THE GEOMETRY OF AN ORTHOGONAL GROUP IN SIX VARIABLES

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THE subject of this paper, a 'simple' group of order $2^7.3^6.5.7$, is familiar (**5**) in its representation as a group of collineations in complex projective space of five dimensions. It has, however, orthogonal and unitary representations in finite projective spaces of five and three dimensions respectively; these have not been studied, and here some account is offered of the geometry of the orthogonal one. The nucleus of the figure is its invariant quadric Q; noteworthy features are (§§ 28–33) the 540 null systems which reciprocate Q into itself and (§§ 18–27) 5184 heptahedra, of two categories, that are circumscribed to Q. The figure also provides clear definitions of the Sylow subgroups S_2 (§§ 7–8) and S_3 (§§ 15–17).

1. The geometry of quadrics in projective spaces of 2, 3, 4 dimensions over the Galois field $\mathscr{K}$ of three marks has been described in earlier papers (**2, 3**). A non-singular conic σ consists of four of the 13 points that compose the finite plane, and the quadrilateral of tangents thereat has six vertices; hence there are only three points that do not lie on any tangent, and they form a triangle whose sides are the only lines of the plane skew to σ and which is self-polar for σ. Elementary though this figure is (see the diagram on p. 264 of (**2**)) reference to it is often helpful because, as well as being a section of non-singular quadrics in spaces of higher dimension, it is the section of a quadric cone in [3]—for instance, of the intersection of a non-singular quadric in [4] with a tangent solid—by any plane not passing through its vertex.

There are two kinds of non-singular quadric in [3]; two kinds, that is, in that neither can be transformed into the other without extending $\mathscr{K}$. One, the ruled quadric, is a hyperboloid H whose two complementary reguli each consist of four lines. Of the lines in [3] 18 are skew to H; they are edges of two triads of desmic tetrahedra. These six tetrahedra are all self-polar for H but the two triads cannot be transposed by any quaternary orthogonal transformation with coefficients in $\mathscr{K}$. The other kind of quadric has no lines on it and is an ellipsoid F; it involves two systems of six pentahedra whose edges all touch F. Other properties of H and F are given in (**2**); their orthogonal groups of projectivities have orders 288 and 360. The former has a subgroup of index 2 isomorphic to the direct product of two alternating groups $\mathscr{A}_4$ which act as permutation groups on the lines in its two reguli, while the latter is isomorphic to $\mathscr{A}_6$ which acts as a permutation group on either system of pentahedra.

A non-singular quadric ω in [4] consists of 40 points and has on it 40 lines; there are 27 self-polar pentagons whose edges are all skew to ω whose equation, referred to any such pentagon, is given by equating the unit quinary quadratic form to zero. The polar solids of 45 of the 81 points off ω meet it in hyperboloids—it is these 45 points that are vertices of the 27 pentagons—while those of the remaining 36 points meet ω in ellipsoids. The orthogonal group of projectivities that leave ω invariant is of order 51840 and is isomorphic to the cubic surface group; it acts as a permutation group on the 27 pentagons, the 40 lines on ω, and the batches of 40, 45, 36 points just mentioned, and affords a direct access to the study of the many properties of this group (**3**, **4**).

There are two kinds of non-singular quadric in [5]. One is the Klein quadric Ω whose 130 points represent the lines of [3]; it is ruled in the sense that planes lie on it, and the presence of these planes helps to disclose the properties of Ω. But the concern of this paper is the other kind of quadric Q on which there are, over the field $\mathscr{K}$, no planes and which consists of 112 points m. It is Q, not Ω, that is given by equating the unit senary quadratic form to zero, and so we study the geometry, over $\mathscr{K}$, of the quadric

$$x^2+y^2+z^2+u^2+v^2+w^2 = 0,$$

and use this geometry to investigate the corresponding projective orthogonal group. The marks of $\mathscr{K}$ are 0, 1, -1; they obey the usual multiplication rule, but $1+1 = -1$ as with residues *modulo* 3.

2. The finite projective space [5] wherein Q lies consists of $\frac{1}{2}(3^6-1) = 364$ points; each of the six constituents of the vector of homogeneous coordinates of a point can be any of 0, 1, -1, with the sole exception that not all six are 0, and each point answers to two vectors, negatives of each other. Points are of three categories k, l, m according as the sum of the squares of the six coordinates is 1, -1, 0; points k, l are off Q, points m on Q. The corresponding capitals K, L, M denote their polar primes; primes M are tangent primes of Q. Since the square of either non-zero mark is 1 each k has either one or four, each l two or five, each m three or six squares of its coordinates equal to 1, so that the number

$$\begin{aligned} \text{of } k \text{ is } \quad & 6+15.2^3 = 126,\\ \text{of } l \text{ is } \quad & 15.2+6.2^4 = 126,\\ \text{of } m \text{ is } \quad & 20.2^2+1.2^5 = 112.\end{aligned}$$

3. Q is invariant for all substitutions whose matrices M satisfy either $M'M = I$ or $M'M = -I$; but, in the latter contingency, although Q is left invariant in the sense that its $112\,m$ are permuted among themselves, the categories k and l are transposed, so that only those substitutions for which $M'M = I$ will be discussed. $|M|$ may be either 1 or -1; those substitutions with $|M| = 1$ form the orthogonal group and impose permutations of like parity on the two batches of 126 points, whereas those with $|M| = -1$ impose permutations of opposite parity. For example: the harmonic inversion in a point k and its polar prime imposes an even permutation on the $126\,k$, transposing each of 40 pairs, but an odd permutation on the $126\,l$, transposing each of 45 pairs. The substitutions of matrices M and $-M$ impose the same projectivity and have the same determinant; the projective orthogonal group has half the order of the orthogonal group of substitutions. In the projective group there is a subgroup of index 2 which is simple and consists of those projectivities which

(i) leave Q invariant,

(ii) do not transpose the categories k, l,

(iii) have determinant 1,

(iv) permute each batch of 126 points evenly.

This simple group is called G^* here because it is isomorphic to the group so designated by Miss Hamill at the end of (**5**). The group of the title of (**5**) has order double that of G^* and is isomorphic to the group which, while obeying (i) and (ii) and including only projectivities that permute one batch of 126 points, say the k, evenly is permitted to include projectivities, of determinant -1, that impose odd permutations on the other batch of 126 points.

The group G^* of projectivities is found by Dickson ((**1**), p. 183) in a text illustrious for the flood of its new contributions to the subject of linear groups and, at the same time, notorious for the absence of any geometrical account of them. The corresponding group of substitutions is what Dickson there denotes by $O_1'(6, 3)$.

4. Primes K, L meet Q in non-singular quadrics such as the one described in §§ 12–15 of (**3**). Such a section of Q has ((**3**), § 12) 27 self-polar pentagons, so that there are $126 \times 27/6 = 567$ 'positive' simplexes Σ^+, self-polar for Q, whose vertices are all k and faces all K; among these is Σ_0^+, the simplex of reference. There are, likewise, 567 'negative' simplexes Σ^-, self-polar for Q whose vertices are all l and faces all L. A prime M, however, meets Q in a singular quadric, a cone with m as vertex. An instance is

$$x+y+z+u+v+w = 0,$$

the tangent prime at the unit point m_0; the points of Q herein, other than m_0 itself, are manifestly those of whose coordinates three are 1 and the others -1, together with the two further points (these have three co-ordinates zero and the other three equal) on each of the lines that join the ten to m_0. The cone consists of ten lines g and projects an ellipsoid from a point outside the solid that contains it.

5. The conditions for a matrix M to satisfy $M'M = I$ are, precisely, that its columns are the coordinate vectors of 6 points k which are vertices of a Σ^+; these conditions are necessary and sufficient. Since there are 6! permutations of the columns, and since each column may be replaced by its negative without changing the k of which it is the coordinate vector, the number of orthogonal matrices of determinant $\Delta = 1$ is

$$\tfrac{1}{2}.567.6!\,2^6 = 2^9.3^6.5.7,$$

and so, since each is here accompanied by its negative, there is a group of $2^8.3^6.5.7$ orthogonal projectivities. This, with all its members having $\Delta = 1$, is not the group of this order that provides the title of (**5**); but both these groups have G^* as a subgroup of index 2. The above group includes projectivities that permute both batches of 126 points oddly; an instance is the harmonic inversion whose fundamental spaces are any chord c of Q and its polar solid C (which meets Q in an ellipsoid). For, of the points k, one is on c and 15 in C, leaving 110 to undergo transpositions of 55 pairs; and likewise with the points l. Thus G^*, wherein the permutations have to be even, has order $2^7.3^6.5.7$. It includes the harmonic inversions whose fundamental spaces are any edge and opposite solid of any Σ^+ for, of the points k, two, namely vertices of Σ^+, are on the edge and 12 in the solid, leaving 112 to undergo transpositions of 56 pairs. These inversions are, in fact, the only involutions in G^*.

6. G^* is transitive on the 567 Σ^+. For any Σ^+ is obtained from Σ_0^+ by using a matrix whose columns are the coordinate vectors of the required simplex. Should this matrix, with its columns arranged to make $\Delta = 1$, subject the k to an odd permutation it is only necessary to multiply it by another which, also having $\Delta = 1$ and subjecting the k to an odd permutation, does not change the former matrix except in so far that it may permute the columns and may multiply any of them by -1. Such an auxiliary is, for instance,

$$\begin{bmatrix} . & 1 & & & & \\ -1 & . & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \tag{6.1}$$

which leaves invariant those 12 k for which $x = y = 0$, transposes the two vertices of Σ_0^+ that lie outside this solid, and permutes the remaining 112 k in 28 cycles of four. G^* is, indeed, transitive on the plane faces of the Σ^+; it is now only necessary to establish transitivity on the plane faces of Σ_0^+, or triple transitivity on the primes of reference. Since, as we shall see at once, all the even permutation matrices belong to G^* the statement is proved.

Since G^* is transitive on the Σ^+ the stabilizer of Σ_0^+ has order $2^7.3^2.5$; the projectivities which constitute it are imposed by monomial matrices—by those matrices, that is, with only a single non-zero element, and that either 1 or -1, in any row or column. There are $2^6.6!$ monomial matrices, of which $2^5.6!$ have $\Delta = 1$; since each of these is accompanied by its negative they impose $2^4.6!$ projectivities. This is double the order of the stabilizer, the reason being that half of them, including (6.1), subject the 126 k to odd permutations and so are extraneous to G^*. The criterion which serves to discard these extraneous matrices is a simple one: those monomial matrices that impose odd permutations on the vertices of Σ_0^+ do so also on the whole set of 126 k. It is enough to show this for any of the 15 transpositions of pairs of vertices of Σ_0^+, and this has just been done. Though one should add that those projectivities in the stabilizer that impose the identity permutation on the vertices of Σ_0^+ all permute the 126 k evenly; the corresponding matrices are diagonal, for example

$$\mathrm{diag}(-1, -1, 1, 1, 1, 1)$$

for which the invariant k are the vertices of Σ_0^+ and eight more k having $x = y = 0$, $z^2 = u^2 = v^2 = w^2 = 1$; the remaining 112 undergo 56 transpositions. The criterion is now fully established.

7. The way is now open to identify the Sylow 2-groups S_2, of order 2^7, of G^*, and to find how many there are. Take the stabilizer of Σ_0^+, of order $2^7.3^2.5$, whose projectivities are imposed by what we may now call even (monomial) matrices. There are 15 ways of selecting three edges of Σ_0^+ that together account for all six vertices; these 15 trios of edges, each edge the polar of the solid spanned by the other two, are permuted transitively by the stabilizer, and so each trio is, while admitting permutations of its three edges and transposition of the vertices on any edge, invariant for a subgroup of $2^7.3$ projectivities. This itself has subgroups of order 2^7 for which a single edge of the trio is invariant, and these are S_2. Each S_2 has a normal subgroup of order 2^6, the intersection of three S_2, for which all three edges of the trio are invariant. There is also a normal abelian subgroup of order 2^4 consisting of those operations for which every vertex of Σ_0^+ is invariant;

this abelian subgroup belongs to 45 S_2 and is elementary, all its operations save identity having period 2.

Each line s skew to Q is an edge of three Σ^+ whose other vertices, as seen in (**2**), form a triad of desmic tetrahedra in the polar solid of s. For $s_0 : x = y = z = u = 0$ these tetrahedra T, Q, R in $v = w = 0$ are displayed on p. 268 of (**2**). Take, as a trio of edges of Σ_0^+, the intersections of the pairs of solids

$$x = y = 0, \qquad z = u = 0, \qquad v = w = 0,$$

and, when a definite S_2 is in question, let it be $\mathscr{S}$ which, in addition to keeping this trio fixed, leaves s_0 invariant. One of its projectivities is imposed by, let us say,

$$\begin{bmatrix} . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & 1 & . & . & . & . \\ -1 & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & -1 & . \end{bmatrix}. \tag{7.1}$$

This transposes Q and R so that, although s_0 is an edge of three Σ^+, no S_2 associated with any of these three can be the same as the one associated with another. Now s_0 belongs to three trios of edges of Σ_0^+, and the S_2 associated with s_0 in these trios are distinct. For each trio is completed by combining with s_0 one of the three pairs of opposite edges of T and while (7.1), as postulated, leaves one pair invariant (transposing its two members) it interchanges the other two pairs and so does not belong to the S_2 associated with s_0 in either of these two trios. Hence the number of S_2 in G^* is

$$567.15.3 = 25515.$$

The simplexes Σ^- also provide an approach to the S_2; but the same S_2 are obtained as from the Σ^+ because any trio of edges of a Σ^+ is also a trio of edges of a Σ^-, and conversely (see § 29 below).

The operation (7.1) belongs to a single S_2, namely $\mathscr{S}$. For, since its characteristic polynomial $(\lambda^4+1)(\lambda^2+1)$ is devoid of roots in $\mathscr{K}$ it does not leave any point of [5] invariant; any pairs of points that it transposes must be invariant for its square which on multiplying

$$(x, \quad y, \quad z, \quad u, \quad v, \quad w)'$$

produces

$$(y, \quad -x, \quad u, \quad -z, \quad -v, \quad -w)'.$$

These non-zero vectors cannot be identical; but they can be negatives of each other when, and only when, $x = y = z = u = 0$; they are then coordinate vectors of a point on s_0. Hence (7.1) transposes in pairs the points of s_0, and only these, and cannot belong to any S_2 other than those defined by s_0 and the trios involving it. It has just been demonstrated that $\mathscr{S}$ is the only such S_2 to which (7.1) does belong.

8. The period of the matrix (7.1) is 8. The number of such operations in G^* was calculated ((**5**), pp. 449–51) by Miss Hamill by a process that was the culmination of a long exposition of steadily increasing elaboration. It is of interest that this other, orthogonal, representation of G^* affords a much more direct route to these operations. All that is now necessary for calculating their number is to calculate the number in an S_2 and then, as each belongs to a unique S_2, multiply by 25515. The calculation is elementary, and can run as follows.

All operations of $\mathscr{S}$ are given by 6×6 matrices partitioned into nine blocks of 2×2 matrices, six of these nine blocks being composed of zeros. The other three blocks are the tail, at the bottom right-hand corner, and two others either both on the diagonal or else, as in (7.1), both off it. Should they be on the diagonal the square is a diagonal matrix and the projectivity of period 2 or 4; furthermore, should they be off the diagonal and both, as 2×2 monomial matrices, of the same kind, either diagonal or non-diagonal, the same situation prevails. Hence the only matrices that can yield operations of period 8 have their non-zero elements either as in (7.1) or in the transpose of (7.1). The fourth power is

$$\operatorname{diag}(\pi, \pi, \pi, \pi, 1, 1),$$

where π is the product of the non-zero elements in the top four rows, and so the period is 8 if and only if $\pi = -1$. This allows 2^3 choices when these non-zero elements are disposed as in (7.1), and then, as the elements in the tail must be off its diagonal to provide an even matrix, and the determinant is prescribed, 2 choices for the tail; hence there are 16 matrices, each being accompanied by its negative. Their transposes provide 16 more, and so $\mathscr{S}$ includes 16 projectivities of period 8. The number of these in G^* is thus (cf. (**5**), p. 451) 408240.

An incidental observation is that an S_2 does not have any operations of period exceeding 8.

9. In the geometry of Q, the mere numbers of points and lines on it are certainly known, but it is the different subspaces in [5] that have to be enumerated and their relations to Q and to one another described. We first examine how Q partitions the lines of [5], and it is convenient to retain the notation of (**3**), denoting

generators, lying wholly on Q, by g,
chords, meeting Q in two points, by c,
tangents, meeting Q in a single point, by t,
lines skew to Q by s.

The t divide among positive and negative categories, p and n, according as the three points on t other than its contact are all k or all l.

Any g or t through m has to lie in the tangent prime M and since, as already noted, there are ten such g the total number of g is $112 \times 10/4 = 280$, each g consisting of four m. Moreover, since there are $15\,p$ and $15\,n$ through m (answering to the two categories of 15 points off an ellipsoid) the number of p, as of n, is 1680. Now there pass through m 121 lines in all because there are, over $\mathscr{K}$, $\frac{1}{2}(3^5-1) = 121$ points in a [4]; of these 10, 15, 15 are g, p, n; 81 remain. These are c, and indeed join m to the $112-1-30 = 81$ points of Q outside M. Hence there are $112 \times 81/2 = 4536\,c$. As for s, it has been noted that each is an edge of three Σ^+ and so, since each of the 567 Σ^+ has 15 edges, the number of s is $567 \times 15/3 = 2835$.

The polars of lines g, c, p, n, s are solids G, C, P, N, S; S meets Q in a ruled quadric or hyperboloid as in § 6 of (**2**) while C meets Q in a non-ruled quadric or ellipsoid as in § 12 of (**2**). Through C pass two M, and the ellipsoid is the section by either of the cone in which the other meets Q. But G, P, N meet Q in singular quadrics.

Consider the section by P, the polar solid of p. If m is the contact of p and M the tangent prime there p lies in M as well as passing through m; hence P passes through m as well as lying in M. P and p are polars not only for Q but also for the cone in which M meets Q. Hence, observing the section by a solid lying in M but not passing through m, the section of Q by P is the cone which projects from m the section of an ellipsoid by the polar plane, with respect to this ellipsoid, of a point k. This plane section (cf. the figure on p. 264 of (**2**)) consists of four points: the diagonal points of this quadrangle are three l forming a self-polar triangle whose sides are s, while the plane is completed by six k, vertices of a quadrilateral of p. Thus P meets Q in four g, the lines of intersection of planes spanned by complementary pairs of these g being n. The section by N is analogous, the roles of p and n being interchanged.

A like argument shows G to meet Q in the cone which projects, from any point of g, the section of an ellipsoid by one of its tangent planes; this section is the point of contact, so that G meets Q in g alone. There are four planes in G through g; the properties of the four tangent lines at a point on an ellipsoid ((**2**), § 14) imply that all points in such a plane γ are of the same category, that two planes γ^+ and two planes γ^- constitute the four, and that each pair of similarly signed γ are polars of each other; γ^+ has k while γ^- has l for all its nine points off g. Such planes occur too in the solids P and N. Through any g in, say, P pass four planes lying in P; of these three join g to the other three generators in P, while the fourth is a γ^+. Likewise there are four γ^-, one through each g, in any N.

10. Q partitions the planes of [5], and γ^+, γ^- have just been recorded.

Since two of either category pass through any g there are $560\gamma^+$ and $560\gamma^-$, each aggregate consisting of 280 polar pairs. For other planes, as for γ, we retain the notation of (**3**). Every intersecting pair of g spans a plane π, of which there are $\frac{1}{2}(280\times 36) = 5040$; their polars are $5040\,\nu$ each meeting Q in a single m through which pass, in ν, two p and two n. The lines of π through m consist of two g, one p, one n. π and ν project, from their intersection m, c and s, polar lines for an ellipsoid. There remain planes e and Γ which meet Q in non-singular conics, i.e. in vertices of a quadrangle. Each e includes a unique triangle of k, as does its polar e', and the two triangles furnish a Σ^+; conversely, each Σ^+ has 20 plane faces e. Hence there are, in all, $567\times 20 = 11340\,e$; and, likewise, $11340\,\Gamma$, plane faces of simplexes Σ^-.

There is another way of calculating how many planes meet Q in non-singular conics. Such a plane is spanned by three points m_1, m_2, m_3, no two of which are conjugate. There are 112 choices for m_1; thereafter m_2 may be any of the 81 m outside M_1, m_3 any m outside both M_1 and M_2. Since M_1, M_2 intersect in a C, wherein lie ten m, there are 21 m in M_1 and outside C, and 21 m in M_2 and outside C; hence the number of m in at least one of M_1, M_2 is 52, and the number outside both M_1, M_2 is 60. The successive choices of m_1, m_2, m_3 thus number 112, 81, 60; the plane $m_1 m_2 m_3$ then automatically provides m_4. Hence, since the points could have been chosen in any of 4! sequences, the number of 'secant planes' of Q is

$$112\times 81\times 60/4! = 22680.$$

This accords with the sum of the numbers of e and Γ.

11. The table shows the number of subspaces in each space of the figure: reciprocation in Q provides the number of spaces through any subspace. Much of the table is already known: the columns headed K, L consist of the numbers along the top of Table 4 of (**3**). It is only columns such as those headed by P, N, M, spaces which give singular sections, that may require a word of explanation; these, incidentally, have been split each into two columns, the right-hand one giving the numbers of spaces through the contact with Q. Moreover one may always calculate a number by the reciprocal process. The number of e in M is the same as the number of e through m which, since each e contains four m, is $11340\times 4/112 = 405$. Or the number of s in P, being the number of S through p, is

$$2835\times 16/1680 = 27.$$

Again: any c in P joins points one on each of two g, since the only points of Q in P are on four concurrent g. Since there are six planes spanned by pairs of these g, and since each of these planes contains nine c, there are 54 c in P.

It may be borne in mind too that, since the total number of subspaces of any space is known, for instance 13 lines in a plane, 130 in a solid, 1210 in a [4], the sum of the entries in a vertical column in any compartment is known. The numbers in the table all have some relevance to a full study of the orthogonal group.

g	n	p	s	c	π	e	γ^+	γ^-	Γ	ν	C	S	P		N		G	K	L	M			
.	.	3	2	1	3	3	9	.	6	6	15	12	18	.	9	.	18	45	36	45	.	k	126
.	3	.	2	1	3	6	.	9	3	6	15	12	9	.	18	.	18	36	45	45	.	l	126
4	1	1	.	2	7	4	4	4	4	1	10	16	12	1	12	1	4	40	40	30	1	m	112
					2	.	1	1	.	.	.	8	.	4	.	4	1	40	40	.	10	g	280
					1	4	.	12	.	2	20	16	.	3	36	6	24	120	240	180	15	n	1680
					1	.	12	.	4	2	20	16	36	6	.	3	24	240	120	180	15	p	1680
					.	3	.	.	3	9	45	18	27	.	27	.	81	270	270	405	.	s	2835
					9	6	.	.	6	.	45	72	54	.	54	.	.	540	540	405	.	c	4536
											.	16	.	6	.	6	.	240	240	.	45	π	5040
											15	12	.	.	27	.	.	270	540	405	.	e	11340
											.	.	.	4	.	.	2	40	.	.	20	γ^+	560
											.	.	.	.	.	4	2	.	40	.	20	γ^-	560
											15	12	27	.	.	.	.	540	270	405	.	Γ	11340
											10	.	.	3	.	3	36	120	120	270	45	ν	5040
																		36	36	81	.	C	4536
																		45	45	.	.	S	2835
																		40	.	.	15	P	1680
																		.	40	.	15	N	1680
																		.	.	.	10	G	280

12. When the same group is represented geometrically in two different spaces the isomorphism will imply many relations and correspondences; and that the numbers in the table should, so many of them, occur in (**5**) is only to be expected. Both spaces have dimension 5, but whereas the geometry here is finite the geometry in (**5**) is of complex projective space; moreover Miss Hamill uses an antipolarity whereas here there is ordinary reciprocation in Q. These are two conspicuous contrasts amid the multitude of similarities between the figures. Miss Hamill's figure is built on 126 vertices which correspond to one of the batches of either k or l; the other batch is not, at first sight, represented although its polar primes, since they include points of both batches, are. Nor are the m represented in Miss Hamill's figure though, again, their polar primes are. If it is the 126 k that correspond to vertices then primes K, L, M correspond to primes π, β, α of (**5**), each of the 112 M answering to six of the 672 α—a word about such multiplicity in a moment—and each of the 126 L to 27 of the 3402 β.

Since Miss Hamill builds only spaces that are spanned by vertices, analogues of k, her figure lacks the analogues of those spaces whose k are inadequate to span them, namely c (lines whereon there is only one k), n, g, γ^- (devoid of k) and π (a plane wherein all k are collinear). But she produces a wider variation in types of solid and prime, and often a single space in the finite geometry has several that correspond to it in the complex geometry. It would be disproportionate to catalogue all the details, but one or two features should be underlined to illustrate the circumstances. Take, then, Miss Hamill's S-solids, polar to κ-lines; they answer to P, polar to p; but the six κ-lines *edges of a tetrahedron* in S answer to six p *concurrent* at the contact m of P. Yet just as ((**5**), p. 404) there are 36 κ-lines in S other than the edges of this tetrahedron, so there are 36 p in P that do not pass through m. Note that whereas a κ-line is skew to all those in its polar S a p is not skew to all those in its polar P; it is skew to those 36 which do not pass through m but meets those six which do. Thus a d-plane, defined in (**5**) as spanned by a κ-line and a vertex on an edge of the tetrahedron (not, that is, a vertex of the tetrahedron but one of the 126 vertices on which the figure is built), has for analogue a plane spanned by p and a k on one of those six p in P that pass through m. This analogue is a ν and can be so defined in six ways, there being a choice of two p in ν and then of three k on the other p. This is why the 5040 ν answer to 30240 d-planes. Also a U-solid, being defined as that spanned by a κ-line and one of those in the polar of κ that is not an edge of the tetrahedron, corresponds to the solid spanned by a p, say p_0, and one of those p in the polar of p_0 that do not intersect p_0. Let p' be one, among the 36 eligible, and consider the solid $[p_0 p']$. It is the polar of the g which joins the contacts of p_0 and p', and so is a solid G. But G is spanned by 108 pairs of skew p; for the 24 p in G lie 12 in each of two planes γ^+, and any p in either γ^+ is skew to nine in the other. Thus the fact of there being 280 G is in accord with there being $280 \times 108 = 30240$ U-solids. And so on.

13. Of the hexahedra in (**5**) those there designated by π correspond to Σ^+, those by β to Σ^-; but those designated by α correspond each to an M—a hexahedron with six coincident faces. Yet the edges and plane faces of the α-hexahedron are mirrored distinctly in the finite geometry: the 15 edges answer to the 15 p in M through m while the 20 plane faces answer to the 20 γ^+ in M. These γ^+ consist of ten polar pairs, one pair through each of the ten g, and a polar pair corresponds to a pair of opposite plane faces of the hexahedron.

Miss Hamill also constructs 2592 heptahedra; their analogues will emerge in § 18 below.

14. The solid spanned by a pair of skew g is an S because the line of intersection of their polar G is skew to Q. Since each g consists of four m through each of which pass nine other g, any given g is skew to $280-1-4.9 = 243$

others. This, since each S meets Q in a hyperboloid whereon lie 12 pairs of skew g, accords with there being $\frac{1}{2}(280\times 243)/12 = 2835$ S.

The above remark indicates that G^* is transitive on the 280 g; for if two g are skew they can be transformed one into the other by the orthogonal group induced, in the S which they span, by those transformations of G^* leaving S invariant; while if two g intersect either can be transformed into the other *via* any g skew to both. Moreover, the stabilizer of a given g, say g_0, in G^* is transitive on the four m thereon as is seen by another appeal to the orthogonal group induced by G^* in any S containing g_0. Thus G^* is transitive on incident pairs (g, m) and so, *a fortiori*, on m simply.

15. The geometry points the way to the identification of the Sylow subgroups S_3 of G^*. Each point of Q has, in G^*, a stabilizer of index 112 (that G^* has subgroups of this index was noted by Todd in (7)). The stabilizer of m_0, of order $2^3.3^6.5$, acts as a permutation group on the ten g through m_0 and any given one, say g_0, of these ten is invariant for $2^2.3^6$ of these permutations. These $2^2.3^6$ operations include some, and therefore 2.3^6, which transpose each similarly signed pair of γ through g_0. For, taking the matrix M_5 from § 12 of (4) and extending it by an extra row and column,

$$\begin{bmatrix} . & 1 & & & & \\ 1 & . & & & & \\ & & . & 1 & & \\ & & -1 & . & & \\ & & & & -1 & . \\ & & & & . & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} y \\ x \\ u \\ -z \\ -v \\ w \end{bmatrix} \tag{15.1}$$

so that the latent column vectors are, with multiplier $+1$, the points on

$$x-y = z = u = v = 0$$

and, with multiplier -1, the points on

$$x+y = z = u = w = 0;$$

both these lines are c, and every point of either is conjugate to every point of the other. Hence the joins of the two intersections of either with Q to the two intersections with Q of the other are four g, all invariant under the operation, each having two of its four points invariant and the remaining two transposed. Take one of these g; say

$$x = v+w, \qquad y = -v+w, \qquad z = u = 0.$$

The two planes γ^+ which contain it are

$$x-y+v = x+y+w = z = 0,$$

$$x-y+v = x+y+w = u = 0.$$

Each of these contains a vertex of Σ_0^+, but these two vertices are transposed by the projectivity and so therefore are the two γ^+. The matrix in (15.1) belongs to G^* because it is, as explained in § 6, in the stabilizer of Σ_0^+.

Thus the group which leaves invariant m_0, g_0 and each γ through g_0 is of order 2.3^6. Each permutation that it imposes on the ten g through m_0 induces one on the points of the ellipsoid F wherein Q is met by any C in M_0, the point A of F on g_0 remaining fixed, as also does each of the four tangents to F at A since it is the section by C of a γ through g_0. This group, of permutations of points of F, is homomorphic to the group of order 2.3^6; it is, as remarked in § 25 of (**2**), of order 18 and it has a subgroup of order 9 which is elementary abelian. This is clear from (**2**); the matrices there called g and M commute and (when the lower sign is taken for M) are both of period 3. This subgroup of order nine consists of, in addition to identity, operations which permute the nine points of F other than A in cycles of three, the plane of each cycle passing through A; the homomorphism maps it on to a group of order 3^6, an S_3 of G^*. Thus, by way of definition, an S_3 is the aggregate of those operations of G^* which leave a generator g_0, a point m_0 thereon, and every γ through g_0 all invariant while they do *not* leave invariant any of the other g through m_0 *unless* they leave all nine invariant. Those which do leave every g through m_0 invariant form, in S_3, a normal subgroup κ of order 3^4, the kernel of the above homomorphism, and the intersection of ten of the 1120 S_3 in G^*.

16. Every operation of S_3 has odd period, so none can impose an odd permutation on the points of g_0; m_0 being fixed, the other three are either all fixed or undergo cyclic permutation. The latter alternative does occur: take, for example, the projectivity in G^* wherein u, v, w are unchanged but $x \to y \to z \to x$; if m_0 is (1, 1, 1, 0, 0, 0) and g_0 is

$$u = y-z = z-x, \qquad v = w = 0$$

then the other points on g_0 undergo cyclic permutation. Thus S_3 is homomorphic to the cyclic group $\mathscr{C}_3$, and the kernel of this homomorphism is a normal subgroup λ, of order 3^5, consisting of those projectivities of S_3 which leave all four m on g_0 invariant. Since the cube of any operation of S_3 not only leaves every point of g_0 invariant but also imposes the identity permutation on the g through m_0 (the operation itself, if it imposes on them some other permutation than identity, permuting them in cycles of three, the solid spanned by each cycle passing through g_0) this cube belongs to both λ and κ. The above projectivity, permuting three points on g_0 cyclically, leaves each g through m_0 invariant and so belongs to κ; thus κ too is homomorphic to $\mathscr{C}_3$ and the kernel, of order 3^3, of this homomorphism is the intersection of κ and λ.

It has been remarked that the cube of every operation of S_3 belongs to this intersection; should the operation belong itself either to κ or to λ its cube is the identity. If the operation belongs to κ it imposes the identity permutation on the ten g through m_0, though it may permute cyclically the three points, other than m_0, on any of these g; in any event its cube leaves invariant every m in M_0, and hence every point in each of the 45 planes spanned by pairs of the ten g, that is every point of M_0. But the invariant points of an orthogonal projectivity span in [5] a space of odd dimension ((**4**), p. 2) so that every point in [5] is invariant when every point of M_0 is. Should the operation belong to λ but not to κ let g_1, g_2, g_3 be a triad of g through m_0 that undergoes cyclic permutation, and A_1, A_2, A_3 points on these respective g such that $A_1 \to A_2 \to A_3$. The plane $A_1 A_2 A_3$ does not contain m_0 but, since the solid $g_1 g_2 g_3$ contains g_0, meets g_0 in m_1; but m_1, as the operation belongs to λ, is invariant. So, therefore, is the plane $A_1 A_2 A_3 m_1$ because it is spanned both by the triad $A_1 A_2 m_1$ and by the triad $A_2 A_3 m_1$ into which $A_1 A_2 m_1$ is transformed. Hence it is the same plane as $A_3 B m_1$ where B is the transform of A_3, and so a point on g_1. But the plane meets g_1 at A_1, with which therefore B coincides: hence $A_1 \to A_2 \to A_3 \to A_1$. This applies to any of the three cycles of generators through m_0, and to the cycle initiated by any point, other than m_0, on any one of them. So, again, the cube of the operation, leaving invariant every m in M_0, is the identity.

17. The operations of S_3 that belong neither to λ nor to κ have period 9; as such an operation is outside κ one can begin to discuss it as above, with the triad such that $A_1 \to A_2 \to A_3$; but the intersection of g_0 and the plane $A_1 A_2 A_3$ now, since the operation is outside λ, changes and the transform B_1 of A_3 is on g_1 but distinct from A_1; the cycle now runs

$$A_1 A_2 A_3 B_1 B_2 B_3 C_1 C_2 C_3,$$

points with suffix i being on g_i. Thus the operations of S_3 have only periods 1, 3, 9 and the number of those of period 9 is

$$3^6 - 3^5 - 3^4 + 3^3 = 432.$$

Since such an operation leaves no point on g_0 invariant save m_0 and no generator through m_0 invariant save g_0 it only belongs to one S_3. Hence there are (cf. (**5**), pp. 429, 448) in G^* $432 \times 1120 = 483840$ operations of period 9.

18. The discriminant D of a quadratic form over $\mathscr{K}$ is 1, -1, or 0; if $D = 0$ the form is singular. A non-singular linear transformation, while altering the form, multiplies D by the square of the determinant of the transformation and so does not change it. Conversely: if $D = 1$ the quadratic form is equivalent to the unit form. Now

$$X_1^2+X_2^2+\ldots+X_n^2+(X_1+X_2+\ldots+X_n)^2$$

has $D = n+1$ so that, over $\mathscr{K}$, the quadric

$$X_1^2+X_2^2+X_3^2+X_4^2+X_5^2+X_6^2+X_7^2 = 0,$$

where $$X_1+X_2+X_3+X_4+X_5+X_6+X_7 \equiv 0,$$

has $D = 1$; this is therefore an eligible equation for Q. Since, over $\mathscr{K}$, $D = 0$ when $n = 5$ the sections of Q by the primes $X_i = 0$ are singular; these primes are all M and Q is inscribed in the heptahedron $\mathscr{H}$. The 35 edges of $\mathscr{H}$ are all tangents: this is seen either by remarking that $D = 0$ when $n = 2$ or by noticing that the 21 vertices of $\mathscr{H}$ do not lie on Q, which meets each edge only in that one of its four points which is not a vertex. These contacts of Q with edges of $\mathscr{H}$ have four of their seven supernumerary coordinates zero and so are different from its contacts with the primes $X_i = 0$, these latter contacts having only one coordinate zero and the other six equal.

When $n = 4$, $D = -1$; the section of Q by a solid $X_i = X_j = 0$ is an ellipsoid: these 21 solids are all C. This form for the equation of an ellipsoid is mentioned in (**2**) on p. 274, as is the relation to the ellipsoid of the pentahedron of planes in which the other five primes of $\mathscr{H}$ meet C. These planes are secant planes of the ellipsoid, and so of Q; each contains four edges and six vertices of $\mathscr{H}$. The heptahedra fall into two categories according as their vertices, edges, and plane faces are k, p, Γ or l, n, e. Each category embraces two equinumerous species in the sense that a member of either can only be transformed into one of the other by a projectivity outside G^*; it will appear that there are $\mathscr{H}$ of the same category and sharing three primes, and that two such $\mathscr{H}$ are of different species.

19. Methods of constructing $\mathscr{H}$ are found by using its polar property: any point common to q of its primes is conjugate to any point common to the other $7-q$. The plane face opposite to any of the 35 edges has to lie in the polar solid of this edge; the polar prime of any vertex has to contain the solid of intersection of the two primes not passing through this vertex. Suppose then that a vertex k of an $\mathscr{H}^+$ is given. The opposite solid C has to lie in K and so is one of 36; when it is chosen the two primes of $\mathscr{H}^+$ that contain it are known because their contacts are on its polar c. Take any of the 15 Γ in C to be its intersection with a third prime M of $\mathscr{H}^+$; the contact of M is in the polar plane Γ' of Γ but is not to be on c; hence it can be either of two points m, m'; suppose that it is m. There are ((**2**), p. 273) two of the six pentahedra in C which include Γ, and either of these can provide, with its other four faces, the intersections of C with the remaining primes of $\mathscr{H}^+$. But these primes are known once the choices of m and the pentahedron have been made, for their line of intersection has to lie in Γ' yet cannot contain the contacts of any of the other three primes; hence it is km'. Since there are 36 choices for C, 15 for Γ, 2 for m, and 2 for the pentahedron, and since any face of the chosen pentahedron can play the role of Γ, there are

$$36 \times 15 \times 2 \times 2/5 = 432$$

$\mathscr{H}^+$ with k as a vertex. And since there are 21 of the 126 k that are vertices of any $\mathscr{H}^+$ the number of $\mathscr{H}^+$ is 2592.

Of the left-hand sides of the equations of the primes of $\mathscr{H}^+$ five vanish at k; the remaining two do not, and so both have 1 as their square. Since the sum of these two non-zero squares is -1 it follows that, in terms of the original homogeneous coordinates,

$$X_1^2+X_2^2+X_3^2+X_4^2+X_5^2+X_6^2+X_7^2 \equiv -(x^2+y^2+z^2+u^2+v^2+w^2).$$

Moreover, when the X_i are suitably signed their sum is identically zero. But for an $\mathscr{H}^-$ the sum of the seven X_i^2 is equal to $x^2+y^2+z^2+u^2+v^2+w^2$.

As an example let c be $u = v = w = y-z = 0$, so that k is the first vertex of Σ_0^+ and C is $x = y+z = 0$. Take Γ to be the intersection of C with $u = 0$; then Γ' is $y-z = v = w = 0$ and m, m' are the points

$$(.\ 1\ 1\ 1\ .\ .) \quad \text{and} \quad (.\ 1\ 1\ -1\ .\ .).$$

Choose the former to be the contact of the third prime of $\mathscr{H}^+$; then the remaining four primes join the other four faces, of one of the two pentahedra which include Γ, to the join of the latter point to k. The two pentahedra (save for change of notation) are given on p. 275 of (**2**); take the one whose faces are the intersections of C with

$$u = 0, \quad y+u+v = 0, \quad y+u-v = 0, \quad y+w-u = 0, \quad y-w-u = 0.$$

Then the seven primes are given by

$$\begin{aligned}
X_1 &\equiv x+y+z \\
X_2 &\equiv -x+y+z \\
X_3 &\equiv -y-z-u \\
X_4 &\equiv y+u+v \\
X_5 &\equiv y+u-v \\
X_6 &\equiv z+u-w \\
X_7 &\equiv z+u+w.
\end{aligned}$$

Had the other pentahedron in C, of which $u = 0$ is a face, been chosen the resulting primes are obtained from these seven by transposing v, w and changing the sign of u.

There are, likewise, and they admit the analogous construction, 2592 $\mathscr{H}^-$ of which 432 have a given vertex l.

20. The partition of each category into two species appears on constructing an $\mathscr{H}$ to have a given edge. Let p_0 be prescribed as an edge of $\mathscr{H}^+$; the plane of intersection of the three primes which do not pass through p_0 is a Γ in the polar solid P_0; choose it to be Γ_0, any plane in P_0 that does not pass through the intersection m of P_0 with p_0. The contacts of the three primes are in the polar plane Γ'_0 (which contains p_0) and are identified as the three points, other than m, of Q in Γ'_0. The intersections p_1, p_2, p_3, p_4 of Γ_0 with the remaining primes of $\mathscr{H}^+$ are known. Now those primes that meet Γ_0 in p_2, p_3, p_4 intersect in a plane Γ_1 through p_0 which has to lie in the polar solid P_1 of p_1 and yet be distinct from Γ'_0; there are two choices for this plane since, of the four planes through p_0 in P_1, three are planes Γ. Once this choice is made the primes which join Γ_1 to p_2, p_3, p_4 are fixed, and so six primes of $\mathscr{H}^+$ are known. The seventh is then determined too. Since there are 27 choices for Γ_0 and two for Γ_1 there are 54 $\mathscr{H}^+$ having p_0 for an edge. And since 35 of the 1680 p are edges of any given $\mathscr{H}^+$ the number of $\mathscr{H}^+$ is, again, $54\times 1680/35 = 2592$.

The two $\mathscr{H}^+$ consequent upon the two choices for Γ_1 are intimately related; they have in common the three primes through Γ_0 and are harmonic inverses of each other in the planes Γ_0 and Γ'_0; for this inversion leaves the three primes all unchanged and transposes the alternative Γ_1 of the choice.

Let us illustrate by constructing a pair of $\mathscr{H}^-$ which have in common the edge

$$n_0\colon u = v = w = x+y+z = 0,$$

and the opposite plane face

$$e_0\colon x = y = z = 0.$$

In this plane lie the lines n_1, n_2, n_3, n_4; namely

$$x = y = z = 0 = u+v+w, \quad u-v-w, \quad -u+v-w, \quad -u-v+w.$$

The polar solid of n_1 is $u = v = w$ and the two secant planes other than the polar plane of e_0 which lie in this solid and contain n_0 are

$$u = v = w = \epsilon(x+y+z)$$

where $\epsilon^2 = 1$; three primes of the required $\mathscr{H}^-$ are to join this plane to n_2, n_3, n_4 respectively. Hence, as the contacts of the three primes through e_0 are

$$(1,-1,-1,0,0,0), \qquad (-1,1,-1,0,0,0), \qquad (-1,-1,1,0,0,0),$$

these being the three points, other than (1, 1, 1, 0, 0, 0), of Q in the polar plane of e_0, we take

$$\begin{aligned} X_1 &= x-y-z \\ X_2 &= -x+y-z \\ X_3 &= -x-y+z \\ X_4 &= x+y+z+\epsilon(u-v-w) \\ X_5 &= x+y+z+\epsilon(-u+v-w) \\ X_6 &= x+y+z+\epsilon(-u-v+w) \end{aligned} \tag{20.1}$$

which imply $\qquad X_7 = x+y+z+\epsilon(u+v+w),$

the prime $X_7 = 0$ containing n_0 and n_1 so that X_7 is a linear combination of $x+y+z$

and $u+v+w$. The two $\mathscr{H}^-$ that answer to the two choices ± 1 of ϵ are transformed into one another by changing the signs of u, v, w; that is by harmonic inversion in e_0 and its polar plane.

Heptahedra $\mathscr{H}_1$ and $\mathscr{H}_2$, of the same category, with three common primes cannot be transformed into one another by any projectivity of G^*; this will now be proved, and the partition of each category into two species follows. Suppose, to fix ideas, that $\mathscr{H}_1$ and $\mathscr{H}_2$ are both $\mathscr{H}^+$; call the harmonic inversion, whose fundamental planes are the plane Γ, of intersection of the three common primes of $\mathscr{H}_1$ and $\mathscr{H}_2$, and the polar plane of Γ with respect to Q, J. J permutes the 126 k oddly, imposing $\frac{1}{2}(126-6-6) = 57$ transpositions, and the 126 l evenly, imposing $\frac{1}{2}(126-3-3) = 60$ transpositions.

All projectivities that transform $\mathscr{H}_1$ into $\mathscr{H}_2$ are obtained by combining J with those projectivities which leave $\mathscr{H}_2$ invariant, and these are products of those projectivities $\mathscr{J}$ that transpose pairs of primes of $\mathscr{H}_2$, namely of harmonic inversions whose fundamental spaces are a vertex k of $\mathscr{H}_2$ and its polar K. Each such inversion transposes $\frac{1}{2}(126-1-45) = 40$ pairs of k and $\frac{1}{2}(126-0-36) = 45$ pairs of l. If a projectivity permutes the primes of $\mathscr{H}_2$ evenly then, being the product of an even number of $\mathscr{J}$, it permutes both the k and the l evenly; hence, when it is combined with J, the resultant imposes permutations of opposite parity on the k and the l and so is outside G^*. If a projectivity permutes the primes of $\mathscr{H}_2$ oddly then, being the product of an odd number of $\mathscr{J}$, it imposes an even permutation on the k and an odd permutation on the l; hence, when it is combined with J, the resultant imposes odd permutations on the k as well as on the l and so is outside G^*.

21. When an $\mathscr{H}$ is given the appropriate harmonic inversion produces any of those 35 $\mathscr{H}$ of the same category but opposite species that share three primes with it. For take the canonical $\mathscr{H}$ of § 18; the polar of the plane $X_1 = X_2 = X_3 = 0$ is $X_4 = X_5 = X_6 = X_7$, and harmonic inversion in these two planes leaves any prime through either, and so the first three primes of $\mathscr{H}$, invariant. The primes $\sum_1^7 u_i X_i = 0$ and $\sum_1^7 v_i X_i = 0$ are inverse when

$$\begin{aligned}
v_1 &= u_1, & v_4 &= u_4-u_5-u_6-u_7,\\
v_2 &= u_2, & v_5 &= -u_4+u_5-u_6-u_7,\\
v_3 &= u_3, & v_6 &= -u_4-u_5+u_6-u_7,\\
& & v_7 &= -u_4-u_5-u_6+u_7.
\end{aligned} \tag{21.1}$$

The form of these relations may be varied because, in virtue of $\sum X_i \equiv 0$, all seven u_i, as likewise all seven v_i, may have the same mark added to them without affecting the geometry; but (21.1) are symmetric as they stand in u and v. They are obtained by stipulating that the primes $\sum (u_i \pm v_i) X_i = 0$ pass one through each of the two fundamental planes of the inversion. It follows that one $\mathscr{H}$ of the opposite species (and same category) to the canonical $\mathscr{H}$ has the primes the left-hand sides of whose equations are

$$\begin{array}{ll} & X_4-X_5-X_6-X_7 \\ X_1 & \\ & -X_4+X_5-X_6-X_7 \\ X_2 & \\ & -X_4-X_5+X_6-X_7 \\ X_3 & \\ & -X_4-X_5-X_6+X_7 \end{array} \tag{21.2}$$

The sum of these seven linear forms is $\sum X_i$, the sum of their squares $\sum X_i^2$. Note the application of this substitution to the two $\mathscr{H}^-$ of (20.1).

This substitution changes the species within the category; its combination with another of the same type therefore conserves the species, and so one can derive $\mathscr{H}$ which belong to the same category and species while having two common primes. The seven linear forms (21.2) were derived by fixing the first three; if the same process is now applied to (21.2) but with the first two and the last of the forms kept fixed the outcome is

$$\begin{array}{ll} & -X_3-X_5-X_6+X_7 \\ X_1 & \\ & -X_3-X_4-X_6+X_7 \\ X_2 & \\ & -X_3-X_4-X_5+X_7 \\ X_3+X_4+X_5+X_6 & \\ & -X_4-X_5-X_6+X_7 \end{array} \tag{21.3}$$

This set of forms is symmetric in X_3, X_4, X_5, X_6 and so is one of five sets obtainable by permutations of X_3, X_4, X_5, X_6, X_7; thus six $\mathscr{H}$ arise, of the same category and species, sharing the two primes $X_1 = 0$ and $X_2 = 0$: the canonical $\mathscr{H}$ and five more. Nor are there any others. For the residual five primes of every such $\mathscr{H}$ meet C_{12}—the solid $X_1 = X_2 = 0$—in the faces of a pentahedron, and six pentahedra are eligible, their faces being Γ for $\mathscr{H}^+$, e for $\mathscr{H}^-$. If, say, an $\mathscr{H}^-$ is required to share $X_1 = 0$ and $X_2 = 0$ with the canonical $\mathscr{H}$ it is determined when one, $\wp$ say, of the six pentahedra in C_{12} whose faces are e is chosen; for its opposite vertex, being the only l on the polar of C_{12}, is fixed, and while there are two M containing the solid joining l to a face of $\wp$ they lead to $\mathscr{H}^-$ of opposite species.

The number of $\mathscr{H}$ of given category and species that include a given M among their seven primes is $1296 \times 7/112 = 81$; if $\mathscr{H}_0$ is any one of these six others, it has just been shown, share with $\mathscr{H}_0$ not only M but any one of its six other primes. Thus $81-36 = 45$ share M only, and so there are

$$1296-1-7.45-21.6 = 854$$

of this category and species that have no prime in common with $\mathscr{H}_0$.

22. The use of supernumerary coordinates whose sum vanishes shows that every heptahedron in [5] yields a quadric Q for which it serves as an $\mathscr{H}$. Of the 112 points of Q 35 are those points, one on each edge of $\mathscr{H}$, that are not vertices; of these 35, 20 lie in each prime of $\mathscr{H}$ and consist of 10 opposite pairs whose joins are concurrent: these points of concurrence, one in each prime, are the contacts of the primes with Q. Each join has on it one further point and these 60 further points, 10 contributed by each prime, make up the tally of 112. It may also be noted that the plane spanned by the contacts of three of the primes meets Q further on the line of intersection of the remaining four: the seven contacts form a heptagon whose 35 plane faces meet each an edge of $\mathscr{H}$.

Since Q is thus determined completely by $\mathscr{H}$ any projectivity which, though permuting its bounding primes, leaves $\mathscr{H}$ invariant also leaves Q invariant; moreover, since a projectivity is uniquely determined when the seven primes corresponding to any given seven primes (no six of either set concurrent) are known, there is a group of 7! projectivities imposing the symmetric group of permutations on the primes of $\mathscr{H}$. The odd permutations however are imposed by indirect projectivities: the harmonic inversion in a vertex V of $\mathscr{H}$ and its polar prime v with respect to Q leaves the five primes through V invariant and transposes the other two—for the C common to these latter is in v, and they are those two primes through C other than v and the join of C to V. This inversion, which has determinant -1, thus imposes the transposition; hence every odd permutation, being the product of an odd number of transpositions, is imposed by a projectivity also of determinant -1, that is, by an indirect projectivity. Thus it is the alternating group of order $\frac{1}{2}.7!$ that consists of direct projectivities; it is a subgroup of G^* because, leaving $\mathscr{H}$ invariant, it cannot transpose the batches of k and l and because, each of its members being the product of an even number of harmonic inversions, all these members impose even permutations on k as well as on l.

23. The presence of the heptahedra, each providing an $\mathscr{A}_7$ which is a subgroup of G^*, explains why G^* has operations of period 7; such an operation is provided by any projectivity which permutes the primes of an $\mathscr{H}$ in a single cycle. Conversely: since $1296 \equiv 1 \pmod 7$ there must, for any operation of period 7 in G^*, be an $\mathscr{H}$ of each category and species that is invariant and so has its primes cyclically permuted. In order to give explicitly the matrix of such an operation take that $\mathscr{H}$ of (20.1) with $\epsilon = 1$. The poles of its faces are given by the columns

$$\begin{array}{rrrrrrr} 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ . & . & . & 1 & -1 & -1 & 1 \\ . & . & . & -1 & 1 & -1 & 1 \\ . & . & . & -1 & -1 & 1 & 1 \end{array}$$

and one demands a matrix μ which, on premultiplying any of these columns, turns it into the one, or the negative of the one, immediately on its right (the last column reverting to the first). Only one projectivity, and so only a pair of matrices $\pm\mu$, will serve; but the foreknowledge of the orthogonality of μ may ease its calculation as well as being a useful check. The outcome is

$$\mu = \pm\begin{bmatrix} . & . & 1 & . & . & . \\ . & -1 & . & 1 & 1 & 1 \\ -1 & . & . & 1 & 1 & 1 \\ 1 & 1 & . & -1 & -1 & . \\ -1 & -1 & . & 1 & . & 1 \\ -1 & -1 & . & . & 1 & 1 \end{bmatrix}.$$

24. An alternative way of finding μ opens by observing that a cyclic permutation of seven objects is the resultant of six successive transpositions:

$$(1234567) = (17)(16)(15)(14)(13)(12)$$

where transpositions to the right act first; hence an operation of period 7 in G^* is the product of six harmonic inversions (cf. (**5**), § 7.2); these inversions do not themselves belong to G^* but the product of an even number does. The centres of these inversions are those vertices of $\mathscr{H}$ that are common to the sets of primes

23456 23457 23467 23567 24567 34567

and so, from (20.1), have the respective coordinate vectors

$$\begin{array}{rrrrrr} . & . & . & . & 1 & 1 \\ 1 & 1 & 1 & 1 & . & -1 \\ 1 & 1 & 1 & 1 & 1 & . \\ -1 & 1 & 1 & -1 & . & . \\ -1 & 1 & -1 & 1 & . & . \\ -1 & -1 & 1 & 1 & . & . \end{array} \qquad (24.1)$$

The matrices for the corresponding inversions are instantly written down (see below) and their product, in this order, is precisely μ.

The harmonic inversion in a point, whose coordinate vector is ξ, and its polar prime with respect to Q has the matrix $\xi\xi'+I$ when ξ is a k, the matrix $\xi\xi'-I$ when ξ is an l. It is enough to substantiate the second statement, which is relevant to the points (24.1). When ξ is an l, $\xi'\xi = -1$ and so

$$\begin{aligned}(\xi\xi'-I)^2 &= \xi\xi'\xi\xi'-\xi\xi'-\xi\xi'+I\\ &= -\xi\xi'-\xi\xi'-\xi\xi'+I = I,\end{aligned}$$

so that $\xi\xi'-I$ is the matrix of an involution and, being symmetric, is orthogonal. Moreover

$$(\xi\xi'-I)\xi = \xi\xi'\xi-\xi = -\xi-\xi = \xi,$$

so that ξ is a latent vector with multiplier 1; while, if η is any point in the polar prime of ξ, and so $\xi'\eta = 0$,

$$(\xi\xi'-I)\eta = \xi\xi'\eta-\eta = -\eta,$$

so that every point of the polar prime of ξ is latent with multiplier -1. The matrices of the inversions centred on the points (24.1) are therefore

$$\begin{bmatrix}-1 & . & . & . & . & .\\ . & . & 1 & -1 & -1 & -1\\ . & 1 & . & -1 & -1 & -1\\ . & -1 & -1 & . & 1 & 1\\ . & -1 & -1 & 1 & . & 1\\ . & -1 & -1 & 1 & 1 & .\end{bmatrix} \begin{bmatrix}-1 & . & . & . & . & .\\ . & . & 1 & 1 & 1 & -1\\ . & 1 & . & 1 & 1 & -1\\ . & 1 & 1 & . & 1 & -1\\ . & 1 & 1 & 1 & . & -1\\ . & -1 & -1 & -1 & -1 & .\end{bmatrix} \begin{bmatrix}-1 & . & . & . & . & .\\ . & . & 1 & 1 & -1 & 1\\ . & 1 & . & 1 & -1 & 1\\ . & 1 & 1 & . & -1 & 1\\ . & -1 & -1 & -1 & . & -1\\ . & 1 & 1 & 1 & -1 & .\end{bmatrix}$$

$$\begin{bmatrix}-1 & . & . & . & . & .\\ . & . & 1 & -1 & 1 & 1\\ . & 1 & . & -1 & 1 & 1\\ . & -1 & -1 & . & -1 & -1\\ . & 1 & 1 & -1 & . & 1\\ . & 1 & 1 & -1 & 1 & .\end{bmatrix} \begin{bmatrix}. & . & 1 & . & . & .\\ . & -1 & . & . & . & .\\ 1 & . & . & . & . & .\\ . & . & . & -1 & . & .\\ . & . & . & . & -1 & .\\ . & . & . & . & . & -1\end{bmatrix} \begin{bmatrix}. & -1 & . & . & . & .\\ -1 & . & . & . & . & .\\ . & . & -1 & . & . & .\\ . & . & . & -1 & . & .\\ . & . & . & . & -1 & .\\ . & . & . & . & . & -1\end{bmatrix}.$$

25. It is now to be shown that an operation of period 7 in G^* leaves invariant a unique $\mathscr{H}$ of each category and species. It will then follow, since each of the 1296 $\mathscr{H}$ of given category and species is invariant for 6! operations of period 7 in G^*, that there are (cf. (**5**), p. 449)

$$1296\times 6! = 933120$$

such operations in all. Furthermore: precise formulae will be found which make the passage from one invariant $\mathscr{H}$ to the other three.

First a word or two concerning the primes M referred to the seven coordinates X_i. If $\sum_{i=1}^{7} a_i X_i = 0$ touches $\sum X_i^2 = 0$ at $X_i = \xi_i$, $a_i = \xi_i+\lambda$ where λ is any mark of $\mathscr{K}$. Then

$$\sum a_i^2 = \sum \xi_i^2-\lambda\sum \xi_i+\lambda^2 = 0+0+\lambda^2 = \lambda^2,$$
$$\sum a_i = \lambda+\sum \xi_i = \lambda,$$

so that $$\sum a_i^2 = (\sum a_i)^2.$$

One may take $\lambda = 1$, as has happened with all linear forms in the X_i already used for faces of $\mathscr{H}$, and so

$$\sum a_i^2 = \sum a_i = 1; \tag{25.1}$$

the contact is then given by $X_i = a_i - 1$. Each M admits two representations that conform to (25.1); instances are

$$\begin{aligned} &\text{(i) } X_1 \quad \text{and} \quad X_1 - X_2 - X_3 - X_4 - X_5 - X_6 - X_7, \\ &\text{(ii) } X_1 - X_2 - X_3 - X_4 \quad \text{and} \quad X_1 - X_5 - X_6 - X_7, \\ &\text{(iii) } X_1 + X_2 + X_3 + X_4 \quad \text{and} \quad X_1 + X_2 + X_3 + X_4 - X_5 - X_6 - X_7. \end{aligned} \tag{25.2}$$

All the M are thus identified: $112 = 7 + \frac{1}{2}.7.{}^6C_3 + {}^7C_4$. Those headed (i) consist only of the seven primes of the canonical $\mathscr{H}$ to which all other primes are, in this coordinate system, referred.

There are, on the line $X_1 = X_2 = X_3 = X_4 = 0$, three vertices of the canonical $\mathscr{H}$; at each of them

$$\sum X_i^2 = X_5^2 + X_6^2 + X_7^2 = 0 + 1 + 1 = -1.$$

There is also a fourth point of the line, and there

$$\sum X_i^2 = X_5^2 + X_6^2 + X_7^2 = 1 + 1 + 1 = 0.$$

In other words, the edge of $\mathscr{H}$ is a tangent of Q, being p or n according to the category of $\mathscr{H}$. The 21 vertices of $\mathscr{H}$ leave over 105 more points whereat $\sum X_i^2$ is -1; these lie five in each of the 21 solids C_{ij}: $X_i = X_j = 0$. The five points in C_{ij} are those which do not lie either on the ellipsoid section of Q or in any face of the pentahedron wherein C_{ij} is cut by the five primes of $\mathscr{H}$ that do not contain it wholly. The 126 points whereat $\sum X_i^2$ is $+1$ are also partitioned as $21 + 105$; there are 21 which do not lie in any face of $\mathscr{H}$, such as the point $(1, 1, 1, 1, 1, -1, -1)$, and three in each of the 35 plane faces of $\mathscr{H}$, such as $(0, 0, 0, 1, 1, -1, -1)$. Whenever two $\mathscr{H}$ are of opposite categories $\sum X_i^2 + \sum Y_i^2$ is zero at every point in [5].

26. Consider now the projectivity π that permutes the suffixes of the X_i in the cycle (*1234567*); it belongs to G^* and leaves the canonical $\mathscr{H}$ invariant. Does it leave any other $\mathscr{H}$ invariant and, if so, which?

Let one face of an $\mathscr{H}$ invariant under π be

$$X_1' \equiv aX_1 + bX_2 + cX_3 + dX_4 + eX_5 + fX_6 + gX_7 = 0, \tag{26.1}$$

so that, by (25.1), $$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 = 1, \tag{26.2}$$

$$a + b + c + d + e + f + g = 1. \tag{26.3}$$

Then, by (26.3), $\sum X_i' \equiv \sum X_i \equiv 0$; but, in order that $\sum X_i'^2 = \sum X_i^2$ and so the $\mathscr{H}$ be of the same category as the canonical one as well as being invariant, not only must (26.2) hold but also the three relations

$$\begin{aligned} ab+bc+cd+de+ef+fg+ga &= 0, \\ ac+bd+ce+df+eg+fa+gb &= 0, \\ ad+be+cf+dg+ea+fb+gc &= 0. \end{aligned} \tag{26.4}$$

The question to be resolved is whether any of the forms (25.2), other than those under (i), satisfy (26.4). It may therefore be presumed that three of the seven coefficients are zero and all the other four non-zero; furthermore, since one is at liberty to impose a power of π, one may take $a = 1$ while, of the other coefficients, three are zero and the others equal but non-zero. Thereupon (26.4) become

$$\begin{aligned} b+bc+cd+de+ef+fg+g &= 0, \\ c+bd+ce+df+eg+f+gb &= 0, \\ d+be+cf+dg+e+fb+gc &= 0. \end{aligned} \tag{26.5}$$

Were $b = g = 0$ the first of these would demand that $cd+de+ef = 0$, which cannot happen with only one of c, d, e, f zero; were, on the other hand $b = g \neq 0$ then three of c, d, e, f would be zero and another condition in (26.5) contradicted, just as the first would have been by $b = g = 0$. Hence it must be that

$$b \neq g, \qquad c \neq f, \qquad d \neq e$$

which, under the prevalent restrictions, imply that

$$\begin{gathered} bg = cf = de = 0, \\ b+g = c+f = d+e. \end{gathered}$$

Now subtract the third from the first of the relations (26.5):

$$d(c-g)+e(f-b)+(b-g)(c-f) = 0.$$

The third term here is non-zero; were $b = f$, and so also $c = g$, the first two terms would both be zero and the condition violated. Each pair of relations (26.5) may be so combined, and it follows that

$$b = c = e, \qquad d = f = g.$$

One of these two unequal marks is zero; (26.5) then requires the other to be -1; the two choices for the zero triad yield however the same M, as (ii) of (25.2) asserted. Hence there is a single $\mathscr{H}$, distinct from but belonging to the same category as the canonical $\mathscr{H}$, invariant under π; namely that whose primes are

$$\begin{array}{rl}
X_1-X_2-X_3 \quad -X_5 & = 0,\\
X_2-X_3-X_4 \quad -X_6 & = 0,\\
X_3-X_4-X_5 \quad -X_7 & = 0,\\
-X_1 \quad +X_4-X_5-X_6 & = 0,\\
-X_2 \quad +X_5-X_6-X_7 & = 0,\\
-X_1 \quad -X_3 \quad +X_6-X_7 & = 0,\\
-X_1-X_2 \quad -X_4 \quad +X_7 & = 0.
\end{array}$$

27. It being now established that, in one category, there is a single $\mathscr{H}$ of either species invariant under π it follows, on applying an outer automorphism to G^* that transposes k and l, that the same is true for the opposite category: there is a single $\mathscr{H}$ therein of either species invariant under a projectivity of period 7. It is, however, informative to inquire precisely which $\mathscr{H}$ these are under π. Equations (26.1), (26.2), (26.3) still hold but now, for the opposite category, $\sum X_i'^2 = -\sum X_i^2$. This is not incompatible with (26.2) since any multiple of the vanishing form $\sum X_i$ can be added to either side; as the forms now to occur are to have cyclic symmetry the linear form that multiplies $\sum X_i$ will be $\sum X_i$ itself, and the identity to be satisfied is

$$\sum X_i'^2 = -(\sum X_i)^2-\sum X_i^2.$$

The relations (26.4) and (26.5) are now to be replaced by those with the same left-hand sides but with -1 instead of zero on their right; one again takes $a = 1$, three of the other coefficients zero and the remaining three equal but not zero. If, now, $b = g = 0$, then

$$cd+de+ef = -1 = c+ce+df+f = d+cf+e;$$

one of c, d, e, f is zero and this cannot be d or e. Hence

either $\qquad c = 0, \quad d = e = f = 1,$

or $\qquad f = 0, \quad c = d = e = 1.$

These are the only solutions occurring when $b = g = 0$. Hence the two $\mathscr{H}$ of the category opposite to the canonical $\mathscr{H}$ that are invariant under π have for their primes

$$\begin{array}{rl}
X_1+X_2+X_3 \quad +X_5 & = 0,\\
X_2+X_3+X_4 \quad +X_6 & = 0,\\
X_3+X_4+X_5 \quad +X_7 & = 0,\\
X_1 \quad +X_4+X_5+X_6 & = 0,\\
X_2 \quad +X_5+X_6+X_7 & = 0,\\
X_1 \quad +X_3 \quad +X_6+X_7 & = 0,\\
X_1+X_2 \quad +X_4 \quad +X_7 & = 0;
\end{array}
\qquad
\begin{array}{rl}
X_1+X_2+X_3 \quad +X_6 & = 0,\\
X_2+X_3+X_4 \quad +X_7 & = 0,\\
X_1 \quad +X_3+X_4+X_5 & = 0,\\
X_2 \quad +X_4+X_5+X_6 & = 0,\\
X_3 \quad +X_5+X_6+X_7 & = 0,\\
X_1 \quad +X_4 \quad +X_6+X_7 & = 0,\\
X_1+X_2 \quad +X_5 \quad +X_7 & = 0.
\end{array}$$

This information is available for any projectivity of period 7 and any of its four invariant $\mathscr{H}$. Take the $\mathscr{H}^-$ of (20.1) with $\epsilon = +1$, and the projectivity whose matrix μ has already been calculated twice. The invariant $\mathscr{H}^+$ have for their primes

$$\begin{array}{ll} -u+v-w = 0, & -u-v+w = 0, \\ -y-z+v = 0, & -y-z-u = 0, \\ -x-y+z+u+v-w = 0, & -x-z+w = 0, \\ x-y-z-u-v-w = 0, & -x+y-z-u-v-w = 0, \\ -x+y-z-u+v+w = 0, & -x-y+z-u+v+w = 0, \\ -x-z-w = 0, & x-y-z+u-v+w = 0, \\ -x-y-u = 0; & -x-y-v = 0. \end{array}$$

28. There are, in any projective space of odd dimension, null-systems or screws—correlations whose matrices are skew and non-singular. If x is the column vector of coordinates of a point and u the row vector of coordinates of a prime the correlation is

$$u' = Bx$$

where $B' = -B$. Since the prime equation of Q is $uu' = 0$ the locus of those points that are polars, in the screw, of the primes M is $x'B'Bx = 0$, so that the screw reciprocates Q into itself whenever $B'B = \pm I$; each m then has, in the screw, a polar M and each M a pole m. Now the primes K are those satisfying $uu' = 1$, so that their poles in the screw satisfy $x'B'Bx = 1$; hence these poles are k if $B'B = I$ and l if $B'B = -I$. We confine the discussion to screws which not only, as leaving Q invariant, turn each m into an M, but which also turn each k into a K and so too each l into an L. That is, we take $B'B = I$; B is not only skew, but orthogonal as well, and the symbol B will henceforward denote only a matrix with these attributes. The diagonal of B consists of zeros and the sum of the squares of the other five elements in any row is 1; hence, in any row, either

(*a*) there is only one non-zero element, or
(*b*) there is only one zero beside the diagonal one.

Moreover all rows of B are alike; there are no hybrids. For let row r consist of zeros save for ± 1 in column s so that, since $B' = -B$, column r has ∓ 1 in row s and all its other elements zero. Then, because of the orthogonality, every element save the ± 1 in column s is zero, as is every element save the ∓ 1 in row s. Hence each row has two zeros off the diagonal and is of type (*a*).

In a matrix of this type there are five places in the top row each of which can be occupied either by 1 or by -1; if this mark is in column s row s is thereby fixed. In any row other than *1* and s there are now three places available, namely those off the diagonal but in neither column *1* nor column s; each of these can be occupied by either 1 or -1; if this mark is in column t row t is thereby fixed. In either of the two remaining rows only one place is available for the non-zero mark; hence there are

$$5.2.3.2.1.2 = 120$$

matrices under (a).

The calculation of the number of matrices under (b) is but slightly more elaborate. There are five choices for the column r that the zero element off the diagonal in the top row occupies, and 2^4 choices for the other four elements of the top row; the choice fixes the first column too. Next write in row r, which already has zeros in columns *1* and r; the orthogonality requires that rows *1* and r must have zero splice and this allows six choices for the four non-zero elements of row r; the choice fixes column r too. Every row other than *1* and r now has two non-zero elements and these contribute zero to the splice of this row with both row *1* and row r; the other two non-zero elements still to be inserted must therefore contribute zero too to both splices, and this double condition determines which two of the three remaining places off the diagonal they occupy as well as determining, save for a common multiplier -1, what they are. Thus two choices are available, and the choice fixes the corresponding column too. The matrix can then be filled in one, and only one, way, and so there are

$$5.2^4.6.2 = 960$$

matrices under (b). The whole tally of matrices is thus 1080 so that, as each matrix is here accompanied by its negative, there are 540 screws. Their separation into 60 and 480 in the two types has no geometrical significance but is merely a consequence of the choice of Σ_0^+. The 60 monomial $\pm B$ reciprocate the vertices of Σ_0^+ into its own faces but, whereas Q turns each vertex into the opposite face, each of the 60 screws is seen, in virtue of the monomial form of B, to superimpose a triple transposition; to transpose, that is, each of the three pairs of a syntheme of the six faces. Each of the 15 synthemes provides (eight matrices and so) four of the 60 screws, the non-zero marks occupying the same positions in the matrices and differing only in sign. The product $B_1 B_2$ of any two of these eight matrices is diagonal and so has square I:

$$B_1 B_2 B_1 B_2 = I$$

or, since $B_1^2 = B_2^2 = -I$,

$$B_2 B_1 = B_1 B_2.$$

This shows that the reciprocations in the four screws are mutually commutative: the screws being

$$u' = B_1 x, \qquad u' = B_2 x,$$

the pole y in the second of the polar prime of x in the first is

$$y = B_2^{-1} B_1 x,$$

so that there is commutation whenever

$$B_2^{-1} B_1 = B_1^{-1} B_2$$

which is, since $B_1^2 = B_2^2 = -I$,

$$B_2 B_1 = B_1 B_2.$$

Each of the 15 synthemes from each of the 567 Σ^+ provides such a tetrad of four mutually commutative screws, so that there are 8505 of these tetrads. The number of tetrads to which a given screw belongs is

$$8505 \times 4/540 = 63.$$

An equivalent statement is that the number of Σ^+ which are self-polar for a given screw as well as for Q is

$$567 \times 60/540 = 63.$$

29. Similar reasoning can be founded on the negative simplexes Σ^-, but it is the same tetrads of mutually commuting screws that emerge. For take any syntheme of faces of a Σ^+, with the polar syntheme of the vertices; the join s of each pair of vertices has for its polar in Q the solid S spanned by the joins of the other two pairs. Now through S pass four primes, namely two K that are faces of Σ^+ and two L; the six L so arising, two from each S spanned by two of three joins of vertices of Σ^+, are the faces of a Σ^- whose vertices, poles of these L, are manifestly two on each of the three joins of the pairs of the syntheme of vertices of Σ^+. All this is clear on taking Σ_0^+ and, say, the syntheme

$$x = 0, \quad y = 0; \qquad z = 0, \quad u = 0; \qquad v = 0, \quad w = 0;$$

whereupon the six L are

$$x+y = 0, \quad x-y = 0; \qquad z+u = 0, \quad z-u = 0; \qquad v+w = 0, \quad v-w = 0.$$

There is a (15, 15) correspondence between the 567 Σ^+ and the 567 Σ^-; two simplexes correspond when they share three edges, joins of pairs of a syntheme of vertices.

30. Since $B^2 = -I$ the matrix B, when regarded as the instrument not of a correlation but of a projectivity, imposes an involution; but it has not appeared among the operations of G^*. This is because it violates the prescription (iv) of § 3; it has no latent root in $\mathscr{K}$, its characteristic polynomial being λ^6+1, and so does not leave any point invariant; it transposes the $126\,k$ as 63 pairs, and so subjects them to an odd permutation, as it likewise does the $126\,l$.

B serves to impose an outer automorphism on G^* and its geometrical interpretation is salient once $\mathscr{K}$ is extended to $GF(3^2)$ by adjoining a mark j satisfying $j^2 = j+1$. There then appear planes on Q and they fall into pairs that are conjugate in the automorphism of period 2 of the extended field. The Grassmann coordinates of such a plane are the conjugates, that is the cubes, of those of the companion plane. Any operation that involves the two planes symmetrically is expressible in terms of the marks of $\mathscr{K}$, and B is simply the harmonic inversion whose fundamental spaces are such a pair of conjugate planes. This inversion transposes the m as 56 pairs, and the polar of any given m_0 in the corresponding screw is the tangent prime of Q at that point m' which is paired with m_0. Since m_0 lies in this polar, $m'm_0$ is a g; each of the 30 m', three on each of the g through m_0, is the pole of M_0 in 18 of the 540 screws.

Each k has, in each screw, a polar K that passes through it; hence, since there are only 45 K through k, each K that passes through a given k_0 is the polar of k_0 in 12 of the 540 screws. Suppose, to fix ideas, that k_0 is X, the first vertex of Σ_0^+; $w = 0$ is the polar of X in any screw whose matrix has every element in its first column zero save the bottom one. Such matrices are all of type (a), wherein the 60 screws consist of five sets of 12 that reciprocate X into the five faces of Σ_0^+ which pass through it; those 12 that reciprocate X into $w = 0$ consist of three commuting tetrads, one tetrad for each of the three synthemes of faces of Σ_0^+ that include the pair $x = 0$, $w = 0$. The 480 screws of type (b) correspond 12 to each of the 40 K through X which are not faces of Σ_0^+.

31. A screw in [5] has null planes; planes, that is, which lie in the polar primes of all their points. The conditions on B which ensure that $x = y = z = 0$ is a null plane are that the bottom right-hand quadrant consist wholly of zeros. This not only causes B to be of type (a); it prohibits any two of the faces $u = 0$, $v = 0$, $w = 0$ forming a pair of the associated syntheme. Thus no two of $x = 0$, $y = 0$, $z = 0$ can form a pair either, so that the top left-hand quadrant of B consists wholly of zeros too and $u = v = w = 0$ is another null plane. This is to be expected: if a secant plane of Q is a null plane of a screw that reciprocates Q into itself, so is the polar, in Q, of this secant plane. Now, of the 15 synthemes among the faces of Σ_0^+, six have the property of pairing each of $u = 0$, $v = 0$, $w = 0$ with one of $x = 0$, $y = 0$, $z = 0$; hence there are 48 matrices B and so 24 screws. And since there are, among the 540 screws, 24 which have a given polar pair of planes e as null planes the number of polar pairs of planes e, as also of polar pairs of planes Γ, that are null planes of a given screw is

$$5760 \times 24/540 = 256.$$

32. G^* is certainly transitive on the pairs of polar planes e, and in order to establish that it is transitive on the screws it is therefore only necessary to show that it is transitive on those 24 screws which have a given pair of polar planes e as null planes. Given a correlation and a projectivity

$$u' = Bx, \qquad y = Hx$$

then, if the polar plane of y in the correlation has coordinates given by the row vector v, the relations

$$u = vH^{-1} = y'B'H^{-1} = x'H'B'H^{-1}$$

show that the projectivity transforms the screw whose matrix is B into that whose matrix is the transpose of $H'B'H^{-1}$ which, when H is orthogonal and B skew, is $-HBH$. Hence, in order that the projectivity transform a screw of matrix B_1 into one of matrix B_2, it is necessary that

$$B_2 = HB_1H.$$

This relation is $H'B_2 = B_1H$, and it is necessary to show that it is satisfied, by some H that imposes a projectivity of G^*, when

$$B_2 = \begin{bmatrix} . & I \\ -I & . \end{bmatrix}, \qquad B_1 = \begin{bmatrix} . & P \\ -P & . \end{bmatrix}$$

where P is a three-rowed monomial matrix. Since $PP' = I$ the relation is satisfied by

$$H = \begin{bmatrix} P' & . \\ . & I \end{bmatrix};$$

of the two matrices $\pm B_1$ we can use that for which $|P| = 1$, and so $|H| = 1$. This choice for H is adequate provided that the resulting projectivity permutes each batch of 126 points, k and l, evenly; and this it certainly does if P, and therefore H, has odd period. This disposes of those P where the ± 1 occupy the positions of the units in a cyclic permutation matrix, but in other instances the above choice for H is not adequate. However, there are always adequate choices available: to give just two instances, for

$$P = \begin{bmatrix} 1 & . & . \\ . & -1 & . \\ . & . & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & . & . \\ . & . & 1 \\ . & 1 & . \end{bmatrix}$$

respectively, take

$$H = \begin{bmatrix} . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ . & 1 & . & . & . & . \\ -1 & . & . & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & -1 \end{bmatrix}, \quad \begin{bmatrix} . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ . & 1 & . & . & . & . \\ 1 & . & . & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{bmatrix}$$

belonging to the stabilizer of Σ_0^+ in G^* (see § 6).

33. Since G^* is transitive on the 540 screws it has subgroups of index 540 and order 6048, namely the stabilizers of the screws. These are the subgroups found by Miss Hartley in (**6**), and the apparatus required to identify them in this finite geometry seems simpler than that elaborated there. Yet there can be little doubt that the most appropriate setting in which to place these subgroups is neither there nor here. For G^* is isomorphic to one of the hyperorthogonal groups, namely to that group of projectivities imposed by unitary matrices, of four rows and columns and unit determinant, whose elements are marks of $GF(3^2)$; that this group has the same order as G^* is seen on p. 310 of (**1**), where Dickson calls it $HO(4, 3^2)$. But there is no mention there, or perhaps for that matter elsewhere, of the geometrical setting: a [3] wherein the points, with homogeneous coordinates all marks of $GF(3^2)$, fall into two classes; isotropic points for which the unit Hermitian form

$$\mathbf{H} = x_1\bar{x}_1+x_2\bar{x}_2+x_3\bar{x}_3+x_4\bar{x}_4$$

is zero, and non-isotropic points for which, when their coordinate vectors are normalized, $\mathbf{H} = 1$. Of these non-isotropic points there are 540, and $HO(4, 3^2)$ permutes them transitively so that, in this setting, the occurrence of the subgroups is patent and taken in at a glance. So, indeed, is their isomorphism with $HO(3, 3^2)$, an isomorphism established by Miss Hartley at the close of (**6**). The partitioning of the 6048 operations described in § 2.4 of (**6**), and accomplished there by a final appeal to group characters by way of a respite from the perhaps over-elaborate geometry, is tantamount to the separation of $HO(3, 3^2)$ into conjugate sets. This separation is most expeditiously achieved by treating $HO(3, 3^2)$ as a group of unitary projectivities in a plane consisting of 28 isotropic and 63 non-isotropic points.

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