A SETTING FOR THE GROUP OF THE BITANGENTS

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1. THE group Γ, as it will be denoted here, 'of the bitangents' has order 1451520. It was decomposed into its 30 conjugate sets by Dickson ((**5**) 318) who used its representation as the symplectic group on 6 variables over **F**—the Galois field of 2 marks 0, 1 with $1+1=0$. Since Γ admits this representation it is isomorphic ((**9**) 17) to an orthogonal group, also over **F**, in 7 variables and it is this latter representation, as the group of projectivities in a finite space [6] that leave a non-singular quadric Q therein invariant, that will be described here. The work is analogous to that on the cubic surface group G in (**7**); just as G has a representation as the group of automorphisms of the symmetric quadratic form $\sum_{i<j} x_i x_j$ in 6 variables, so Γ has a representation as the group of automorphisms of the symmetric quadratic form $\sum_{i<j} x_i x_j$ in 7 variables—the base field being always **F**. The 7-rowed matrices in the representation of Γ have attributes analogous to those of the 6-rowed matrices in the representation of G; each column is the coordinate vector of a point on Q and the points of Q that correspond so to the columns of a matrix are such that no pair of them is conjugate with respect to Q. In other words, the columns of the matrix answer to the vertices of a simplex Σ whose 7 vertices all lie on Q and whose 21 edges are all chords (not generators) of Q. One consequence is that the number of units occurring in any column has to be one of 1, 4, 5. Moreover, Q has ((**7**), § 9) a kernel k; when referred to Σ k is the unit point. Since k cannot change under any operation that leaves Q invariant the 7 marks in any row of any matrix in Γ will sum to 1; the number of units in any row is odd, the number of zeros even.

It may well be easier, in deciding whether or not a given matrix belongs to Γ, to use the above restrictions on a 7-rowed matrix rather than Dickson's 'Abelian conditions' ((**4**) 89–90; (**5**) 247–9) on a 6-rowed matrix. Dickson, it need hardly be said, surveys far wider horizons than any limited by Γ; his subject is the partitioning into conjugate classes of the symplectic group over any field and in any (even) number of variables, so that little surprise is caused either by the prolixity of his calculations or by his protestations that they have been convincingly checked. Yet this breadth of view is perhaps too sweeping for a proper study of Γ some, at least, of whose properties are seen to advantage in this orthogonal representation.

Another representation of Γ, as a subgroup (of index 2) of the group of symmetries of a regular polytope in euclidean space of 7 dimensions, has been used by Frame in (**8**) not only to rediscover, independently and presumably without knowledge of (**5**), the partitioning of Γ into conjugate sets but also, with the aid of powerful theorems of Schur and Brauer, to compile the whole table of group characters. Frame deduces therefrom certain permutation characters: their degrees must be interpretable, in any representation of Γ, as numbers of objects undergoing permutation. The representation to be used here discloses instantly (§ 3) sets of 28, 36, 63 primes (i.e. hyperplanes, spaces of dimension 5); one notes, too, that there are, answering to the last permutation character in (**8**), 135 planes on Q.

2. A summary of matters treated here now follows.

§§ 3–8 are concerned to describe the whole figure; to enumerate and catalogue the spaces in it, and their various relations with Q. All this information is displayed in Table 1, wherein every entry has some relevance to the study of Γ. The coordinate system is well adapted to discriminate between the 3 kinds of prime, and the polarity set up by Q helps greatly in the calculation of the numbers of different spaces.

The Klein quadric in [5], whose points map the 35 lines of a [3] over **F** as base field, has a set of 8 heptagons associated with it; these have scarcely been noticed since their discovery (in **3**) 50 years ago. Each of the 36 Klein sections of Q affords such a set of heptagons, so that the figure includes 288; these are intimately linked with those 288 simplexes in regard to which Q has the equation $\sum_{i<j} x_i x_j = 0$. This is explained in §§ 9–11, and there is a passing allusion to symmetric subgroups of degree 8. There follows, in § 12, a description of the 3 types of involution in Γ, and § 13 notes the various types of operation in a symmetric subgroup of degree 7. Section 14 gives the 3 types of operations of period 3 which, like the involutions, are all products of 1, 2, or 3 commutative operations, the factors having a very simple geometrical definition.

Among information that this orthogonal representation of Γ affords readily are explicit forms for the normalizers of some operations; instances are operations of periods 5, 7, 10, 12 whose normalizers are found in §§ 15–17. The number in the conjugate class to which any such operation belongs follows instantly. An incidental disclosure is the matrix form of certain operations of period 15; this is obtained again, in § 18, by permuting 8 heptagons.

The concluding §§ 19–20 describe the basic features of the mapping, in this figure, of half-integer period and theta characteristics. There is no danger here of the confusion, so justly deprecated in (**2**), between the two kinds of characteristic, and although one does not lengthen the paper by giving details of how to pass from one fundamental system to another these matters offer no difficulty.

The description of the figure

3. The $2^7-1 = 127$ primes in [6] are, relative to Q, of three kinds: they may

(*a*) pass through the kernel k,
(*b*) meet Q in ruled quadrics $\mathscr{K}$,
(*c*) meet Q in non-ruled quadrics $\mathscr{L}$.

A non-singular quadric in [5] is said to be *ruled* when it contains planes; it is a Klein quadric, its points mapping the 35 lines of a [3], and it contains two systems each of 15 planes. A non-ruled quadric $\mathscr{L}$ in [5] does not contain any planes; it is described in § 16 and § 17 of (**7**).

The number of primes through k is the number of [4]'s in a [5], which is 63. This is seen, too, on remarking that a prime passes, or does not pass, through k according as an even or an odd number of the coordinates x_i appears in its equation, so that there are $\binom{7}{2}+\binom{7}{4}+\binom{7}{6} = 21+35+7$ primes T through k. There remain 64 primes not through k and having an odd number of the x_i in their equations; if this odd number is 3 or 7 the section is ruled, if it is 1 or 5 the section is not ruled. For example

$$x_0 = x_4+x_5, \qquad x_1 = x_5+x_3, \qquad x_2 = x_3+x_4, \qquad x_6 = x_3+x_4+x_5$$

is a plane on Q; it lies in the unit prime as well as in primes whose equations include only 3 coordinates. Thus $1+35 = 36$ prime sections of Q are ruled. That the 7 sections by bounding primes of the simplex of reference are not ruled is known from (**7**); that the 21 sections by primes into whose equations 5 of the coordinates enter are not ruled follows because, for instance, the quadratic form is invariant over **F** when x_0, x_1, x_2, x_3 are left unchanged while each of the other x_i is replaced (cf. 16.2 below) by $x_0+x_1+x_2+x_3+x_i$. Thus $7+21 = 28$ sections of Q are not ruled.

A prime will be labelled T, C, S according as it is in category (*a*), (*b*), (*c*). This nomenclature is not capricious: it is used with an eye on a figure in [7] of which the figure here is a section. In [7] the polar of a [5] is a line, and as lines are denoted by t, c, s according as they are tangents of, chords of, or skew to a quadric it will, in this larger figure, be convenient if one is able to use the same letter, small and capital, to label polar spaces.

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4. Q consists of 63 points m, namely, those with 1, 4, 5 of the x_i non-zero. Their joins t_0 to k, whose coordinates are all non-zero, each contain one further point p with 6, 3, 2 of the x_i non-zero; these 63 p are not on Q. The 63 T are the tangent primes of Q.

There are [4]'s J, not containing k, meeting Q in non-singular quadrics; each C contains 28, and each S 36, polars therein, with respect to the sections of Q, of the points that do not lie on these sections. When J is given the 3 primes through it are a C, an S, and a T, the latter joining J to k; the number of J is $36 \times 28 = 1008$. Other [4]'s F, H meet Q in cones, with point vertices, whose sections by solids not containing the vertex are ruled and non-ruled respectively. The tangent [4] to the Klein section of Q by C at any of its 35 m is an F, and all [4]'s in C are accounted for by these 35 F and the 28 J. The tangent [4] to the section of Q by S at any of its 27 m is an H, and all [4]'s in S are accounted for by these 27 H and the 36 J. Of the 3 primes through F one, T, joins it to k while the others are both C; of the 3 primes through H one, T, joins it to k while the others are both S. There are $36 \times 35/2 = 630$ F and $28 \times 27/2 = 378$ H. Each T includes 10 F, 6 H, 16 J, and these account for the 32 [4]'s in T that do not contain k.

The figure also includes another type E of [4] that will be seen always to contain k and to meet Q in 3 planes.

5. The notation used in (7) will be incorporated here. A line is labelled g, c, t, s according as 3, 2, 1, 0 of its points are m. One appends a suffix, as in t_0, if a space contains k (unless every space in the category contains k). The symbols for planes, and their relations to Q, are

d: lying wholly on Q,
e: meeting Q in a single line,
f: meeting Q in two lines,
h: meeting Q in a single point,
j: meeting Q in 3 non-collinear points,

and there occur planes e_0, j_0. The symbols for solids, and their relations to Q, are

γ: meeting Q in a single plane,
ϕ: meeting Q in two planes,
χ: meeting Q in a single line,
ψ: meeting Q in 3 concurrent non-planar lines,
κ: meeting Q in a non-singular ruled quadric
λ: meeting Q in a non-singular non-ruled quadric,

Every γ contains k, as do some solids ψ_0.

The bounding primes, [4]'s, solids, planes, lines, vertices of the simplex of reference Σ_0 are, respectively,

$$S,\ H,\ \lambda,\ j,\ c,\ m,$$

the equation of the section of Q being $\Sigma\, x_i x_j = 0$ in the appropriate set of variables.

6. The prerequisite to a full description of the figure is a knowledge of how many spaces of different categories occur in it; the numbers of T, C, S, J, F, H, p, m have been found. The specifications and numbers of other spaces are readily obtained once one knows in detail how spaces are paired in the polarity (degenerate because k is conjugate to every point of [6]) set up by Q.

Let $\mathscr{S}$ be an $[n]$ that does not contain k; its polar $\mathscr{S}'$ is a $[5-n]$ that does. Every point of $\mathscr{S}$ is conjugate to every point of $\mathscr{S}'$, as indeed, since points collinear with k have the same polar, is every point of the $[n+1]$ that joins $\mathscr{S}$ to k. Thus $\mathscr{S}'$ is the polar not only of $\mathscr{S}$ but of all those $[n]$, 2^{n+1} of them, that lie in the join of $\mathscr{S}$ to k but do not contain k. And this join is the polar of all those $[4-n]$, 2^{5-n} of them, that lie in $\mathscr{S}'$ but do not contain k. If, for instance, $\mathscr{S}$ is m, $\mathscr{S}'$ is the tangent prime T of Q at m and is also the polar of p on km: this line t_0 is the polar of every [4] in T that does not contain k. These 32 [4]'s consist, as has been said already, of 16 J, 10 F, 6 H; as there are 63 lines t_0 it is seen again that the figure includes 1008 J, 630 F, 378 H.

Now take $\mathscr{S}$ to be g: an instance is the line

$$x_0+x_1 = x_2+x_3 = x_4+x_5 = x_0+x_2+x_4 = x_6 = 0, \tag{6.1}$$

whose polar [4], common to the tangent primes at

$$(1, 1, 1, 1, 0, 0, 0), \qquad (1, 1, 0, 0, 1, 1, 0), \qquad (0, 0, 1, 1, 1, 1, 0),$$

is
$$x_0+x_1 = x_2+x_3 = x_4+x_5. \tag{6.2}$$

This is an E, passing through k and meeting Q in 3 planes through g, namely

$$\left.\begin{array}{ll} x_0+x_1 = x_2+x_3 = x_4+x_5 = x_0+x_2+x_4 = x_6 & \\ x_0+x_1 = x_2+x_3 = x_4+x_5 = x_6, & x_0+x_2+x_4 = 0 \\ x_0+x_1 = x_2+x_3 = x_4+x_5 = 0, & x_0+x_2+x_4 = x_6 \end{array}\right\}. \tag{6.3}$$

The join of (6.1) to k is a plane e_0 wherein are 3 lines t not containing k; these also have (6.2) for their polar, and so there are thrice as many t in the figure as there are g. On the other hand, e_0 is the polar of each of those 16 solids in (6.2) that do not contain k; of these 3, spanned by pairs of the planes (6.3) are ϕ and one, namely,

$$x_0+x_1 = x_2+x_3 = x_4+x_5, \qquad x_6 = 0$$

is a χ, meeting Q only in g. The other 12 are all ψ, meeting Q in 3 lines, one in each plane (6.3), concurring on g. There are thrice as many ϕ and 12 times as many ψ as there are χ or g.

Now take $\mathscr{S}$ to be c, say the edge $X_0 X_1$ of Σ_0. Its polar is

$$x_0 = x_1 = x_2+x_3+x_4+x_5+x_6,$$

a [4] J_0, containing k and meeting Q in a non-singular quadric. The lines in the plane $j_0 \equiv kX_0 X_1$ that do not contain k consist ((7), § 10) of an s and 3 c; they all have J_0 for their polar, as does j_0. Thus there are as many s, and thrice as many c, as there are j_0. The 16 solids in J_0 that do not contain k consist ((7), § 13) of 10 κ and 6 λ, all having j_0 for their polar. Thus there are 10 times as many κ and 6 times as many λ as there are j_0.

Every point in a plane d on Q is conjugate to every other, as well as to k; hence the polar solid of d is its join γ to k. Since the join t_0 of k to any m does not meet Q save at m, γ does not meet Q save on d. The 8 planes of γ that do not contain k all have γ for their polar; they consist of d and 7 e, so that there are 7 times as many e as there are d.

7. The polar of $X_0 X_1 X_2$ is

$$x_0 = x_1 = x_2, \qquad x_3+x_4+x_5+x_6 = 0.$$

All m save one, namely $m^*(0, 0, 0, 1, 1, 1, 1)$, in this solid ψ_0 have 5 non-zero coordinates; they lie in pairs on 3 g concurrent at m^*. Moreover the join, $x_3 = x_4 = x_5 = x_6$, of $X_0 X_1 X_2$ to k is a solid ψ_0' wherein too the 3 g concur at m^*; ψ_0 and ψ_0' meet in the line $t_0 = km^*$ and span T, the tangent prime at m^*; they are polars of one another, and each is also the pole of every plane in the other that does not contain k. The g in ψ_0, as in ψ_0', form a trihedron, and t_0 is the only line through its vertex m^* that does not lie in any of its faces. Thus the 8 planes in ψ_0, as in ψ_0', that do not contain k consist of 3 f (the faces of the trihedron), an h through m^*, and 4 j. Once the number of ψ_0 is known so are the numbers of f, h, j. This information is found on taking the section, by a [4] J not containing m^*, of the figure in T.

J meets Q in q, the quadric described in §§ 13–15 of (7); its kernel is either k (if $J \equiv J_0$ contains k) or the projection of k from m^* on J—for every point of J, as lying in T, is conjugate to m^* as well as to k, and so to the intersection of m^*k with J whether this be k itself or not. As q consists of 15 m and 15 g there are 15 g and 15 d on Q which contain m^*; hence there are on Q $63\times 15/3 = 315$ g and $63\times 15/7 = 135$ d. Moreover there are 10 pairs of polar planes j_0, j_0' through the kernel of q, so that there are 10 pairs of polar solids ψ_0, ψ_0' through m^*. The number of such pairs in the whole figure is thus 630, and of solids ψ_0 1260.

The information now gathered suffices to begin the construction of Table 1, once it is supplemented by the number of chords c of Q. Since the

TABLE 1

	k	p	m	g	t_0	t	c	s	d	e	e_0	f	h	j_0	j
1 k				.	1	.	.	.	.	.	1	.	.	1	.
63 p				.	1	2	1	3	.	4	3	2	6	3	4
63 m				3	1	1	2	.	7	3	3	1+4	1	3	3
315 g	.	.	15						7	1	1	2	.	.	.
63 t_0	63	1	1						.	.	3	.	.	3	.
945 t	.	30	15						.	6	3	1	3	.	3
1008 c	.	16	32						.	.	.	4	.	3	3
336 s	.	16	.						.	.	.	.	4	1	1
135 d	.	.	15	3	.	.	.	.							
945 e	.	60	45	3	.	6	.	.							
315 e_0	315	15	15	1	15	1	.	.							
3780 f	.	120	60+240	24	.	4	15	.							
1260 h	.	120	20	.	.	4	.	15							
336 j_0	336	16	16	.	16	.	1	1							
5040 j	.	320	240	.	.	16	15	15							
135 γ	135	15	15	3	15	3	.	.	1	1	3	.	.	.	.
945 ϕ	.	60	45+120	3+36	.	6	15	.	14	1	.	3	.	.	.
315 χ	.	60	15	1	.	6	.	15	.	1	.	.	3	.	.
3780 ψ	.	480	60+360	36	.	16+48	45	45	.	12	.	3	3	.	6
1260 ψ_0	1260	140	20+120	12	20+120	4+8	15	15	.	.	12	1	1	15	1
3360 κ	.	320	480	64	.	32	60	20	.	.	.	8	.	.	4
2016 λ	.	320	160	.	.	32	20	60	.	.	.	.	8	.	4
1008 J	.	256	240	48	.	64	60	60	.	16	.	12	12	.	16
378 H	.	120	6+60	6	.	4+24	15	45	.	6	.	1	3+12	.	6
630 F	.	120	10+180	18+48	.	4+24	45	15	28	6	.	3+12	1	.	6
315 E	315	75	15+60	1+18	15+60	7+12	15	15	7	1+6	1+18	3	3	15	3
336 J_0	336	80	80	16	80	16	20	20	.	.	16	4	4	20	4
63 T	63	31	1+30	3+12	1+30	1+14	15	15	7	7	15	1+6	4+3	15	7
28 S	.	16	12	4	.	8	6	10	.	4	.	2	6	.	4
36 C	.	16	20	12	.	8	10	6	8	4	.	6	2	.	4

	γ	ϕ	χ	ψ	ψ_0	κ	λ	J	H	F	E	J_0	T	S	C
1 k	1	.	.	.	1	.	.	.	.	.	1	1	1	.	.
63 p	7	4	12	8	7	6	10	16	20	12	15	15	31	36	28
63 m	7	3+8	3	1+6	1+6	9	5	15	1+10	1+18	3+12	15	1+30	27	35
315 g	7	1+12	1	3	3	6	.	15	5	9+24	1+18	15	15+60	45	105
63 t_0	7	.	.	.	1+6	.	.	.	.	.	3+12	15	1+30	.	.
945 t	21	6	18	4+12	3+6	9	15	60	10+60	6+36	21+36	45	15+210	270	210
1008 c	.	16	.	12	12	18	10	60	40	72	48	60	240	216	280
336 s	.	.	16	4	4	2	10	20	40	8	16	20	80	120	56
135 d	1	2	.	.	.	.	.	.	.	6	3	.	15	.	30
945 e	7	1	3	3	.	.	.	15	15	9	3+18	.	105	135	105
315 e_0	7	.	.	.	3	.	.	.	.	.	1+18	15	75	.	.
3780 f	.	12	.	3	3	9	.	45	10	18+72	36	45	60+360	270	630
1260 h	.	.	12	1	1	.	5	15	10+40	2	12	15	80+60	270	70
336 j_0	.	.	.	.	4	.	.	.	.	.	16	20	80	.	.
5040 j	.	.	.	8	4	6	10	80	80	48	48	60	560	720	560
135 γ								.	.	.	3	.	15	.	.
945 ϕ								.	.	9	3	.	45	.	105
315 χ								.	5	.	1	.	15	45	.
3780 ψ								15	10	6	12	.	60+120	270	210
1260 ψ_0								.	.	.	12	15	20+120	.	.
3360 κ								10	.	16	.	10	160	120	280
2016 λ								6	16	.	.	6	96	216	56
1008 J	.	.	.	4	.	3	3						16	36	28
378 H	.	.	6	1	.	.	3						6	27	.
630 F	.	6	.	1	.	3	.						10	.	35
315 E	7	1	1	1	3	.	.						15	.	.
336 J_0	.	.	.	.	4	1	1						16	.	.
63 T	7	3	3	1+2	1+6	3	3	1	1	1	3	3			
28 S	.	.	4	2	.	1	3	1	2	.	.	.			
36 C	.	4	.	2	.	3	1	1	.	2	.	.			

c through a given m are those 32 lines through m that lie outside the tangent prime this number is $\frac{1}{2}(63\times 32) = 1008$.

8. The entries above the main diagonal in Table 1 record the number of subspaces (symbol at left of row) in any given space (symbol at head of column); those below record the number of spaces containing a given subspace; thus any entry determines its opposite, or mirror image in the diagonal. For instance: if it is known that 45 of the 3780 f lie in any one of the 336 J_0 it follows that $336\times 45/3780 = 4$ J_0 contain any one f. An entry above the diagonal shown as the sum of two numbers implies that the section of Q is singular and that the subspaces in question can be distinguished by their different relations to the vertex of the section. For instance: ϕ meets Q in 2 planes; of the m in ϕ 3 belong to the line-vertex of the section whereas 8 (4 in each plane) do not; of the g in ϕ one is the vertex of the section whereas 12 (6 in each plane) are not. The entry $45+120$ opposite to, and deducible from, $3+8$ tells that there are 165 ϕ containing a given m and that in 45 of these m is on the line-vertex of the section; the entry $3+36$ opposite to, and deducible from, $1+12$ tells that there are 39 ϕ containing a given g and that in 3 of these g is the vertex of the section.

The entries above the main diagonal as far as column H are known from (**7**); columns H, J correspond to those headed respectively by M, P in Table 2 of (**7**). Those in column F are readily found from the fact that the section of Q is a cone, vertex m^* say (this asterisk will be used also to indicate vertices of other singular sections of Q), whose section is a hyperboloid. Thus there are $9+24$ g in F; 9 of these concur at m^* while 24 lie 4 in each of 6 planes d through m^*. The 16 solids in F that do not contain m^* are all κ, and each contains 6 j and 9 f. Since there are 2 κ in F through any j or f (namely, the solids that do not join the plane to m^*, it being presumed that, when the plane is f, it does not itself contain m^*) there are, in F, $16\times 6/2 = 48$ j and $16\times 9/2 = 72$ f. In addition there are, in correspondence with the 18 c in κ, 18 f through m^*. There are 2 h through m^* in F, in correspondence with the Dandelin lines in κ. Moreover F includes 72 c (4 in each f that contains m^*) and 8 s (4 in each h that contains m^*); and so on. And each entry in this column headed by F determines its opposite in the row led by F, or vice versa.

The entries under S appear along the top of Table 2 of (**7**); those under C are deducible from the line geometry in [3] over this field (see (**3**) and (**6**)). Column J_0 is a manifest modification of column J, and it only remains to say a word or two about columns E and T. A useful check on the calculations is provided by the known total number of spaces of any dimension in a space of larger dimension, for this total is the sum of the entries in some stratum of a column.

The number of E through any of ϕ, χ, ψ is, by reciprocation, the number of g in e_0, and so 1. There are, likewise, 3 E through any of f, h, j; 7 through d, e, γ; 15 through c, s, j_0. As for subspaces of T the number of E, as of g through m^*, is 15; other [4]'s in T were encountered in § 4, and each of the 10 F includes 16 κ, each of the 6 H includes 16 λ. Since the cone in which T meets Q projects, from its vertex m^*, a non-singular quadric q in [4] it follows, since q is accompanied by 45 f, 60 j, 20 j_0, 60 c, 20 s and consists of 15 m and 15 g, that through m^* there pass 45 ϕ, 60 ψ, 20 ψ_0, 60 f, 20 h, 15 g, and 15 d; furthermore there are 240 c and 80 s in T lying in fours in planes through m^*, in those planes namely that project the 60 c and 20 s that accompany q. Lastly, to pass further details over, the 7 primes through γ and the 15 through either e_0 or j_0 have all to be T; these three facts provide the opposite entries 15, 75, 80 of numbers of γ, e_0, j_0 in T. And so forth.

288 heptagons and the simplexes linked with them

9. Each C includes 28 p lying 3 on each of 56 s, so that, in C, 6 s pass through any p. These points and lines can ((3) 68) be distributed as vertices and joins of 8 heptagons $\mathfrak{H}$; all 7 vertices of an $\mathfrak{H}$ are p, all 21 joins of an $\mathfrak{H}$ are s. Any pair of p whose join is an s are vertices of a single $\mathfrak{H}$ in any C through s and each p can, once the 8 $\mathfrak{H}$ are given, be identified as the unique vertex common to some pair, and to no other pair ((3) 68).

When C is the unit prime the system of supernumerary coordinates induced therein by the homogeneous coordinates in the ambient [6] serves to identify the 8 $\mathfrak{H}$ readily. One notes, in passing, that the condition for the join of 2 p to be an s is that the third point on this join should be a p too, and that the coordinate vector of this last point is the sum of those of the other two. The points in the unit prime are those, and only those, of whose 7 coordinates an even number are 1; in particular, the only points of Q in this prime are the 35 with 4 non-zero coordinates. The 7 columns in any of

$$\mathfrak{H}_1:\ \begin{array}{ccccccc} . & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & . & . & . & . & . \\ 1 & . & 1 & . & . & . & . \\ 1 & . & . & 1 & . & . & . \\ 1 & . & . & . & 1 & . & . \\ 1 & . & . & . & . & 1 & . \\ 1 & . & . & . & . & . & 1 \end{array} \qquad \mathfrak{H}_2:\ \begin{array}{ccccccc} 1 & 1 & . & . & . & . & . \\ 1 & . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & . & . & . & . \\ . & 1 & . & 1 & . & . & . \\ . & 1 & . & . & 1 & . & . \\ . & 1 & . & . & . & 1 & . \\ . & 1 & . & . & . & . & 1 \end{array} \qquad \mathfrak{H}_3:\ \ldots \tag{9.1}$$

are coordinate vectors of vertices of an $\mathfrak{H}$, and (9.1) indicates clearly 7 of the 8 $\mathfrak{H}$, every pair sharing a single vertex; the vertices of $\mathfrak{H}_0$ are the first vertex of $\mathfrak{H}_1$, the second of $\mathfrak{H}_2$, and so on. The common vertex of $\mathfrak{H}_i$ and $\mathfrak{H}_j$ has, if neither i nor j is 0, $x_{i-1} = x_{j-1} = 1$ and every other coordinate zero; the common vertex of $\mathfrak{H}_i$ and $\mathfrak{H}_0$ has $x_{i-1} = 0$ and every other coordinate 1.

10. Take, now, any $\mathfrak{H}$ and join its vertices to k; the other point on each join is on Q, and the join of any 2 of these 7 m is c, not g, since its remaining point, being on a join of $\mathfrak{H}$, is a p. These m are vertices of a simplex Σ; when C is the unit prime and $\mathfrak{H}$ is $\mathfrak{H}_0$ Σ is Σ_0, the simplex of reference; but whichever of the 36 C is used and whichever of the 8 $\mathfrak{H}$ therein is chosen, Σ has all its vertices m and all its joins c, while k is the sole point of [6] not in any bounding prime of Σ. The equation of Q referred to Σ is, again, $\sum_{i<j} x_i x_j = 0$, and the change from Σ_0 to Σ as a new simplex of reference leaves the quadratic form invariant.

As there are 288 Σ, with 288 $\mathfrak{H}$, Γ has a permutation representation of degree 288. The stabilizer of Σ_0 in Γ consists of the permutation matrices and so has order 7 !; hence the order of Γ, if transitive on Σ, is 288×7 !. This transitivity is a consequence of that on C combined with that on the 8 Σ associated with the $\mathfrak{H}$ in the unit prime; and clearly Σ_0 can be transformed into any of those Σ associated with $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, $\mathfrak{H}_4$, $\mathfrak{H}_5$, $\mathfrak{H}_6$, $\mathfrak{H}_7$, by using matrices

$$\begin{bmatrix} 1 & . & . & . & . & . & . \\ . & . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & 1 \\ . & 1 & 1 & . & 1 & 1 & 1 \\ . & 1 & 1 & 1 & . & 1 & 1 \\ . & 1 & 1 & 1 & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & . \end{bmatrix}, \quad \begin{bmatrix} . & . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & . & . & . & . & . \\ 1 & . & . & 1 & 1 & 1 & 1 \\ 1 & . & 1 & . & 1 & 1 & 1 \\ 1 & . & 1 & 1 & . & 1 & 1 \\ 1 & . & 1 & 1 & 1 & . & 1 \\ 1 & . & 1 & 1 & 1 & 1 & . \end{bmatrix}, \quad \ldots \qquad (10.1)$$

11. The stabilizer of a C has order 8 ! and acts as the symmetric group on the 8 $\mathfrak{H}$ in C. Thus is disclosed a representation of $\mathscr{S}_8$ by 7-rowed matrices over **F**: namely, by all matrices that arise on permuting the columns of the unit matrix and of the 7 matrices (10.1). The marks of each column of a matrix that leaves the unit prime invariant have to sum to 1 so that, if the matrix is in Γ, each column includes either 1 or 5 units. The prohibition of conjugacy, with respect to Q, of any pair of columns insists that the row containing any unit that is isolated in a column must contain a zero in every column wherein 5 units appear, and this restriction on the matrix of Γ selects, when combined with the requirement of non-singularity, precisely the 8 ! matrices described above.

Operations of periods 2 and 3

12. The transposition $x_0 \longleftrightarrow x_1$ belongs to Γ; it is an involution $\mathscr{I}$ for which every point in the T $x_0 = x_1$ is invariant. The points outside T are transposed in pairs, the join of each such pair containing $(1, 1, 0, 0, 0, 0, 0)$ which one may call the *centre* of $\mathscr{I}$. There are 63 such involutions in Γ, one centred at each p, and it is seen, as in § 25 of (7), that $\mathscr{I}_1$, $\mathscr{I}_2$ commute or do not commute according as the join of their centres is a t or an s. $\mathscr{I}$ can be identified not only by its centre p but by its *pole* m on kp, and its united points constitute the tangent prime of Q at the pole; $\mathscr{I}_1$, $\mathscr{I}_2$ commute or do not commute according as the join of their poles is a g or a c. When a set of $\mathscr{I}$ is such that every two of its members commute their poles span a linear space on Q, so that the maximum number in such a set is 7, with poles in a plane d; thus the 135 d correspond to the 135 sets of 7 mutually orthogonal hyperplanes of symmetry of the regular polytope in euclidean space of 7 dimensions ((8) 105) and just as each such hyperplane belongs to 15 sets of 7 so each m lies in 15 d.

Since 2 m whose join is a g are accompanied thereon by a third, each pair of commutative $\mathscr{I}$ determines a third; these 315 *triples* are Frame's 'cubic sets' ((8) 105). The entry $3+12$ below E in Table 1 is relevant to Frame's statements at this juncture. The circumstances in the present setting are as follows. There are 945 products of pairs of commuting $\mathscr{I}$ and 315 triples. There are $135 \times 28 = 3780$ other sets of 3 mutually commuting $\mathscr{I}$ whose poles are not, as with a triple, collinear but span a plane d. As the product of 7 $\mathscr{I}$ with poles in d is identity (see below for an example) there are no other products of mutually commuting $\mathscr{I}$ that provide involutions in Γ.

From 7 mutually commuting $\mathscr{I}$ one derives an elementary abelian group consisting of (i) identity, (ii) 7 $\mathscr{I}$, (iii) 21 products of pairs of $\mathscr{I}$, (iv) 7 products of a triple of $\mathscr{I}$, (v) 28 products of 3 $\mathscr{I}$ not belonging to a triple. As an instance take d to be the plane

$$x_0 = x_1, \qquad x_2 = x_3, \qquad x_4 = x_5, \qquad x_0+x_2+x_4 = 0. \tag{12.1}$$

Then the coordinate vector of every one of its points is a latent column vector for each of the $2^6 = 64$ matrices

$$\begin{pmatrix} \xi & \eta & \zeta \\ \alpha & \beta & \gamma \end{pmatrix} \equiv \begin{bmatrix} \xi+1 & \xi & \alpha & \alpha & \alpha & \alpha & \cdot \\ \xi & \xi+1 & \alpha & \alpha & \alpha & \alpha & \cdot \\ \beta & \beta & \eta+1 & \eta & \beta & \beta & \cdot \\ \beta & \beta & \eta & \eta+1 & \beta & \beta & \cdot \\ \gamma & \gamma & \gamma & \gamma & \zeta+1 & \zeta & \cdot \\ \gamma & \gamma & \gamma & \gamma & \zeta & \zeta+1 & \cdot \\ \beta+\gamma & \beta+\gamma & \gamma+\alpha & \gamma+\alpha & \alpha+\beta & \alpha+\beta & 1 \end{bmatrix}.$$

Clearly
$$\begin{pmatrix}\xi & \eta & \zeta\\ \alpha & \beta & \gamma\end{pmatrix}\begin{pmatrix}\xi' & \eta' & \zeta'\\ \alpha' & \beta' & \gamma'\end{pmatrix} = \begin{pmatrix}\xi+\xi' & \eta+\eta' & \zeta+\zeta'\\ \alpha+\alpha' & \beta+\beta' & \gamma+\gamma'\end{pmatrix}$$
so that every matrix save I has period 2 and every pair of matrices commutes. Whatever any of ξ, η, ζ, α, β, γ, whether 0 or 1, each column has 1, 4, or 5 of its members non-zero, and the other conditions on a matrix of Γ are readily verified; this elementary abelian group of order 2^6 is a normal subgroup of the stabilizer of (12.1) of order $288\times 7!/135 = 2^9.3.7$, the quotient group being Klein's group of order 168.

The points of (12.1) are the vertex X_6 of Σ_0 together with

$$\begin{array}{lll}(0, 0, 1, 1, 1, 1, 1); & (1, 1, 0, 0, 1, 1, 1); & (1, 1, 1, 1, 0, 0, 1);\\ (0, 0, 1, 1, 1, 1, 0); & (1, 1, 0, 0, 1, 1, 0); & (1, 1, 1, 1, 0, 0, 0).\end{array}$$

The involutions whose poles are the first three are the transpositions

$$x_0 \longleftrightarrow x_1, \qquad x_2 \longleftrightarrow x_3, \qquad x_4 \longleftrightarrow x_5;$$

their matrices are

$$\begin{pmatrix}1, 0, 0\\ 0, 0, 0\end{pmatrix} \quad \begin{pmatrix}0, 1, 0\\ 0, 0, 0\end{pmatrix} \quad \begin{pmatrix}0, 0, 1\\ 0, 0, 0\end{pmatrix}.$$

The involutions whose poles are the second three have matrices

$$\begin{pmatrix}0, 0, 0\\ 1, 0, 0\end{pmatrix} \quad \begin{pmatrix}0, 0, 0\\ 0, 1, 0\end{pmatrix} \quad \begin{pmatrix}0, 0, 0\\ 0, 0, 1\end{pmatrix}.$$

The product of the 6 (commutative) matrices is

$$\begin{pmatrix}1, 1, 1\\ 1, 1, 1\end{pmatrix},$$

and this is the matrix of that $\mathscr{I}$ having X_6 for pole.

13. The 7-rowed permutation matrices Π constitute the stabilizer of Σ_0 in Γ, and this is one of 288 conjugate subgroups $\mathscr{S}_7$. The space filled by those points that are invariant under a projectivity imposed by a permutation matrix has to include k; the dimension of the space, and its relation to Q, depend on the cycle type of the permutation and so on a partition of 7. Take, for example, the partition $2^2 3$; the space of invariant points has equations like

$$x_0 = x_1, \qquad x_2 = x_3, \qquad x_4 = x_5 = x_6;$$

these represent a plane, in fact an e_0, the m therein being

$$(1, 1, 1, 1, 0, 0, 0), \qquad (0, 0, 1, 1, 1, 1, 1), \qquad (1, 1, 0, 0, 1, 1, 1),$$

and so collinear. That $1^3 4$ leads to a solid ψ_0 has been noted in § 7; so in fact does $1^2 2 3$, which implies equations like $x_2 = x_3$, $x_4 = x_5 = x_6$. And so on. If the partition is $\lambda_1 \lambda_2 \ldots$ the space of invariant points has dimension

$6-\sum_i(\lambda_i-1)$, one less than the number of parts in the partition. The complete list of partitions, with the dimension and category of each space of invariant points, is given in Table 2.

TABLE 2

Partition	*Number of parts*	*Dimension of invariant space*	*Category of invariant space*
1^7	7	6	[6]
$1^5 2$	6	5	T
$1^4 3$	5	4	J_0
$1^3 4$	4	3	ψ_0
$1^3 2^2$	5	4	E
$1^2 23$	4	3	ψ_0
$1^2 5$	3	2	j_0
16	2	1	t_0
124	3	2	e_0
12^3	4	3	γ
13^2	3	2	j_0
$2^2 3$	3	2	e_0
25	2	1	t_0
34	2	1	t_0
7	1	0	k

Of the 3 types of involutions in this table those labelled $1^5 2$ are $\mathscr{I}$, those labelled $1^3 2^2$ products of pairs of commuting $\mathscr{I}$, those labelled 12^3 products of 3 commuting $\mathscr{I}$ that are not a triple. The 315 products of triples of $\mathscr{I}$, each having an E of invariant points, do not have representatives in any of the 288 subgroups $\mathscr{S}_7$ of Γ.

14. The cyclic permutation $(x_0 x_1 x_2)$ belongs to $\mathscr{S}_7$ and so to Γ; it has period 3, with every point in the J_0, $x_0 = x_1 = x_2$ invariant. If P is outside J_0, and not on the s of which J_0 is the polar, the plane of the cycle generated by P contains this s, whose equations are

$$x_0+x_1+x_2 = x_3 = x_4 = x_5 = x_6 = 0; \tag{14.1}$$

hence, in order to identify this cycle, one takes the plane Ps and omits its intersection with J_0. One may thus speak of an *axial* operation, s being the axis. Any operation in Γ that has period 3 leaves at least one of the 28 S invariant and so belongs to the 'cubic surface' group that is the stabilizer of this S in Γ.

The permutations $(x_0 x_1 x_2)$ and $(x_4 x_5 x_6)$ commute; the axis of the first is given by (14.1), that of the second by

$$x_0 = x_1 = x_2 = x_3 = x_4+x_5+x_6 = 0.$$

They span the solid

$$x_0+x_1+x_2 = x_3 = x_4+x_5+x_6 = 0$$

and are the Dandelin lines therein. Table 1 tells that there are 20 κ through s; each of these provides a companion s' such that operations with s, s' for axes commute. Thus there are, in Γ,

$$\tfrac{1}{2}(336\times 20)\times 4 = 13440$$

operations of period 3 that are products of two commutative axial operations.

In each S there are ((7), § 17) 40 trios, each consisting of 3 skew lines s, s', s'' such that the κ spanned by any two is the polar (with respect to the section of Q by S) of the third. The corresponding axial operations are commutative, and the trio thus affords 8 operations of period 3, each the product of 3 mutually commutative axial operations. But although there are 28×40 trios there are not $28\times 40\times 8$ operations of this last kind, but only one-quarter of this number, namely 2240, since each such operation arises from 4 different trios. This was not mentioned, let alone proved, in (7), so that one may give the explanation now. The trios in S fall into 40 quadruplets, each trio belonging to 4 quadruplets; each trio accounts, by 3 points on each of 3 s, for 9 p, and the trios in a quadruplet together account for all 36 p in S. The 8 operations afforded, as just described, by a trio τ consist of 4 pairs of inverse operations; each pair is associated with one of the quadruplets of which τ is a member in the sense that the pair is afforded by each trio of this quadruplet.

The lines s, s', s'' are all of them invariant under the operations of an elementary abelian group $\mathscr{E}$ of order 27. If one permits them to undergo cyclic permutation one obtains a Sylow subgroup of Γ (and of one of its 'cubic surface' subgroups) of order 81 with $\mathscr{E}$ as a normal subgroup. The Sylow 2-subgroups of Γ are of order 2^9; their index is 945, and they are the stabilizers of e and/or ϕ.

The normalizers of certain operations

15. The number of matrices in Γ conjugate to a given one M is the index in Γ of the normalizer of M. This normalizer consists of those matrices in Γ that commute with M; when their number is small, and an explicit form for them is available, the (large) number in the conjugate set that M belongs to is found at once.

Take, as a first instance, M to be a permutation matrix Π of period 7; it commutes only with matrices

$$a_0 I+a_1\Pi+a_2\Pi^2+a_3\Pi^3+a_4\Pi^4+a_5\Pi^5+a_6\Pi^6. \qquad (15.1)$$

This is seen either, as in the examples that follow this, by writing out the general form of the commuting matrix or else by observing that Π has, in the extension $GF(2^6)$ of $\mathbf{F}$, all its latent roots distinct. Each matrix (15.1) is a circulant, and, since every row is composed of all the a_i

$$a_0+a_1+a_2+a_3+a_4+a_5+a_6 = 1,$$

when (15.1) belongs to Γ. Thus an odd number, 1 or 5, of the a_i are units. The latter alternative violates the condition that no two columns of the circulant are coordinate vectors of conjugate points on Q; only the former alternative is therefore allowable and so Π commutes, in Γ, only with its own powers. Thus, its normalizer having order 7, Π is one of a conjugate set of 207360.

16. The matrix

$$\begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix} \equiv \Pi_1 \oplus \Pi_2 \tag{16.1}$$

is of period 12 and commutes only with matrices of the form

$$\begin{bmatrix} a & b & c & \alpha & \alpha & \alpha & \alpha \\ c & a & b & \alpha & \alpha & \alpha & \alpha \\ b & c & a & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & \beta & d & e & f & g \\ \beta & \beta & \beta & g & d & e & f \\ \beta & \beta & \beta & f & g & d & e \\ \beta & \beta & \beta & e & f & g & d \end{bmatrix}.$$

Such a matrix does not belong to Γ unless

$$a+b+c = 1, \qquad d+e+f+g = 1+\beta,$$

nor, since no two columns (say the third and fourth) can represent conjugate points on Q, unless

$$(a+b+c+\beta)(d+e+f+g) = 1.$$

Consistency prohibits $\beta = 1$; hence $\beta = 0$ and only one of a, b, c is 1 because of the restrictions on the number of units in a column. Then, by these same restrictions,

either $\alpha = 0$ and only one of d, e, f, g is 1,

or $\alpha = 1$ and only one of d, e, f, g is 1.

Thus some power of Π_1 occupies the top left-hand, some power of Π_2 the bottom right-hand block and, there being two choices for α, the normalizer has order $3\times4\times2 = 24$.

The cyclic group $\mathscr{C}_{12}$ of powers of (16.1) is a subgroup of this normalizer, which indeed is a direct product $\mathscr{C}_{12} \times \mathscr{C}_2$, the generator of $\mathscr{C}_2$ being

$$\begin{bmatrix} 1 & . & . & 1 & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 1 & 1 & 1 & 1 \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{bmatrix}. \tag{16.2}$$

This generator is the involution $\mathscr{I}$ whose pole (0, 0, 0, 1, 1, 1, 1) is on that line t_0 whose points are invariant under (16.1). The conjugate set that includes (16.1) has 60480 members.

17. The normalizer of

$$\begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & . & . & . & . \end{bmatrix} \tag{17.1}$$

consists of matrices having the form

$$\begin{bmatrix} a & b & \alpha & \alpha & \alpha & \alpha & \alpha \\ b & a & \alpha & \alpha & \alpha & \alpha & \alpha \\ \beta & \beta & c & d & e & f & g \\ \beta & \beta & g & c & d & e & f \\ \beta & \beta & f & g & c & d & e \\ \beta & \beta & e & f & g & c & d \\ \beta & \beta & d & e & f & g & c \end{bmatrix}$$

wherein, to obviate singularity, $b = a+1$. Hence, since the marks in a row sum to 1, $\alpha = 0$; also, to obviate a superfluity of units in either of the first two columns, $\beta = 0$, this implying further that $c+d+e+f+g = 1$. Since c, d, e, f, g cannot (again to obviate singularity) all be equal these marks fill one of 5 permutation matrices and the normalizer has order 10 since a can be 0 or 1. Thus (17.1) is one of a conjugate set in Γ of 145152 matrices, of period 10.

If the first component of the direct sum (17.1) is replaced by $\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}$ the matrices of the normalizer take the form

$$\begin{bmatrix} a & b & \alpha & \alpha & \alpha & \alpha & \alpha \\ a' & b' & \beta & \beta & \beta & \beta & \beta \\ \gamma & \delta & c & d & e & f & g \\ \gamma & \delta & g & c & d & e & f \\ \gamma & \delta & f & g & c & d & e \\ \gamma & \delta & e & f & g & c & d \\ \gamma & \delta & d & e & f & g & c \end{bmatrix}. \tag{17.2}$$

If $\gamma = 1$ then, by the restriction on units, $a = a' = 0$. Then, to obviate singularity, $\delta = 0$; this allows only a single unit in the second column, so that $b' = b+1 = \alpha = \beta+1$. Since $c+d+e+f+g = 1+\gamma+\delta = 0$ an even number of c, d, e, f, g are units; this even number is, to give the proper quota of units to a column, 4. Hence, with 2 choices for b and 5 for the zero among c, d, e, f, g there are 10 such matrices in the normalizer. Another 10 have $\delta = 1, \gamma = 0$. There remains only the possibility that $\gamma = \delta = 0$, implying

$$a' = a+1 = b = b'+1, \qquad \alpha = \beta = 0,$$
$$c+d+e+f+g = 1,$$

and so there occur the 10 matrices of the normalizer $\mathscr{C}_{10}$ of (17.1). This occurrence is inevitable, since if a matrix commutes with (17.1) it commutes too with its powers (the relevant power here is 6). The whole normalizer N has order 30, so that there is in Γ a conjugate set of 48384 operations of period 5.

N includes, outside $\mathscr{C}_{10}$, a pair of inverse operations of period 3, namely

$$\begin{bmatrix} . & 1 & . & . & . & . & . \\ . & . & 1 & 1 & 1 & 1 & 1 \\ 1 & . & . & 1 & 1 & 1 & 1 \\ 1 & . & 1 & . & 1 & 1 & 1 \\ 1 & . & 1 & 1 & . & 1 & 1 \\ 1 & . & 1 & 1 & 1 & . & 1 \\ 1 & . & 1 & 1 & 1 & 1 & . \end{bmatrix} \text{ and } \begin{bmatrix} . & . & 1 & 1 & 1 & 1 & 1 \\ 1 & . & . & . & . & . & . \\ . & 1 & . & 1 & 1 & 1 & 1 \\ . & 1 & 1 & . & 1 & 1 & 1 \\ . & 1 & 1 & 1 & . & 1 & 1 \\ . & 1 & 1 & 1 & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & . \end{bmatrix}. \tag{17.3}$$

The involution $x_0 \longleftrightarrow x_1$ in $\mathscr{C}_{10}$ generates, with either of these, a dihedral group $\mathscr{D}_6$ which is a direct factor, not merely a subgroup, of N: $N \cong \mathscr{D}_6 \times \mathscr{C}_5$. For it is at once seen that the square of (17.1) commutes with (17.3). Thus N includes operations of period 15, and as each of these has the same cube as its eleventh power there are (at least) 96768 operations of period 15 in Γ—twice as many as there are of period 5.

18. Operations of period 15 occur in an $\mathscr{S}_8$ in association with the partition *35*; thus those in Γ can be found directly by subjecting the heptagons of one of the 36 sets of 8 to appropriate permutations. Since each column of each matrix of N sums to 1, N belongs to that $\mathscr{S}_8$ which permutes the $\mathfrak{H}_i$ (9.1). Let these suffixes i undergo the permutation (34567)(210). The common vertex of $\mathfrak{H}_3$ and $\mathfrak{H}_4$ is moved to that of $\mathfrak{H}_4$ and $\mathfrak{H}_5$, and so on; that of $\mathfrak{H}_1$ and $\mathfrak{H}_0$ to that of $\mathfrak{H}_0$ and $\mathfrak{H}_2$. Hence one requires to solve

$$M\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 \end{bmatrix}$$

for M. The solution is

$$M = \begin{bmatrix} A+1 & A+1 & A & A & A & A & A \\ B+1 & B & B & B & B & B & B \\ C+1 & C & C & C & C & C & C+1 \\ D+1 & D & D+1 & D & D & D & D \\ E+1 & E & E & E+1 & E & E & E \\ F+1 & F & F & F & F+1 & F & F \\ G+1 & G & G & G & G & G+1 & G \end{bmatrix}$$

wherein, in order that each row sum to 1, all of A, B, C, D, E, F, G are 1 save B, and $B = 0$. These conditions cause M to belong to N (put $g = 0$, $\delta = 1$, etc., in (17.2)).

The half-integer characteristics for genus 3

19. The figure in [6] that is the basis of the geometry, as of the algebra, in the preceding pages is a setting for Γ, 'the group of the bitangents'. These bitangents are the 28 contact ϕ-curves of a canonical curve of genus 3, and one therefore expects the geometry to map, in some way, the period characteristics and theta characteristics for this genus. These concluding paragraphs outline one or two features of this mapping.

There are 63 half-integer period characteristics ξ in addition to the zero; these are mapped by the 63 m or, as there is a (1, 1) correspondence between the m and p set up by collinearity with k, by the 63 p, and so also by the 63 involutions $\mathscr{I}$. Period characteristics ξ_1, ξ_2 can be either syzygetic or azygetic; if they are syzygetic $m_1 m_2$ is g, $p_1 p_2$ is t, $\mathscr{I}_1$ and $\mathscr{I}_2$ commute; if they are azygetic $m_1 m_2$ is c, $p_1 p_2$ is s, $\mathscr{I}_1$ and $\mathscr{I}_2$ do not commute. If $\xi_1 \equiv \xi_2$ there is always syzygy.

ξ_1 and ξ_2 have a 'sum'; this is syzygetic or azygetic to both of them according as they are syzygetic or azygetic to each other. Suppose that there is syzygy. Then the sum ξ_3 is mapped by the other point m_3 on $g \equiv m_1 m_2$, or by p_3 in $e_0 \equiv kp_1 p_2$, or by that involution $\mathscr{I}_3$ that completes a triple with $\mathscr{I}_1$ and $\mathscr{I}_2$. If, on the other hand, there is no syzygy ξ_3 is mapped by m_3 in $j_0 \equiv km_1 m_2$, or by the other point p_3 on $s \equiv p_1 p_2$, or by $\mathscr{I}_3$ whose pole is m_3 and centre p_3. There are 315 syzygetic triples, one for each g and its accompanying e_0, and 336 azygetic triples, one for each j_0. Each period characteristic of a triple is the sum of the other two; the sum of the coordinate vectors of the three mapping points has all its 7 components equal: to 0 if the map is by m and there is syzygy or if the map is by p and there is no syzygy, to 1 in opposite circumstances.

This distinction, between syzygy and its absence, indicates the assembling of sets of period characteristics every two of which bear the same relation to each other. If every two are in syzygy the set can be mapped by a set of m every two of which are joined by a g, so that every point of the space spanned by them is on Q. Thus, as there are planes, but no solids, on Q it is possible for every pair among 7 period characteristics to be in syzygy and, as there are 135 planes d, there are 135 such sets of 7. When one supplements such a set by the zero characteristic, which is in syzygy with every other, one obtains a Göpel group ((**1**) 490) of 2^3 mutually syzygetic characteristics. Should, on the other hand, every two of a set be azygetic the join of any two among the m which map the characteristics is a c. One has encountered above (§ 10) sets of 7 m with this property, namely, the vertices of any of the 288 Σ; these correspond, for $p = 3$, to the fundamental systems of $2p+1$ period characteristics.

20. The 64 theta characteristics fall into two batches, 28 being odd and 36 even; they are mapped by 28 S and 36 C. Here, in contrast to the situation with period characteristics, there is no distinction to be made in the relations between pairs of characteristics; nor, save that it is even, has the zero characteristic any special attribute. It is only between sets of 3 theta characteristics that there is, or is not, syzygy. This is mirrored with crystal clarity in the geometry. The intersection of two S is always H, the ${}^{28}C_2 = 378$ H being the intersections of pairs of 28 S; the intersection of two C is always F, the ${}^{36}C_2 = 630$ F being the intersections of pairs of 36 C; the intersection of S and C is always J, the $28 \times 36 = 1008$ J being the intersections of 28 S with 36 C. But, as is seen from the last two rows of Table 1,

S_1, S_2, S_3 may meet in λ or in χ,
C_1, C_2, C_3 may meet in κ or in ϕ,
S_1, S_2, C_3 may meet in λ or in ψ,
S_1, C_2, C_3 may meet in κ or in ψ.

In the latter contingencies there is syzygy, in the former not. Syzygy occurs when the solid common to the 3 spaces meets Q in a singular quadric; if this section is not singular there is no syzygy.

There pass through any solid 4 [5]'s that do not contain k; thus associated with any 3 theta characteristics there is a fourth, and every 3 of these 4 characteristics are in the same relation. This fourth characteristic is, of course, the 'sum' of the other 3. Each theta characteristic can be represented by the row vector of coordinates of the S or C which maps it; of the 7 components of this vector an odd number are 1. The even characteristics occur when this odd number is 3 or 7, the odd ones when it is 1 or 5 (see § 3). The zero characteristic, being even, can correspond to any of 35 vectors; it is natural, though in no way necessary, to map it, once the coordinate system has been set up, by the unit prime so that its representative vector has every component 1. The sum of the theta characteristics in a set is then represented by the sum of their representative vectors provided that the number in the set is odd. One can always take this number to be odd: were it even one would merely have to add the zero characteristic. But, if this artifice be not resorted to, one obtains the representative vector of the sum of an even number of theta characteristics by adding their representative vectors and then, in the outcome of this addition, replacing each 1 by 0 and each 0 by 1.

The vector sum of the representatives of two theta characteristics has an even number of its components 1 and so is the coordinate vector of a prime T; thus it maps, via the contact m of T with Q, a period characteristic. Details given in (v) on p. 188 of (**2**) are, for genus 3, in accord with the geometry here. Take, say, a prime C_0; the remaining 35 C and the 28 S meet C_0 in 35 F and 28 J; these 63 spaces are joined to k by the 63 T so that each of the 63 (non-zero) period characteristics is obtainable as a 'sum' of the theta characteristic mapped by C_0 with some other. And a like result follows if one takes a prime S_0. On the other hand, to proceed oppositely, choose any point m_0, on Q, and so the tangent prime T_0. The 32 [4]'s in T_0 that do not contain k consist, as explained in § 4, of

6 H, each common to two S,
10 F, each common to two C,
16 J, each common to an S and C.

Thus a given period characteristic can be obtained as a 'sum' of two theta characteristics in 32 ways. In precisely half of these the theta characteristics have opposite parity; of those wherein they have like parity 10 make it even, 6 odd. And these two numbers are the values, for $p = 3$, of $2^{p-2}(2^{p-1}+1)$ and $2^{p-2}(2^{p-1}-1)$.

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