i$ are forms of degree $i$ such that $q_2(0, y, z, t) \neq 0$,
  and $P$ is given by the equations $x = y = z = t = 0$.

One can easily check that $2 \geq \iota(P) \geq 1$. Similarly, one can show that

- $\iota(P) = 1 \iff$ the hyperplane section of $X$ that is singular at $P$ is a cone,
- $\iota(P) = 3/2 \iff$ the threefold $X$ contains no lines passing through $P$.

The proof of Theorem 1.5 is completed on board of IL-96-300 Valery Chkalov while flying from Seoul to Moscow. We thank Aeroflot Russian Airlines for good working conditions.

2. Important lemma

Let $S$ be a normal surface, let $O$ be a smooth point of $S$, let $R$ be an effective divisor on the surface $S$, and let $D$ be a linear system on the surface $S$ that has no fixed components.

Lemma 2.1. Let $D_1$ and $D_2$ be general curves in $D$. Then

\[
\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) \leq \text{mult}_O(R) \cdot \text{mult}_O(D_1 \cdot D_1).
\]

Proof. Put $S_0 = S$ and $O_0 = O$. Let us consider the sequence of blow ups

\[
S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0
\]

such that $\pi_1$ is a blow up of the point $O_0$, and $\pi_i$ is a blow up of the point $O_{i-1}$ that is contained in the curve $E_{i-1}$, where $E_{i-1}$ is the exceptional curve of $\pi_{i-1}$, and $i = 2, \ldots, n$.

Let $D_j^i$ be the proper transform of $D_j$ on $S_i$ for $i = 0, \ldots, n$ and $j = 1, 2$. Then

\[
D_j^i \equiv D_j^i = \pi_i^*(D_j^1 E_i) \equiv \pi_i^*(D_j^2 E_i)
\]

for $i = 1, \ldots, n$. Put $d_i = \text{mult}_{O_{i-1}}(D_1^i E_i) = \text{mult}_{O_{i-1}}(D_2^i E_i)$ for $i = 1, \ldots, n$.

Let $R_j^i$ be the proper transform of $R$ on the surface $S_i$ for $i = 0, \ldots, n$. Then

\[
R_j^i \equiv R_j^i = \pi_i^*(R_j^{i-1} E_i)
\]

for $i = 1, \ldots, n$. Put $r_i = \text{mult}_{O_{i-1}}(R_j^{i-1} E_i)$ for $i = 1, \ldots, n$. Then $r_1 = \text{mult}_O(R)$.
We may chose the blow ups \( \pi_1, \ldots, \pi_n \) in a way such that \( D_1^n \cap D_2^n \) is empty in the neighborhood of the exceptional locus of \( \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n \). Then
\[
\text{mult}_O(D_1 \cdot D_2) = \sum_{i=1}^n d_i^2.
\]

We may chose the blow ups \( \pi_1, \ldots, \pi_n \) in a way such that \( D_1^n \cap R^n \) and \( D_2^n \cap R^n \) are empty in the neighborhood of the exceptional locus of \( \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n \). Then
\[
\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i,
\]
where some numbers among \( r_1, \ldots, r_n \) may be zero. Then
\[
\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i \leq \sum_{i=1}^n d_i r_1 \leq \sum_{i=1}^n d_i^2 r_1 = \text{mult}_O(R) \text{mult}_O(D_1 \cdot D_2),
\]
because \( d_i \leq d_i^2 \) and \( r_i \leq r_1 = \text{mult}_O(R) \) for every \( i = 1, \ldots, n \). \( \square \)

The assertion of Lemma 2.1 is a cornerstone of the proof of Theorem 1.5.

3. Curves

Let \( X \) be a smooth quartic threefold in \( \mathbb{P}^4 \), let \( \mathcal{M} \) be a Halphen pencil on \( X \). Then
\[
\mathcal{M} \sim -nK_X,
\]
since \( \text{Pic}(X) = \mathbb{Z}K_X \). Put \( \mu = 1/n \). Then

- the log pair \( (X, \mu \mathcal{M}) \) is canonical by [3, Theorem A],
- the log pair \( (X, \mu \mathcal{M}) \) is not terminal by [2, Theorem 2.1].

Let \( \mathcal{CS}(X, \mu \mathcal{M}) \) be the set of non-terminal centers of \( (X, \mu \mathcal{M}) \) (see [2]). Then
\[
\mathcal{CS}(X, \mu \mathcal{M}) \neq \emptyset,
\]
because \( (X, \mu \mathcal{M}) \) is not terminal. Let \( M_1 \) and \( M_2 \) be two general surfaces in \( \mathcal{M} \).

**Lemma 3.1.** Suppose that \( \mathcal{CS}(X, \mu \mathcal{M}) \) contains a point \( P \in X \). Then
\[
\text{mult}_P(M) = n \text{mult}_P(T) = 2n,
\]
where \( M \) is any surface in \( \mathcal{M} \), and \( T \) is the surface in \( |-K_X| \) that is singular at \( P \).
Proof. It follows from [5, Proposition 1] that the inequality
\[ \text{mult}_P(M_1 \cdot M_2) \geq 4n^2 \]
holds. Let \( H \) be a general surface in \( |-K_X| \) such that \( P \in H \). Then
\[ 4n^2 = H \cdot M_1 \cdot M_2 \geq \text{mult}_P(M_1 \cdot M_2) \geq 4n^2, \]
which gives \( (M_1 \cdot M_2)_P = 4n^2 \). Arguing as in the proof of [5, Proposition 1], we see that
\[ \text{mult}_P(M_1) = \text{mult}_P(M_2) = 2n, \]
because \( (M_1 \cdot M_2)_P = 4n^2 \). Similarly, we see that
\[ 4n = H \cdot T \cdot M_1 \geq \text{mult}_P(T) \text{mult}_P(M_1) = 2n \text{mult}_P(T) \geq 4n, \]
which implies that \( \text{mult}_P(T) = 2 \). Finally, we also have
\[ 4n^2 = H \cdot M \cdot M_1 \geq \text{mult}_P(M) \text{mult}_P(M_1) = 2n \text{mult}_P(M) \geq 4n^2, \]
where \( M \) is any surface in \( \mathcal{M} \), which completes the proof. \( \square \)

Lemma 3.2. Suppose that \( \mathcal{C}(X, \mu \mathcal{M}) \) contains a point \( P \in X \). Then
\[ M_1 \cap M_2 = \bigcup_{i=1}^{r} L_i, \]
where \( L_1, \ldots, L_r \) are lines on the threefold \( X \) that pass through the point \( P \).

Proof. Let \( H \) be a general surface in \( |-K_X| \) such that \( P \in H \). Then
\[ 4n^2 = H \cdot M_1 \cdot M_2 = \text{mult}_P(M_1 \cdot M_2) = 4n^2 \]
by Lemma 3.1. Then \( \text{Supp}(M_1 \cdot M_2) \) consists of lines on \( X \) that pass through \( P \). \( \square \)

Lemma 3.3. Suppose that \( \mathcal{C}(X, \mu \mathcal{M}) \) contains a point \( P \in X \). Then
\[ n/3 \leq \text{mult}_L(\mathcal{M}) \leq n/2 \]
for every line \( L \subset X \) that passes through the point \( P \).

Proof. Let \( D \) be a general hyperplane section of \( X \) through \( L \). Then we have
\[ M|_D = \text{mult}_L(\mathcal{M})L + \Delta, \]
where \( M \) is a general surface in \( \mathcal{M} \) and \( \Delta \) is an effective divisor such that
\[ \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(\mathcal{M}). \]
On the surface $D$ we have $L \cdot L = -2$. Then
\[ n = (\text{mult}_L(M)L + \Delta) \cdot L = -2 \text{mult}_L(M) + \Delta \cdot L \]
on the surface $D$. But $\Delta \cdot L \geq \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(M)$. Thus, we get
\[ n \geq -2 \text{mult}_L(M) + \text{mult}_P(\Delta) \geq 2n - 3 \text{mult}_L(M), \]
which implies that $n/3 \leq \text{mult}_L(M)$.

Let $T$ be the surface in $|−K_X|$ that is singular at $P$. Then $T \cdot D$ is reduced and
\[ T \cdot D = L + Z, \]
where $Z$ is an irreducible plane cubic curve such that $P \in Z$. Then
\[ 3n = (\text{mult}_L(M)L + \Delta) \cdot Z = 3 \text{mult}_L(M) + \Delta \cdot Z \]
on the surface $D$. The set $\Delta \cap Z$ is finite by Lemma 3.2. In particular, we have
\[ \Delta \cdot Z \geq \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(M), \]
because $\text{Supp}(\Delta)$ does not contain the curve $Z$. Thus, we get
\[ 3n \geq 3 \text{mult}_L(M) + \text{mult}_P(\Delta) \geq 2n + 2 \text{mult}_L(M), \]
which implies that $\text{mult}_L(M) \leq n/2$. \qed

In the rest of this section we prove the following result.

**Proposition 3.4.** Suppose that $\mathcal{C}_S(X, \mu M)$ contains a curve. Then $n = 1$.

Suppose that the set $\mathcal{C}_S(X, \mu M)$ contains a curve $Z$. Then $\mathcal{C}_S(X, \mu M)$ does not contain points of the threefold $X$ by Lemmas 3.2 and 3.3. Then
\[ \text{mult}_Z(M) = n, \quad (3.5) \]
because the log pair $(X, \mu M)$ is canonical. Then $\text{deg}(Z) \leq 4$ by [2, Lemma 2.1].

**Lemma 3.6.** Suppose that $\text{deg}(Z) = 1$. Then $n = 1$.

**Proof.** Let $\pi : V \to X$ be the blow up of $X$ along the line $Z$. Let $B$ be the proper transform of the pencil $M$ on the threefold $V$, and let $B$ be a general surface in $B$. Then
\[ B \sim −nK_V \quad (3.7) \]
by (3.5). There is a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\eta} & \mathbb{P}^2 \\
\downarrow{\pi} & & \\
X & \xrightarrow{\psi} & \mathbb{P}^2,
\end{array}
\]

where \(\psi\) is the projection from the line \(Z\) and \(\eta\) is the morphism induced by the complete linear system \(|-K_V|\). It follows from (3.7) that \(B\) is the pull-back of a pencil \(P\) on \(\mathbb{P}^2\) by \(\eta\).

We see that the base locus of \(B\) is contained in the union of fibers of \(\eta\).

The set \(\mathcal{CS}(V, \mu B)\) is not empty by [2, Theorem 2.1]. Thus, it easily follows from (3.5) that the set \(\mathcal{CS}(V, \mu B)\) does not contain points because \(\mathcal{CS}(X, \mu M)\) contains no points.

We see that there is an irreducible curve \(L \subset V\) such that

\[
\text{mult}_L(B) = n
\]

and \(\eta(L)\) is a point \(Q \in \mathbb{P}^2\). Let \(C\) be a general curve in \(\mathcal{P}\). Then \(\text{mult}_Q(C) = n\). But

\[
C \sim \mathcal{O}_{\mathbb{P}^2}(n)
\]

by (3.7). Thus, we see that \(n = 1\), because general surface in \(\mathcal{M}\) is irreducible. \(\square\)

Thus, we may assume that the set \(\mathcal{CS}(X, \mu M)\) does not contain lines.

**Lemma 3.8.** The curve \(Z \subset \mathbb{P}^4\) is contained in a plane.

**Proof.** Suppose that \(Z\) is not contained in any plane in \(\mathbb{P}^4\). Let us show that this assumption leads to a contradiction. It follows from [2, Lemma 2.1] that \(\deg(Z) \leq 4\). Then

\[
\deg(Z) \in \{3, 4\},
\]

and \(Z\) is smooth if \(\deg(Z) = 3\). If \(\deg(Z) = 4\), then \(Z\) may have at most one double point.

Suppose that \(Z\) is smooth. Let \(\alpha : U \rightarrow X\) be the blow up at \(Z\), and let \(F\) be the exceptional divisor of the morphism \(\alpha\). Then the base locus of the linear system

\[
\left|\alpha^*(-\deg(Z)K_X) - F\right|
\]

does not contain any curve. Let \(D_1\) and \(D_2\) be the proper transforms on \(U\) of two sufficiently general surfaces in the linear system \(\mathcal{M}\). Then it follows from (3.5) that

\[
\left(\alpha^*(-\deg(Z)K_X) - F\right) \cdot D_1 \cdot D_2 = n^2\left(\alpha^*(-\deg(Z)K_X) - F\right) \cdot \left(\alpha^*(-K_X) - F\right)^2 > 0,
\]

because the cycle \(D_1 \cdot D_2\) is effective. On the other hand, we have

\[
\left(\alpha^*(-\deg(Z)K_X) - F\right) \cdot \left(\alpha^*(-K_X) - F\right)^2 = (3\deg(Z) - (\deg(Z))^2 - 2) < 0,
\]

which is a contradiction. Thus, the curve \(Z\) is not smooth.
Thus, we see that $Z$ is a quartic curve with a double point $O$.

Let $\beta : W \rightarrow X$ be the composition of the blow up of the point $O$ with the blow up of the proper transform of the curve $Z$. Let $G$ and $E$ be the exceptional surfaces of the birational morphism $\beta$ such that $\beta(E) = Z$ and $\beta(G) = O$. Then the base locus of the linear system

$$|\beta^*(-4K_X) - E - 2G|$$

does not contain any curve. Let $R_1$ and $R_2$ be the proper transforms on $W$ of two sufficiently general surfaces in $M$. Put $m = \text{mult}_O(M)$. Then it follows from (3.5) that

$$(\beta^*(-4K_X) - E - 2G) \cdot R_1 \cdot R_2 = (\beta^*(-4K_X) - E - 2G) \cdot (\beta^*(-nK_X) - nE - mG)^2 \geq 0,$$

and $m < 2n$, because the set $CS(X, \mu M)$ does not contain points. Then

$$(\beta^*(-4K_X) - E - 2G) \cdot (\beta^*(-nK_X) - nE - mG)^2 = -8n^2 + 6mn - m^2 < 0,$$

which is a contradiction. $\square$

If $\deg(Z) = 4$, then $n = 1$ by Lemma 3.8 and [2, Theorem 2.2].

**Lemma 3.9.** Suppose that $\deg(Z) = 3$. Then $n = 1$.

**Proof.** Let $P$ be the pencil in $|-K_X|$ that contains all hyperplane sections of $X$ that pass through the curve $Z$. Then the base locus of $P$ consists of the curve $Z$ and a line $L \subset X$.

Let $D$ be a sufficiently general surface in the pencil $P$, and let $M$ be a sufficiently general surface in the pencil $M$. Then $D$ is a smooth surface, and

$$M|_D = nZ + \text{mult}_L(M)L + B \equiv nZ + nL,$$  \hspace{1cm} (3.10)

where $B$ is a curve whose support does not contain neither $Z$ nor $L$.

On the surface $D$, we have $Z \cdot L = 3$ and $L \cdot L = -2$. Intersecting (3.10) with $L$, we get

$$n = (nZ + nL) \cdot L = 3n - 2\text{mult}_L(M) + B \cdot L \geq 3n - 2\text{mult}_L(M),$$

which easily implies that $\text{mult}_L(M) \geq n$. But the inequality $\text{mult}_L(M) \geq n$ is impossible, because we assumed that $CS(X, \mu M)$ contains no lines. $\square$

**Lemma 3.11.** Suppose that $\deg(Z) = 2$. Then $n = 1$.

**Proof.** Let $X \rightarrow U$ be the blow up of the curve $Z$. Then $|-K_U|$ is a pencil, whose base locus consists of a smooth irreducible curve $L \subset U$.

Let $D$ be a general surface in $|-K_U|$. Then $D$ is a smooth surface.

Let $B$ be the proper transform of the pencil $M$ on the threefold $U$. Then

$$-nK_U|_D \equiv B|_D \equiv nL,$$
where $B$ is a general surface in $B$. But $L^2 = -2$ on the surface $D$. Then

$$L \in \mathcal{CS}(U, \mu B)$$

which implies that $B = |-K_U|$ by [2, Theorem 2.2]. Then $n = 1$. □

The assertion of Proposition 3.4 is proved.

4. Points

Let $X$ be a smooth quartic threefold in $\mathbb{P}^4$, let $\mathcal{M}$ be a Halphen pencil on $X$. Then

$$\mathcal{M} \sim -nK_X,$$

since $\text{Pic}(X) = \mathbb{Z}K_X$. Put $\mu = 1/n$. Then

- the log pair $(X, \mu \mathcal{M})$ is canonical by [3, Theorem A],
- the log pair $(X, \mu \mathcal{M})$ is not terminal by [2, Theorem 2.1].

Remark 4.1. To prove Theorem 1.5, it is enough to show that $X$ can be given by

$$w^3 x + w^2 q_2(x, y, z, t) + w x p_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where $q_i$ and $p_i$ are homogeneous polynomials of degree $i \geq 2$ such that $q_2(0, y, z, t) \neq 0$.

Let $\mathcal{CS}(X, \mu \mathcal{M})$ be the set of non-terminal centers of $(X, \mu \mathcal{M})$ (see [2]). Then

$$\mathcal{CS}(X, \mu \mathcal{M}) \neq \emptyset,$$

because $(X, \mu \mathcal{M})$ is not terminal. Suppose that $n \neq 1$. There is a point $P \in X$ such that

$$P \in \mathcal{CS}(X, \mu \mathcal{M})$$

by Proposition 3.4. It follows from Lemmas 3.1, 3.2 and 3.3 that

- the equality $\text{mult}_P(T) = 2$ holds, where $T \in |-K_X|$ such that $\text{mult}_P(T) \geq 2$,
- there are finitely many distinct lines $L_1, \ldots, L_r \subset X$ containing $P \in X$,
- the equality $\text{mult}_P(\mathcal{M}) = 2n$ holds, and

$$n/3 \leq \text{mult}_{L_i}(\mathcal{M}) \leq n/2,$$

where $\mathcal{M}$ is a general surface in the pencil $\mathcal{M}$,
- the base locus of the pencil $\mathcal{M}$ consists of the lines $L_1, \ldots, L_r$, and

$$\text{mult}_P(M_1 \cdot M_2) = 4n^2,$$

where $M_1$ and $M_2$ are sufficiently general surfaces in $\mathcal{M}$. 

Please cite this article in press as: I. Cheltsov, I. Karzhemanov, Halphen pencils on quartic threefolds, Adv. Math. (2009), doi:10.1016/j.aim.2009.08.020
Lemma 4.2. The equality $\text{CS}(X, \mu M) = \{P\}$ holds.

Proof. The set $\mathbb{C}S(X, \mu M)$ does not contain curves by Proposition 3.4. Suppose that $\mathbb{C}S(X, \mu M)$ contains a point $Q \in X$ such that $Q \neq P$. Then $r = 1$. Let $D$ be a general hyperplane section of $X$ that passes through $L_1$. Then

$$M|_D = \text{mult}_{L_1}(M)L_1 + \Delta,$$

where $M$ is a general surface in $\mathcal{M}$ and $\Delta$ is an effective divisor such that

$$\text{mult}_P(\Delta) \geq 2n - \text{mult}_{L_1}(M) \leq \text{mult}_Q(\Delta).$$

On the surface $D$, we have $L_1^2 = -2$. Then

$$n = (\text{mult}_{L_1}(M)L_1 + \Delta) \cdot L_1 = -2 \text{mult}_{L_1}(M) + \Delta \cdot L$$

$$\geq -2 \text{mult}_{L_1}(M) + 2(2n - \text{mult}_{L_1}(M)),$$

which gives $\text{mult}_{L_1}(M) \geq 3n/4$. But $\text{mult}_{L_1}(M) \leq n/2$ by Lemma 3.3. □

The quartic threefold $X$ can be given by the equation

$$w^3x + w^2q_2(x, y, z, t) + wq_3(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where $q_i$ is a homogeneous polynomial of degree $i \geq 2$.

Remark 4.3. The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = q_2(x, y, z, t) = q_3(x, y, z, t) = q_4(x, y, z, t) = 0,$$

the surface $T$ is cut out on $X$ by $x = 0$, and $\text{mult}_P(T) = 2 \iff q_2(0, y, z, t) \neq 0$.

Let $\pi : V \to X$ be the blow up of the point $P$, let $E$ be the $\pi$-exceptional divisor. Then

$$\mathcal{B} \equiv \pi^*(-nK_X) - 2nE \equiv -nK_V,$$

where $\mathcal{B}$ is the proper transform of the pencil $\mathcal{M}$ on the threefold $V$.

Remark 4.4. The pencil $\mathcal{B}$ has no base curves in $E$, because

$$\text{mult}_P(M_1 \cdot M_2) = \text{mult}_P(M_1)\text{mult}_P(M_2).$$

Let $\bar{L}_i$ be the proper transform of the line $L_i$ on the threefold $V$ for $i = 1, \ldots, r$. Then

$$B_1 \cdot B_2 = \sum_{i=1}^r \text{mult}_{\bar{L}_i}(B_1 \cdot B_2)\bar{L}_i,$$

where $B_1$ and $B_2$ are proper transforms of $M_1$ and $M_2$ on the threefold $V$, respectively.
Lemma 4.5. Let $Z$ be an irreducible curve on $X$ such that $Z \notin \{L_1, \ldots, Z_r\}$. Then

$$\deg(Z) \geq 2 \text{mult}_P(Z),$$
and the equality $\deg(Z) = 2 \text{mult}_P(Z)$ implies that

$$\tilde{Z} \cap \left( \bigcup_{i=1}^r \tilde{L}_i \right) = \emptyset,$$

where $\tilde{Z}$ is a proper transform of the curve $Z$ on the threefold $V$.

**Proof.** The curve $\tilde{Z}$ is not contained in the base locus of the pencil $B$. Then

$$0 \leq B_i \cdot \tilde{Z} \leq n(\deg(Z) - 2 \text{mult}_P(Z)),$$

which implies the required assertions. $\square$

To conclude the proof of Theorem 1.5, it is enough to show that

$$q_3(x, y, z, t) = x^2 + q_2(x, y, z, t) + q_2(x, y, z, t)p_1(x, y, z, t),$$

where $p_1$ and $p_2$ are some homogeneous polynomials of degree 1 and 2, respectively.

### 5. Good points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$$

is reduced and irreducible. In this section we prove the following result.

**Proposition 5.1.** The polynomial $q_3(0, y, z, t)$ is divisible by $q_2(0, y, z, t)$.

Let us prove Proposition 5.1. Suppose that $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$. Let $\mathcal{R}$ be the linear system on the threefold $X$ that is cut out by quadrics

$$xh_1(x, y, z, t) + \lambda(\omega x + q_2(x, y, z, t)) = 0,$$

where $h_1$ is an arbitrary linear form and $\lambda \in \mathbb{C}$. Then $\mathcal{R}$ does not have fixed components.

**Lemma 5.2.** Let $R_1$ and $R_2$ be general surfaces in the linear system $\mathcal{R}$. Then

$$\sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2) \leq 6.$$
Proof. We may assume that $R_1$ is cut out by the equation

$$wx + q_2(x, y, z, t) = 0,$$

and $R_2$ is cut out by $xh_1(x, y, z, t) = 0$, where $h_1$ is sufficiently general. Then

$$\text{mult}_{L_i}(R_1 \cdot R_2) = \text{mult}_{L_i}(R_1 \cdot T).$$

Put $m_i = \text{mult}_{L_i}(R_1 \cdot T)$. Then

$$R_1 \cdot T = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and $\Delta$ is a cycle, whose support contains no lines passing through $P$.

Let $\bar{R}_1$ and $\bar{T}$ be the proper transforms of $R_1$ and $T$ on $V$, respectively. Then

$$\bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i \bar{L}_i + \Omega,$$

where $\Omega$ is an effective cycle, whose support contains no lines passing through $P$.

The support of the cycle $\Omega$ does not contain curves that are contained in the exceptional divisor $E$, because $q_3(0, y, z, t)$ is not divisible by $q_2(0, y, z, t)$ by our assumption. Then

$$6 = E \cdot \bar{R}_1 \cdot \bar{T} = \sum_{i=1}^{r} m_i (E \cdot \bar{L}_i) + E \cdot \Omega \geq \sum_{i=1}^{r} m_i (E \cdot \bar{L}_i) = \sum_{i=1}^{r} m_i,$$

which is exactly what we want. □

Let $M$ and $R$ be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and $\Delta$ is a cycle, whose support contains no lines passing through $P$.

Lemma 5.3. The cycle $\Delta$ is not trivial.

Proof. Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil. □

We have $\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i$. On the other hand, the inequality

$$\text{mult}_{P}(\Delta) \geq 6n - \sum_{i=1}^{r} m_i$$
holds, because \( \text{mult}_P(M) = 2n \) and \( \text{mult}_P(R) \geq 3 \). It follows from Lemma 4.5 that
\[
\deg(\Delta) = 8n - \sum_{i=1}^{r} m_i \geq 2 \text{mult}_P(\Delta) \geq 2 \left( 6n - \sum_{i=1}^{r} m_i \right),
\]
which implies that \( \sum_{i=1}^{r} m_i \geq 4n \). But it follows from Lemmas 2.1 and 3.3 that
\[
m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2
\]
for every \( i = 1, \ldots, r \), where \( R_1 \) and \( R_2 \) are general surfaces in \( \mathcal{R} \). Then
\[
\sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2)n/2 \leq 3n
\]
by Lemma 5.2, which is a contradiction.

The assertion of Proposition 5.1 is proved.

6. Bad points

Let us use the assumptions and notation of Section 4. Suppose that the conic
\[
q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2
\]
is reduced and reducible. Therefore, we have
\[
q_2(x, y, z, t) = (\alpha_1 y + \beta_1 z + \gamma_1 t)(\alpha_2 y + \beta_2 z + \gamma_2 t) + xp_1(x, y, z, t)
\]
where \( p_1(x, y, z, t) \) is a linear form, and \((\alpha_1 : \beta_1 : \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2 : \beta_2 : \gamma_2)\).

**Proposition 6.1.** The polynomial \( q_3(0, y, z, t) \) is divisible by \( q_2(0, y, z, t) \).

Suppose that \( q_3(0, y, z, t) \) is not divisible by \( q_2(0, y, z, t) \). Then without loss of generality, we may assume that \( q_3(0, y, z, t) \) is not divisible by \( \alpha_1 y + \beta_1 z + \gamma_1 t \).

Let \( Z \) be the curve in \( X \) that is cut out by the equations
\[
x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0.
\]

**Remark 6.2.** The equality \( \text{mult}_P(Z) = 3 \) holds, but \( Z \) is not necessarily reduced.

Hence, it follows from Lemma 4.5 that \( \text{Supp}(Z) \) contains a line among \( L_1, \ldots, L_r \).

**Lemma 6.3.** The support of the curve \( Z \) does not contain an irreducible conic.

**Proof.** Suppose that \( \text{Supp}(Z) \) contains an irreducible conic \( C \). Then
\[
Z = C + L_i + L_j
\]
for some $i \in \{1, \ldots, r\} \ni j$. Then $i = j$, because otherwise the set
\[(C \cap L_i) \cup (C \cap L_j)\]
contains a point that is different from $P$, which is impossible by Lemma 4.5. We see that
\[Z = C + 2L_i,
\]
and it follows from Lemma 4.5 that $C \cap L_i = P$. Then $C$ is tangent to $L_i$ at the point $P$.
Let $\tilde{C}$ be a proper transform of the curve $C$ on the threefold $V$. Then
\[\tilde{C} \cap \tilde{L}_i \neq \emptyset,
\]
which is impossible by Lemma 4.5. The assertion is proved. □

**Lemma 6.4.** The support of the curve $Z$ consists of lines.

**Proof.** Suppose that $\text{Supp}(Z)$ does not consist of lines. It follows from Lemma 6.3 that
\[Z = L_i + C,
\]
where $C$ is an irreducible cubic curve. But $\text{mult}_P(Z) = 3$. Then
\[\text{mult}_P(C) = 2,
\]
which is impossible by Lemma 4.5. □

We may assume that there is a line $L \subset X$ such that $P \notin P$ and
\[Z = a_1L_1 + \cdots + a_kL_k + L,
\]
where $a_1, a_2, a_3 \in \mathbb{N}$ are such that $a_1 \geq a_2 \geq a_3$ and $\sum_{i=1}^k a_i = 3$.

**Remark 6.5.** We have $L_i \neq L_j$ whenever $i \neq j$.

Let $H$ be a sufficiently general surface of $X$ that is cut out by the equation
\[\lambda x + \mu(\alpha_1 y + \beta_1 z + \gamma_1 t) = 0,
\]
where $(\lambda: \mu) \in \mathbb{P}^1$. Then $H$ has at most isolated singularities.

**Remark 6.6.** The surface $H$ is smooth at the points $P$ and $L \cap L_i$, where $i = 1, \ldots, k$.

Let $\tilde{H}$ and $\tilde{L}$ be the proper transforms of $H$ and $L$ on the threefold $V$, respectively.

**Lemma 6.7.** The inequality $k \neq 3$ holds.
Proof. Suppose that the equality $k = 3$ holds. Then $H$ is smooth. Put

$$B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega,$$

where $B$ is a general surface in $\mathcal{B}$, and $\Omega$ is an effective divisor on $\bar{H}$ whose support does not contain any of the curves $\bar{L}_1$, $\bar{L}_2$ and $\bar{L}_3$. Then

$$\bar{L} \notin \text{Supp}(\Omega) \cap \bar{H} \cap E,$$

because the base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Then

$$n = \bar{L} \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega) = \sum_{i=1}^{3} m_i + \bar{L} \cdot \Omega \geq \sum_{i=1}^{3} m_i,$$

which implies that $\sum_{i=1}^{3} m_i \leq n$. On the other hand, we have

$$-n = \bar{L}_i \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega) = -3m_i + \bar{L}_i \cdot \Omega \geq -3m_i,$$

which implies that $m_i \geq n/3$. Thus, we have $m_1 = m_2 = m_3 = n/3$ and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,$$

which implies that $\text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \text{Supp}(\Omega) \cap \bar{L}_3 = \emptyset$.

Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that

$$B'|_{\bar{H}} = \frac{n}{3}(\bar{L}_1 + \bar{L}_2 + \bar{L}_3) + \Omega',$$

where $\Omega'$ is an effective divisor on the surface $\bar{H}$ such that

$$\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \text{Supp}(\Omega') \cap \bar{L}_3 = \emptyset.$$

One can easily check that $\Omega \cdot \Omega' = n^2 \neq 0$. Then

$$\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,$$

because $|\text{Supp}(\Omega) \cap \text{Supp}(\Omega')| < +\infty$ due to generality of the surfaces $B$ and $B'$.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\bar{L}_1, \ldots, \bar{L}_r$. Hence, we have

$$\text{Supp}(\mathcal{B}) \cap \text{Supp}(B') = \bigcup_{i=1}^{r} \bar{L}_i,$$

but $\bar{L}_i \cap \bar{H} = \emptyset$ whenever $i \notin \{1, 2, 3\}$. Hence, we have

$$\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup (\text{Supp}(\Omega) \cap \text{Supp}(\Omega')) = \text{Supp}(B) \cap \text{Supp}(B') \cap \bar{H} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3,$$
which implies that \( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \). In particular, we see that
\[
\text{Supp}(\Omega) \cap (\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3) \neq \emptyset,
\]
because \( \text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset \). But \( \text{Supp}(\Omega) \cap \bar{L}_i = \emptyset \) for \( i = 1, 2, 3 \). \( \Box \)

**Lemma 6.8.** The inequality \( k \neq 2 \) holds.

**Proof.** Suppose that the equality \( k = 2 \) holds. Then \( Z = 2L_1 + L_2 + L \). Put
\[
B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega,
\]
where \( B \) is a general surface in \( B \), and \( \Omega \) is an effective divisor on \( \bar{H} \) whose support does not contain the curves \( \bar{L}_1 \) and \( \bar{L}_2 \). Then \( \bar{L} \not\subseteq \text{Supp}(\Omega) \not\subseteq \bar{H} \cap E \) and
\[
n = \bar{L} \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = m_1 + m_2 + \bar{L} \cdot \Omega \geq m_1 + m_2,
\]
which implies that \( m_1 + m_2 \leq n \). On the other hand, we have
\[
\bar{T}|_{\bar{H}} = 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}} \equiv (\pi^*(\omega X_K) - 2E)|_{\bar{H}},
\]
where \( \bar{T} \) is the proper transform of the surface \( T \) on the threefold \( V \). Then
\[
-1 = \bar{L}_1 \cdot (2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}}) = 2(\bar{L}_1 \cdot \bar{L}_1) + 2,
\]
which implies that \( \bar{L}_1 \cdot \bar{L}_1 = -3/2 \) on the surface \( \bar{H} \). Then
\[
-n = \bar{L}_1 \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = -3m_1/2 + \bar{L}_1 \cdot \Omega \geq -3m_1/2,
\]
which gives \( m_1 \geq 2n/3 \). Similarly, we see that \( \bar{L}_2 \cdot \bar{L}_2 = -3 \) on the surface \( \bar{H} \). Then
\[
-n = \bar{L}_2 \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = -3m_2 + \bar{L}_2 \cdot \Omega \geq -3m_2,
\]
which implies that \( m_2 \leq n/3 \). Thus, we have \( m_1 = 2m_2 = 2n/3 \) and
\[
\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = 0,
\]
which implies that \( \text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \emptyset \).

Let \( B' \) be another general surface in \( B \). Arguing as above, we see that
\[
B'|_{\bar{H}} = \frac{2n}{3} \bar{L}_1 + \frac{n}{3} \bar{L}_2 + \Omega',
\]
where \( \Omega' \) is an effective divisor on \( \bar{H} \) whose support does not contain \( \bar{L}_1 \) and \( \bar{L}_2 \) such that
\[
\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \emptyset,
\]
which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that
\[
\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,
\]
and arguing as in the proof of Lemma 6.7 we obtain a contradiction. \(\square\)

It follows from Lemmas 6.7 and 6.8 that $Z = 3L_1 + L$. Put
\[
B|_{\tilde{H}} = m_1 \tilde{L}_1 + \Omega,
\]
where $B$ is a general surface in $\mathcal{B}$, and $\Omega$ is a curve such that $\tilde{L}_1 \not\in \text{Supp}(\Omega)$. Then
\[
\tilde{L} \not\in \text{Supp}(\Omega) \not\in \tilde{H} \cap E,
\]
because the base locus of $\mathcal{B}$ consists of the curves $\tilde{L}_1, \ldots, \tilde{L}_r$. Then
\[
\begin{align*}
    n &= \tilde{L} \cdot (m_1 \tilde{L}_1 + \Omega) = m_1 + \tilde{L} \cdot \Omega \\ &= m_1 \leq n,
\end{align*}
\]
which implies that $m_1 \leq n$. On the other hand, we have
\[
\tilde{T}|_{\tilde{H}} = 3\tilde{L}_1 + \tilde{L} + E|_{\tilde{H}} \equiv (\pi^*(-K_X) - 2E)|_{\tilde{H}},
\]
where $\tilde{T}$ is the proper transform of the surface $T$ on the threefold $V$. Then
\[
-1 = \tilde{L}_1 \cdot (3\tilde{L}_1 + \tilde{L} + E|_{\tilde{H}}) = 3\tilde{L}_1 \cdot \tilde{L}_1 + 2,
\]
which implies that $\tilde{L}_1 \cdot \tilde{L}_1 = -1$ on the surface $\tilde{H}$. Then
\[
-n = \tilde{L}_1 \cdot (m_1 \tilde{L}_1 + \Omega) = -m_1 + L_1 \cdot \Omega \geq -m_1,
\]
which gives $m_1 \geq n$. Thus, we have $m_1 = n$ and $\Omega \cdot \tilde{L} = \Omega \cdot \tilde{L}_1 = 0$. Then $\text{Supp}(\Omega) \cap \tilde{L}_1 = \emptyset$.

Let $B'$ be another general surface in $\mathcal{B}$. Arguing as above, we see that
\[
B'|_{\tilde{H}} = n\tilde{L}_1 + \Omega',
\]
where $\Omega'$ is an effective divisor on $\tilde{H}$ whose support does not contain $\tilde{L}_1$ such that
\[
\text{Supp}(\Omega') \cap \tilde{L}_1 = \emptyset,
\]
which implies that $\Omega \cdot \Omega' = n^2$. In particular, we see that $\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset$.

The base locus of the pencil $\mathcal{B}$ consists of the curves $\tilde{L}_1, \ldots, \tilde{L}_r$. Hence, we have
\[
\text{Supp}(\mathcal{B}) \cap \text{Supp}(\mathcal{B}') = \bigcup_{i=1}^{r} \tilde{L}_i,
\]
but $\tilde{L}_i \cap \tilde{H} = \emptyset$ whenever $\tilde{L}_i \neq \tilde{L}_1$. Then $\text{Supp}(\Omega) \cap \tilde{L}_1 = \emptyset$, because
\[
\tilde{L}_1 \cup (\text{Supp}(\Omega) \cap \text{Supp}(\Omega')) = \text{Supp}(\mathcal{B}) \cap \text{Supp}(\mathcal{B}') \cap \tilde{H} = \tilde{L}_1,
\]
which is a contradiction. The assertion of Proposition 6.1 is proved.
7. Very bad points

Let us use the assumptions and notation of Section 4. Suppose that \( q_2 = y^2 \).
The proof of Proposition 6.1 implies that \( q_3(0, y, z, t) \) is divisible by \( y \). Then

\[
q_3 = yf_2(z, t) + xh_2(z, t) + x^2a_1(x, y, z, t) + xyb_1(x, y, z, t) + y^2c_1(y, z, t)
\]

where \( a_1, b_1, c_1 \) are linear forms, \( f_2 \) and \( h_2 \) are homogeneous polynomials of degree two.

**Proposition 7.1.** The equality \( f_2(z, t) = 0 \) holds.

Let us prove Proposition 7.1 by reductio ad absurdum. Suppose that \( f_2(z, t) \neq 0 \).

**Remark 7.2.** By choosing suitable coordinates, we may assume that \( f_2 = zt \) or \( f_2 = z^2 \).

We must use smoothness of the threefold \( X \) by analyzing the shape of \( q_4 \). We have

\[
q_4 = f_4(z, t) + xu_3(z, t) + yv_3(z, t) + x^2a_2(x, y, z, t) + xyb_2(x, y, z, t) + y^2c_2(y, z, t),
\]

where \( a_2, b_2, c_2 \) are homogeneous polynomials of degree two, \( u_3 \) and \( v_3 \) are homogeneous polynomials of degree three, and \( f_4 \) is a homogeneous polynomial of degree four.

**Lemma 7.3.** Suppose that \( f_2(z, t) = zt \) and

\[
f_4(z, t) = t^2g_2(z, t)
\]

for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(z, 0) \neq 0 \).

**Proof.** Suppose that \( v_3(z, 0) = 0 \). The surface \( T \) is given by the equation

\[
w^2y^2 + yzt + y^2c_1(x, y, z, t) + t^2g_2(z, t) + yv_3(z, t) + y^2c_2(x, y, z, t) = 0
\]

\[
\subset \text{Proj}(\mathbb{C}[y, z, t, w]) \cong \mathbb{P}^3
\]

because \( T \) is cut out on \( X \) by the equation \( x = 0 \). Then \( T \) has non-isolated singularity along the line \( x = y = t = 0 \), which is impossible because \( X \) is smooth. \( \square \)

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

**Corollary 7.4.** Suppose that \( f_2(z, t) = zt \) and

\[
f_4(z, t) = z^2g_2(z, t)
\]

for some \( g_2(z, t) \in \mathbb{C}[z, t] \). Then \( v_3(0, t) \neq 0 \).

**Lemma 7.5.** Suppose that \( f_2(z, t) = zt \). Then \( f_4(0, t) = f_4(z, 0) = 0 \).
Proof. We may assume that $f_4(z, 0) \neq 0$. Let $\mathcal{H}$ be the linear system on $X$ that is cut out by

$$\lambda x + \mu y + \nu t = 0,$$

where $(\lambda : \mu : \nu) \in \mathbb{P}^2$. Then the base locus of $\mathcal{H}$ consists of the point $P$.

Let $\mathcal{R}$ be the proper transform of $\mathcal{H}$ on the threefold $V$. Then the base locus of $\mathcal{R}$ consists of a single point that is not contained in any of the curves $\bar{L}_1, \ldots, \bar{L}_r$.

The linear system $\mathcal{R}|_B$ has no base points, where $B$ is a general surface in $\mathcal{B}$. But

$$R \cdot R \cdot B = 2n > 0,$$

where $R$ is a general surface in $\mathcal{R}$. Then $\mathcal{R}|_B$ is not composed from a pencil, which implies that the curve $R \cdot B$ is irreducible and reduced by the Bertini theorem.

Let $H$ and $M$ be general surfaces in $\mathcal{H}$ and $\mathcal{M}$, respectively. Then $M \cdot H$ is irreducible and reduced. Thus, the linear system $\mathcal{M}|_H$ is a pencil.

The surface $H$ contains no lines passing through $P$, and $H$ can be given by

$$w^3 x + w^2 y^2 + w(y^2 l_1(x, y, z) + x l_2(x, y, z)) + l_4(x, y, z) = 0$$

$$\subset \text{Proj}(\mathbb{C}[x, y, z, w]) \cong \mathbb{P}^3,$$

where $l_i(x, y, z)$ is a homogeneous polynomial of degree $i$.

Arguing as in Example 1.4, we see that there is a pencil $Q$ on the surface $H$ such that

$$Q \sim \mathcal{O}_{\mathbb{P}^3}(2)|_H,$$

general curve in $Q$ is irreducible, and $\text{mult}_P(Q) = 4$. Arguing as in the proof of Lemma 3.1, we see that $\mathcal{M}|_H = Q$ by [2, Theorem 2.2]. Let $M$ be a general surface in $\mathcal{M}$. Then

$$M \equiv -2K_X,$$

and $\text{mult}_P(M) = 4$. The surface $M$ is cut out on $X$ by the equation

$$\lambda x^2 + x(A_0 + A_1(y, z, t)) + B_2(y, z, t) + B_1(y, z, t) + B_0 = 0,$$

where $A_i$ and $B_i$ are homogeneous polynomials of degree $i$, and $\lambda \in \mathbb{C}$.

It follows from $\text{mult}_P(M) = 4$ that $B_1(y, z, t) + B_0 = 0$.

The coordinates $(y, z, t)$ are also local coordinates on $X$ near the point $P$. Then

$$x = -y^2 - y(zt + yp_1(y, z, t)) + \text{higher order terms},$$

which is a Taylor power series for $x = x(y, z, t)$, where $p_1(y, z, t)$ is a linear form.

The surface $M$ is locally given by the analytic equation

$$\lambda y^4 + (-y^2 - yzt - y^2 p_1(y, z, t))(A_0 + A_1(y, z, t)) + B_2(y, z, t) + \text{higher order terms} = 0,$$

and $\text{mult}_P(M) = 4$. Hence, we see that $B_2(y, z, t) = A_0y^2$ and

$$A_1(y, z, t)y^2 + A_0y(zt + yp_1(y, z, t)) = 0,$$
which implies that $A_0 = A_1(y, z, t) = B_2(y, z, t) = 0$. Hence, we see that a general surface in the pencil $\mathcal{M}$ is cut out on $X$ by the equation $x^2 = 0$, which is absurd.

Arguing as in the proof of Lemma 7.5, we obtain the following corollary.

**Corollary 7.6.** Suppose that $f_2(z, t) = z^2$. Then $f_4(0, t) = 0$.

Let $\mathcal{R}$ be the linear system on the threefold $X$ that is cut out by cubics

$$xh_2(x, y, z, t) + \lambda(w^2 x + wy^2 + q_3(x, y, z, t)) = 0,$$

where $h_2$ is a form of degree 2, and $\lambda \in \mathbb{C}$. Then $\mathcal{R}$ has no fixed components.

Let $M$ and $R$ be general surfaces in $\mathcal{M}$ and $\mathcal{R}$, respectively. Put

$$M \cdot R = \sum_{i=1}^{r} m_i L_i + \Delta,$$

where $m_i \in \mathbb{N}$, and $\Delta$ is a cycle, whose support contains no lines among $L_1, \ldots, L_r$.

**Lemma 7.7.** The cycle $\Delta$ is not trivial.

**Proof.** Suppose that $\Delta = 0$. Then $\mathcal{M} = \mathcal{R}$ by [2, Theorem 2.2]. But $\mathcal{R}$ is not a pencil.

We have $\text{mult}_P(\Delta) \geq 8n - \sum_{i=1}^{r} m_i$, because $\text{mult}_P(\mathcal{M}) = 2n$ and $\text{mult}_P(\mathcal{R}) \geq 4$. Then

$$\deg(\Delta) = 12n - \sum_{i=1}^{r} m_i \geq 2 \text{mult}_P(\Delta) \geq 2 \left(8n - \sum_{i=1}^{r} m_i\right)$$

by Lemma 4.5, because $\text{Supp}(\Delta)$ does not contain any of the lines $L_1, \ldots, L_r$.

**Corollary 7.8.** The inequality $\sum_{i=1}^{r} m_i \geq 4n$ holds.

Let $R_1$ and $R_2$ be general surfaces in the linear system $\mathcal{R}$. Then

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \cdot \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every $1 \leq i \leq 4$ by Lemmas 2.1 and 3.3. Then

$$4n \leq \sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2)n/2.$$

**Corollary 7.9.** The inequality $\sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8$ holds.

Now we suppose that $R_1$ is cut out on the quartic $X$ by the equation

$$w^2 x + wy^2 + q_3(x, y, z, t) = 0,$$
and $R_2$ is cut out by $xh_2(x, y, z, t) = 0$, where $h_2$ is sufficiently general. Then

$$\sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot T) = \sum_{i=1}^{r} \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8,$$

where $T$ is the hyperplane section of the hypersurface $X$ that is cut out by $x = 0$. But

$$R_1 \cdot T = Z_1 + Z_2,$$

where $Z_1$ and $Z_2$ are cycles on $X$ such that $Z_1$ is cut out by $x = y = 0$, and $Z_2$ is cut out by

$$x = wy + f_2(z, t) + yc_1(x, y, z, t) = 0.$$

**Lemma 7.10.** The equality $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_1) = 4$ holds.

**Proof.** The lines $L_1, \ldots, L_r \subset \mathbb{P}^4$ are given by the equations

$$x = y = q_4(x, y, z, t) = 0,$$

which implies that $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_1) = 4$. \qed

Hence, we see that $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2) \geq 4$. But $Z_2$ can be considered as a cycle

$$wy + f_2(z, t) + yc_1(y, z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, w]) \cong \mathbb{P}^3,$$

and, putting $u = w + c_1(y, z, t)$, we see that $Z_2$ can be considered as a cycle

$$uy + f_2(z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,$$

and we can consider the set of lines $L_1, \ldots, L_r$ as the set of curves in $\mathbb{P}^3$ given by $y = f_4(z, t) = 0$.

**Lemma 7.11.** The inequality $f_2(z, t) \neq zt$ holds.

**Proof.** Suppose that $f_2(z, t) = zt$. Then it follows from Lemma 7.5 that

$$f_4(z, t) = zt(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$$

for some $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$. Then $Z_2$ can be given by

$$uy + zt = yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,$$

which implies $Z_2 = Z_2^1 + Z_2^2$, where $Z_2^1$ and $Z_2^2$ are cycles in $\mathbb{P}^3$ such that $Z_2^1$ is given by

$$y = uy + zt = 0,$$

and $Z_2^2$ is given by $uy + zt = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0$. 

Please cite this article in press as: I. Cheltsov, I. Karzhemanov, Halphen pencils on quartic threefolds, Adv. Math. (2009), doi:10.1016/j.aim.2009.08.020
We may assume that $L_1$ is given by $y = z = 0$, and $L_2$ is given by $y = t = 0$. Then

$$Z_2^1 = L_1 + L_2,$$

which implies that $\sum_{i=1}^r \text{mult}_{L_i} (Z_2^2) \geq 2$.

Suppose that $r = 4$. Then $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\alpha_2 \neq 0$, $\beta_2 \neq 0$. Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_2,$$

because $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$ does not vanish on $L_1$ and $L_2$. But

$$L_3 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_4,$$

because $zt$ does not vanish on $L_3$ and $L_4$. Then $\sum_{i=1}^r \text{mult}_{L_i} (Z_2^2) = 0$, which is impossible.

Suppose that $r = 3$. We may assume that $(\alpha_1, \beta_1) = (1, 0)$, but $\alpha_2 \neq 0 \neq \beta_2$. Then

$$L_2 \not\subseteq \text{Supp}(Z_2^2),$$

because $v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t)$ does not vanish on $L_2$. We have

$$f_4(z, t) = z^2 t (\alpha_2 z + \beta_2 t),$$

which implies that $v_3(0, t) \neq 0$ by Corollary 7.4. Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_3,$$

because $v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t)$ and $zt$ do not vanish on $L_1$ and $L_3$, respectively, which implies that $\sum_{i=1}^r \text{mult}_{L_i} (Z_2^2) = 0$. The latter is a contradiction.

We see that $r = 2$. We may assume that $(\alpha_1, \beta_1) = (1, 0)$, and either $\alpha_2 = 0$ or $\beta_2 = 0$.

Suppose that $\alpha_2 = 0$. Then $f_4(z, t) = \beta_2 z^2 t^2$. By Lemma 7.3 and Corollary 7.4, we get

$$v_3(0, t) \neq 0 \neq v_3(z, 0),$$

which implies that $v_3(z, t) + yc_2(y, z, t) - \beta_2zt$ does not vanish on neither $L_1$ nor $L_2$. Then

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_2,$$

which implies that $\sum_{i=1}^r \text{mult}_{L_i} (Z_2^2) = 0$, which is a contradiction.

We see that $\alpha_2 \neq 0$ and $\beta_2 = 0$. We have $f_4(z, t) = \alpha_2 z^3 t$. Then

$$v_3(0, t) \neq 0$$

by Corollary 7.4. Then $L_1 \not\subseteq \text{Supp}(Z_2^2)$ because the polynomial

$$v_3(z, t) + yc_2(y, z, t) - \alpha_2 z^2$$

does not vanish on $L_1$. 

Please cite this article in press as: I. Cheltsov, I. Karzhemanov, Halphen pencils on quartic threefolds, Adv. Math. (2009), doi:10.1016/j.aim.2009.08.020
The line $L_2$ is given by the equations $y = t = 0$. But $Z_2$ is given by the equations
\[ uy + zt = v_3(z, t) + yc_2(y, z, t) - \alpha_2 uz^2 = 0, \]
which implies that $L_2 \not\subseteq \text{Supp}(Z_2^2)$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction. \(\square\)

Therefore, we see that $f_2(z, t) = z^2$. It follows from Corollary 7.6 that
\[ f_4(z, t) = zg_3(z, t) \]
for some $g_3(z, t) \in \mathbb{C}[z, t]$. We may assume that $L_1$ is given by $y = z = 0$.

**Lemma 7.12.** The equality $g_3(0, t) = 0$ holds.

**Proof.** Suppose that $g_3(0, t) \neq 0$. Then $\text{Supp}(Z_2) = L_1$, because $Z_2$ is given by
\[ uy + z^2 = zg_3(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0, \]
and the lines $L_2, \ldots, L_r$ are given by the equations $y = g_3(z, t) = 0$.

The cycle $Z_2 + L_1$ is given by the equations
\[ uy + z^2 = z^2g_3(z, t) + zv_3(z, t) + zy^2c_2(y, z, t) = 0, \]
which implies that the cycle $Z_2 + L_1$ can be given by the equations
\[ uy + z^2 = zyv_3(z, t) + zy^2c_2(y, z, t) - uyg_3(z, t) = 0. \]

We have $Z_2 + L_1 = C_1 + C_2$, where $C_1$ and $C_2$ are cycles in $\mathbb{P}^3$ such that $C_1$ is given by
\[ y = uy + z^2 = 0, \]
and the cycle $C_2$ is given by the equations
\[ uy + z^2 = vz_3(z, t) + zyc_2(y, z, t) - ug_3(z, t) = 0. \]

We have $C_1 = 2L_2$. But $L_1 \not\subseteq \text{Supp}(C_2)$ because the polynomial
\[ zv_3(z, t) + zyc_2(y, z, t) - ug_3(z, t) \]
does not vanish on $L_1$, because $g_3(0, t) \neq 0$. Then
\[ Z_2 + L_1 = 2L_2, \]
which implies that $Z_2 = L_1$. Then $\sum_{i=1}^r \text{mult}_{L_i}(Z_2) = 1$, which is a contradiction. \(\square\)

Thus, we see that $r \leq 3$ and
\[ f_4(z, t) = z^2(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) \]
for some \((\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)\). Then
\[
v_3(0, t) \neq 0
\]
by Corollary 7.4. But \(Z_2\) can be given by the equations
\[
uy + z^2 = yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t)
\]
\[
= 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,
\]
which implies \(Z_2 = Z_2^1 + Z_2^2\), where \(Z_2^1\) and \(Z_2^2\) are cycles on \(\mathbb{P}^3\) such that \(Z_2^1\) is given by
\[
y = uy + z^2 = 0,
\]
and the cycle \(Z_2^2\) is given by the equations
\[
uy + z^2 = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t) = 0,
\]
which implies that \(Z_2^1 = 2L_1\). Thus, we see that \(\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) \geq 2\).

**Lemma 7.13.** The inequality \(r \neq 3\) holds.

**Proof.** Suppose that \(r = 3\). Then \(\beta_1 \neq 0 \neq \beta_2\), which implies that
\[
L_1 \not\subseteq \text{Supp}(Z_2^2),
\]
because \(v_3(z, t) + yc_2(y, z, t) - u(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t)\) does not vanish on \(L_1\). But
\[
L_2 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_3,
\]
because \(\beta_1 \neq 0 \neq \beta_2\). Then \(\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0\), which is a contradiction. \(\square\)

**Lemma 7.14.** The inequality \(r \neq 2\) holds.

**Proof.** Suppose that \(r = 2\). We may assume that

- either \(\beta_1 \neq 0 = \beta_2\),
- or \(\alpha_1 = \alpha_2\) and \(\beta_1 = \beta_2 \neq 0\).

Suppose that \(\beta_2 = 0\). Then \(f_4(z, t) = \alpha_2z^3(\alpha_1z + \beta_1t)\) and
\[
L_1 \not\subseteq \text{Supp}(Z_2^2),
\]
because \(v_3(z, t) + yc_2(y, z, t) - \alpha_2uz(\alpha_1z + \beta_2t)\) does not vanish on \(L_1\). But \(L_2\) is given by
\[
y = \alpha_1z + \beta_1t = 0,
\]
which implies that $z^2$ does not vanish on $L_2$, because $\beta_1 \neq 0$. Then

$$L_2 \notin \text{Supp}(Z_2^2),$$

which implies that $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

Hence, we see that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2 \neq 0$. Then $L_1 \notin \text{Supp}(Z_2^2)$, because

$$v_3(z,t) + yc_2(y,z,t) - u(\alpha_1 z + \beta_1 t)^2$$

do not vanish on $L_1$. But $L_2 \notin \text{Supp}(Z_2^2)$, because $z^2$ does not vanish on $L_2$. Then

$$\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0,$$

which is a contradiction. $\Box$

We see that $f_2(z,t) = z^2$ and $f_4(z,t) = \mu z^4$ for some $0 \neq \mu \in \mathbb{C}$. Then $Z_2^2$ is given by

$$uy + z^2 = v_3(z,t) + yc_2(y,z,t) - \mu z^2 = 0,$$

where $v_3(0,t) \neq 0$ by Corollary 7.4. Thus, we see that $L_1 \notin \text{Supp}(Z_2^2)$, because

$$v_3(z,t) + yc_2(y,z,t) - \mu z^2$$

do not vanish on $L_1$. Then $\sum_{i=1}^{r} \text{mult}_{L_i}(Z_2^2) = 0$, which is a contradiction.

The assertion of Proposition 7.1 is proved.

The assertion of Theorem 1.5 follows from Propositions 3.4, 5.1, 6.1, 7.1.

References