zero disorder. Thus, for example, $e_2{}^2e_1{}^2 = e_2(e_2e_1{}^2)$, $e_3e_2e_1 = e_3(e_2e_1)$. The latter example shows all that is left of the " Jacobi identity " in this special case.

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THE ISOMORPHISM BETWEEN $LF(2, 3^2)$ AND A_6

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The isomorphism of the linear fractional group $LF(2, 3^2)$, of order 360, and the alternating group A_6 is on record ([1], 309; and [5], with PSL for LF, 8 and 9). If one seeks not merely for record but for proof one might take the conclusion to Chapter XII of [1], in which chapter the subgroups of $LF(2, q)$ for any Galois field $GF(q)$ are obtained and catalogued; representations of $LF(2, q)$ as permutation groups have degrees equal to the indices of these subgroups whenever $LF(2, q)$ is simple, as it is known to be if $q > 3$. For $q = 3^2$ there are subgroups of index 6. But while the fact of the isomorphism has been common knowledge for so long, and while it cannot be gainsaid that a proof has been available, it may be questioned whether an essential reason for the existence of these subgroups of index 6 has been perceived. $LF(2, q)$ may be handled as

a group of projectivities on a line L, being constituted by those projectivities that can be imposed by matrices of determinant 1; L consists of $q+1$ points whose parameters are ∞ and the marks of the Galois field. Dickson indeed opens his chapter by representing $LF(2, q)$ as a permutation group on precisely these " letters "; he does not call them points, still less does he speak of projectivities or cross-ratios, and it is submitted that much can be gained by doing so, perhaps especially when q is a power of 3.

We now give a summary of the contents of the paper. In §1 we note the line L consisting of ten points, and enumerate the 720 projectivities thereon. In §2 we introduce cross-ratios and emphasize the peculiar property possessed by harmonic sets whenever the base field has characteristic 3: *all* permutations of their four members can be achieved by projectivities. §3 is concerned with the sextuples—three pairs every two of which are harmonic—of points on L, and §4 with the anharmonic pentads—five points no four of which are harmonic. $LF(2, 3^2)$, a subgroup of index 2 of the group of 720 projectivities on L, appears in §5, and we explain how, in regard to it, the harmonic sets on L fall into two separate opposed batches. In §6 the automorphism of period 2 of $GF(3^2)$ serves to define Hermitian forms which, like the harmonic sets, fall into two separate opposed batches, and in §7 we prove that every harmonic set is the set of zeros of a pair of Hermitian forms that are negatives of each other, and conversely. In §8 we define trisections based on a point of L and then, in §9, explain how they enable us to distribute the harmonic sets in quintuples. All the quintuples are displayed in §10 and it is they which afford the most compelling and illuminating proof, in §11, of the isomorphism.

$\mathcal{A}_6$ is isomorphic not only to $LF(2, 3^2)$ but also to $PO_2(4, 3)$, the second projective orthogonal group on four variables over $GF(3)$; thus $LF(2, 3^2)$ and $PO_2(4, 3)$ are isomorphic to each other. This is the special case, for $p^n = 3$, of the isomorphism established by Dickson ([1], 194) between $PO_2(4, p^n)$ and $LF(2, p^{2n})$, but his proof, as he concedes ([1], 308), does not apply in this special case and so does not establish the isomorphism with which we are concerned. There can be little doubt that van der Waerden's proof ([5], 25-26), had it been concluded accurately, would have been preferable; his instinct to geometrize was wise and a suggested completion of his argument is offered in the footnote to p. 277 of [2]. We give in §§12, 13 below some salient details of the correspondence between the two figures wherein $LF(2, 3^2)$ and $PO_2(4, 3)$ act as groups of projectivities: §12 is of course the application in this context of the transformation which van der Waerden ([5], 25) calls C_s. In §13 we compare the features of the figure on L with those of the figure built up in [2], and the comparison almost reduces the proof of the isomorphism to a mere glance at the two figures. The quintuples of harmonic sets on L are paralleled by certain pentahedra, but although these latter have already appeared

in [2] their existence was first suggested by the prior knowledge of the quintuples which were encountered when composing a set of lectures in the summer of 1952.

[☛]. During the revision of these paragraphs (July 1954) there has appeared Dieudonné's paper, on exceptional isomorphisms among the classical finite groups: *Canadian Journal of Mathematics* 6, 305-315; §3 thereof contains a proof of the isomorphism between $LF(2, 3^2)$ and $\mathcal{A}_6$. He does not mention cross-ratio, but does refer to pairs of points being harmonic. His *triplets* are obviously our sextuples, and his argument rests on the phenomenon that, in the nomenclature used here, the fifteen sextuples that are complements of harmonic sets of either batch together account for every pair of points of L once and once only.

1. The Galois Field $GF(3)$ consists of the marks 1, 0, -1, with the law of addition $1+1 = -1$; the residues of the integers to modulus 3 are an instance. It can be extended to the field $GF(3^2)$, or J as we call it here, by adjoining a root of any quadratic which is irreducible over $GF(3)$ and has its coefficients in this latter field. If, as in ([1], 7), $j^2 = j+1$ the marks

$$0,\ j,\ j^2,\ j^3,\ j^4,\ j^5,\ j^6,\ j^7,\ j^8$$

are the same as

$$0,\ j,\ j+1,\ -j+1,\ -1,\ -j,\ -j-1,\ j-1,\ 1;$$

we shall use the monomial marks

$$0,\ j,\ j^2,\ j^3,\ -1,\ -j,\ -j^2,\ -j^3,\ 1.$$

The non-zero marks are all powers of the primitive mark j; the four even powers are squares of marks of J, but the four odd powers are non-squares.

The marks of J can serve as homogeneous coordinates x_1, x_2 of points on a line L; x_1 and x_2 can each be any of the nine marks save that they are not permitted simultaneously to be 0, and since mx_1, mx_2 is the same point as x_1, x_2 for each mark $m \neq 0$ the number of points on L is $(9^2-1)/8 = 10$. We label each point by the mark $x = x_1/x_2$ whenever $x_2 \neq 0$; if $x_2 = 0$ we label the point ∞. The situation is a standard one in projective geometry ([4a], 59) and the projectivities on L form a triply transitive group of order $10.9.8 = 720$. These projectivities can be imposed by premultiplying the column vector $(x_1, x_2)'$ by non-singular two-rowed matrices whose elements all belong to J: either column of the matrix may be any of the eighty non-zero columns and, this column having been chosen, the other column can be a coordinate vector of any of those nine points of L other than that point of which the column first chosen is a coordinate vector. Hence there are altogether $80 \times 72 = 5760$ non-

singular matrices.   But since, if M is any one of these, the 8 matrices mM all impose the same projectivity, the number of distinct projectivities is only 720.

2. The group of projectivities is not quadruply transitive; two tetrads of points on L are projectively equivalent only when they have the same cross-ratios.   And there is an aspect of this topic of cross-ratio peculiar to fields of characteristic 3.   Any cross-ratios calculated with four of the ten marks of K, the aggregate consisting of J and the mark ∞, must themselves all belong to K, while the six cross-ratios of a tetrad arise from any one of them, say λ, by combinations of the two operations of taking the reciprocal $1/\lambda$ and the complement $1-\lambda$.   There are tetrads on L whose six cross-ratios are

$$j, \ -j, \ j^2, \ -j^2, \ j^3, \ -j^3 \, ;$$

the marks 0, 1, ∞ can never occur as cross-ratios of four distinct marks but there are tetrads, harmonic sets or, briefly, h-sets, whose six cross-ratios *are all* -1.   This is the feature peculiar to fields of characteristic 3: the occurrence of sets of four points all of whose six cross-ratios are equal. In such a field -1 is not merely its own reciprocal; it is also its own complement.   And whereas, over the real or complex field, as indeed over any field of characteristic other than 3, an h-set admits a dihedral group of 8 self-projectivities, any h-set on L admits the whole symmetric group $\mathcal{S}_4$; *every* permutation of its four points can be achieved by a projectivity. The standard condition

$$(a+b)(c+d) = 2(ab+cd) \tag{2.1}$$

for a, b to be harmonic to c, d becomes, over any field of characteristic 3,

$$(a+b)(c+d)+(ab+cd) = 0 \tag{2.2}$$

and is *symmetric* in a, b, c, d; the harmonic property is not now dependent on the anterior separation into complementary pairs.   This occurrence of an $\mathcal{S}_4$ of self-projectivities of four collinear points was all but recorded by Fano [3] in 1892.   His remark, on p.116, that the hypothesis of $ABCD$ and $ADBC$ being simultaneously harmonic is not intrinsically absurd, is made only to be instantly superseded, on p. 117, by the *assumption* that it is not satisfied.   Where it satisfied the tetrad would admit a self-projectivity of period 3 in addition to the usual dihedral group, and this amplifies the dihedral group to the whole $\mathcal{S}_4$.

3. The two pairs of marks given by

$$a_1 x^2 - h_1 x + b_1 = 0 \quad \text{and} \quad a_2 x^2 - h_2 x + b_2 = 0 \tag{3.1}$$

are, by (2.2), harmonic when

$$a_1 b_2 + h_1 h_2 + a_2 b_1 = 0. \tag{3.2}$$

The pair harmonic to both these pairs is given by

$$\begin{vmatrix} x^2 & -x & 1 \\ b_1 & h_1 & a_1 \\ b_2 & h_2 & a_2 \end{vmatrix} = 0; \tag{3.3}$$

it is, of course, understood that all coefficients of powers of the indeterminate x are in J. Suppose now that both quadratics (3.1) have their roots in J, and that (3.2) holds; then the quadratic (3.3) also has its roots in J. This is so because, over J,

$$(a_1 b_2 - a_2 b_1)^2 + (h_1 a_2 - h_2 a_1)(h_1 b_2 - h_2 b_1)$$

$$\equiv (a_1 b_2 + h_1 h_2 + a_2 b_1)^2 - (h_1{}^2 - a_1 b_1)(h_2{}^2 - a_2 b_2),$$

so that the discriminant of (3.3) is, if (3.2) is satisfied, a square in J provided that both discriminants of (3.1) are (note that if m is a square, say of n, then $-m$ is a square too, of nj^2). It follows that if A, B, C, D form an h-set on L, and they are partitioned into complementary pairs, say BC, AD, then there are two points E, F on L which form h-sets both with B, C and with A, D; and the pair E, F is uniquely determined by the partitioning. Each two of the three pairs

$$B, \; C; \quad A, \; D; \quad E, \; F$$

form an h-set. We call such a set of three pairs a *sextuple*.

Let us take now any h-set $ABCD$ on L. Then

$$BC, \; AD \text{ have a common harmonic pair } E, \; F;$$

$$CA, \; BD \text{ have a common harmonic pair } G, \; H;$$

$$AB, \; CD \text{ have a common harmonic pair } X, \; Y.$$

We show that all these points are distinct, and so account for the whole line. Suppose that we impose the projectivity I_{AB}, replacing each point of L by its companion in the involution whose foci are A and B or, as we may say, operating with the harmonic inversion in A and B. The points B, C, A, D become, in this order, B, D, A, C and so E, F become, in one order or other, G, H. But E, being distinct from A and B, is not invariant; neither can it be changed to F because, since $ADEF$ is an h-set, $ABEF$ is not. Thus G, H are distinct from E, F; this establishes the result.

Harmonic inversions, here as in other contexts, commute when their pairs of foci are harmonic. The proof is standard; if A, B, C, D are

harmonic then

$$I_{AB} I_{CD}(ABCD) = I_{AB}(BACD) = BADC,$$

$$I_{CD} I_{AB}(ABCD) = I_{CD}(ABDC) = BADC,$$

so that the product in either order is the projectivity wherein $ABCD \barwedge BADC$, that is I_{XY}. Moreover, since I_{XY} transposes B with A and C with D it must leave E, F, the common harmonic pair of BC and AD, invariant and so transpose its members. Hence $XYEF$ is an h-set, as likewise is the set composed of any two of the pairs EF, GH, XY. The complement of any h-set on L is a sextuple.

4. We call a set of five points on L an *anharmonic pentad* when no set of four points belonging to it is harmonic. The pentad q' complementary to an anharmonic pentad q is also anharmonic. For were it not so it would consist of some h-set and one point of the sextuple σ residual thereto; this would imply that σ contained the anharmonic pentad q, which no sextuple can do.

The point which completes an h-set with any three given points of L is, by (2.2), unique, so that there are $\frac{1}{4} \cdot {}^{10}C_3 = 30$ h-sets altogether. Of these $30 \times 4 \div 10 = 12$ include a given point.

Of the ${}^{10}C_5 = 252$ pentads on L there are 6 including any given h-set; since it is impossible for the same anharmonic pentad to contain two h-sets, for these would have three points of the pentad in common, the number of anharmonic pentads is $252 - 180 = 72$, and they consist of 36 complementary pairs.

The points of complementary anharmonic pentads q, q' are in (1, 1) correspondence. For suppose that $q \equiv \alpha\beta\gamma\delta\epsilon$, $q' \equiv \alpha'\beta'\gamma'\delta'\epsilon'$. Those points that complete h-sets one with each of the six triads of q that include α all belong to q', so that at least one point of q', which we label α', must occur twice; it cannot occur three times because some two of any three pairs of β, γ, δ, ϵ must share a member, whereas the h-set which includes this member and α, α' is unique. For the same reason, that three points of L belong to only one h-set, those pairs of β, γ, δ, ϵ that form h-sets with α, α' are complementary. We may suppose that

$$\alpha\beta\epsilon\alpha' \qquad \alpha\gamma\delta\alpha'$$

are h-sets; the other two h-sets that include α and α' may then be labelled

$$\alpha'\beta'\epsilon'\alpha \qquad \alpha'\gamma'\delta'\alpha,$$

so that of the six triads of q' that include α' two are amplified to h-sets by α.

This defines a (1, 1) correspondence between q and q' provided we can demonstrate that only α', and no second point of q', occurs in more than one of the six h-sets which contain α and two of β, γ, δ, ϵ. Suppose that β' so occurs. Then of the four h-sets which include α and β' two are

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completed by complementary pairs from β, γ, δ, ϵ and two by complementary pairs from γ', δ', ϵ', α'; that one of these latter two which includes α' must be $\alpha' \beta' \epsilon' \alpha$, so that the other is $\beta' \gamma' \delta' \alpha$. This is incompatible with $\alpha' \gamma' \delta' \alpha$ being an h-set, and therefore β' cannot so occur.

An example of an anharmonic pentad is

$$0 \quad 1 \quad j \quad j^2 \quad \infty,$$

and the complementary pentad, when its members are written each vertically beneath the one to which it corresponds, is

$$-1 \quad j^3 \quad -j \quad -j^3 \quad -j^2.$$

5. It has been remarked that the matrices mM impose the same projectivity whatever non-zero mark of J m may be. It is important to record that, since $|mM| = m^2|M|$ when M has two rows, the determinants of these eight matrices are either all squares or all non-squares in J, and so there are two distinct types of projectivity on L: direct ([4b], 38) projectivities, of square determinants, and indirect projectivities, of non-square determinants. Indirect projectivities certainly occur: for instance

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & \cdot \\ \cdot & j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

thus the direct projectivities form a subgroup of index 2 of the whole group of order 720. It is this group, of direct projectivities, that is the linear fractional group $LF(2, 3^2)$, of order 360. Any projectivity whose period is odd must belong to $LF(2, 3^2)$, for were its determinant a non-square no odd power thereof could be a square. The harmonic inversions too all belong to $LF(2, 3^2)$. For that whose foci are 0, ∞ is imposed by diag $(1, -1)$ of determinant $-1 = j^4$, while that whose foci are A, B is imposed by $S \cdot \text{diag}\,(1, -1) \cdot S^{-1}$, where S is a matrix imposing any projectivity that turns 0, ∞ into A, B (in either order). This implies that the $\mathcal{S}_4$ for which any h-set is invariant is a subgroup of $LF(2, 3^2)$ because it can be generated by the harmonic inversions in the pairs of the points that constitute it. Now an $\mathcal{S}_4$ has only 15 cosets in a group of order 360; thus each h-set is one among only 15 into which it can be transformed by operations of $LF(2, 3^2)$. The 30 h-sets fall into two disjoint categories: h-sets of opposite categories are certainly projectively equivalent, but they can only be transformed into one another by projectivities of non-square determinant.

6. Every mark of J is its own ninth power, and the replacement of each mark by its cube is an automorphism of J of period 2; that it is an automorphism follows because $(ab)^3 = a^3 b^3$ over any field and

$$(a+b)^3 = a^3 + b^3$$

over a field of characteristic 3. We therefore speak of the cube of a mark as its conjugate. The marks $0, 1, -1$ of $GF(3)$ are self-conjugate; the other six marks of J are conjugate to one another in pairs, each pair being roots of a quadratic whose coefficients all belong to $GF(3)$ and which is irreducible over that field. The automorphism is extended to cover K by the stipulation that ∞ is self-conjugate.

We can now introduce Hermitian forms, H-forms we shall call them, over J. For take the bilinear form $\bar{x}' H x$ where the components of the column vector $x = (x_1, x_2)'$ are independent indeterminates over J and $\bar{x}_1, \bar{x}_2$ are their cubes. This bilinear form is self-conjugate provided that $\bar{H}' = H$, that is

$$H = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$$

where a, c are both self-conjugate and so belong to $GF(3)$. The discriminant is

$$D = |H| = ac - b\bar{b} = \bar{D};$$

we assume that D is not zero, the H-form non-singular. But then D, since it is self-conjugate, must be 1 or -1; it can be either, as

$$\bar{x}_1 x_1 + \bar{x}_2 x_2 \quad \text{and} \quad \bar{x}_1 x_1 - \bar{x}_2 x_2$$

exemplify.

Suppose now that we make a linear transformation:

$$x = M\xi, \quad \bar{x} = \bar{M}\bar{\xi}.$$

Then $\bar{x}' H x = \bar{\xi}' \bar{M}' H M \xi = \bar{\xi}' N \xi$, and $\bar{N}' = N$ if $\bar{H}' = H$; an H-form remains an H-form under linear transformation. The discriminant of the new form is

$$\Delta = |\bar{M}' H M| = |\bar{M}'||H||M| = |\bar{M}||H||M| = |M|^3 D |M| = D|M|^4.$$

Now $|M|^4$ is 1 or -1 according as M is an even or an odd power of j, so that *no direct projectivity can ever change the discriminant* of an H-form.

7. When an H-form is written *in extenso* it is

$$\bar{x}' H x = [\bar{x}_1, \bar{x}_2] \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= a\bar{x}_1 x_1 + b\bar{x}_1 x_2 + \bar{b}x_1 \bar{x}_2 + c\bar{x}_2 x_2$$

$$= ax_1^4 + bx_1^3 x_2 + \bar{b}x_1 x_2^3 + cx_2^4,$$

a binary quartic without the term in $x_1^2 x_2^2$; its four zeros are therefore, by (2.2), harmonic, and should these zeros belong to K they represent

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an h-set on L. We shall see that the zeros do belong to K, and that the h-sets are simply the zeros of H-forms, and conversely.

The unit H-form $\bar{x}_1 x_1 + \bar{x}_2 x_2 \equiv x_1{}^4 + x_2{}^4$ has zeros j, $-j$, j^3, $-j^3$; they form an h-set. This can be transformed by operations of $LF(2, 3^2)$, into (including itself) 15 h-sets on L. These are sets of zeros of binary forms which, as transforms of an H-form, are all H-forms and which, since only direct projectivities are being used, all have $D = 1$. The other 15 h-sets on L arise likewise by direct projectivities from the zeros 1, -1, j^2, $-j^2$ of $\bar{x}_1 x_1 - \bar{x}_2 x_2 \equiv x_1{}^4 - x_2{}^4$. Thus every h-set on L is the set of zeros of an H-form, and an h-set belongs to one or other of the two categories according to the value of the discriminant of that H-form of which it provides the zeros. We may call the h-set positive or negative according as this discriminant is 1 or -1.

That the h-sets on L account for *all* non-singular H-forms over J follows by an easy enumeration. For how many H-forms have $D = ac - b\bar{b} = 1$? The possibilities are

(i) $ac = 1$, $b\bar{b} = 0$; either a or c is known when the other is given, and $b = 0$.

(ii) $ac = -1$, $b\bar{b} = 1$; either a or c is known when the other is given, and b is one of 1, -1, j^2, $-j^2$.

(iii) $ac = 0$, $b\bar{b} = -1$; there are now five choices $(0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$ for (a, c) while b can be any of j, $-j$, j^3, $-j^3$.

Thus the numbers of H-forms under headings (i), (ii), (iii) are 2, 8, 20; total 30. Each form $\bar{x}' H x$ is accompanied here by its negative $\bar{x}'(-H)x$ which has the same zeros and the same discriminant; each of the 15 pairs $\pm \bar{x}' H x$ of H-forms of positive discriminant provides by its zeros one of the 15 positive h-sets. And likewise for the negative h-sets, the H-forms with $D = -1$ also consisting of 15 pairs.

Enumeration shows too that, of the 12 h-sets that include any given point of L, 6 are of one sign and 6 of the other. Suppose the point is labelled ∞: it belongs to those h-sets that are zeros of H-forms wherein $a = 0$. If these have $D = 1$ they all occur under (iii) above; clearly there are 6 pairs $\pm \bar{x}' H x$ that fulfil the condition.

8. During this and the next section it is convenient to label the points of L by the italicised digits *0* to *9* inclusive.

Let *0123* be an h-set and *4* a fifth point of L. Harmonic inversion in *0* and *4* turns *1*, *2*, *3*, into, say, *7*, *8*, *9*; the remaining points *5* and *6* must be harmonic inverses in *0* and *4*. We thus obtain a *trisection*, as we will call it,

$$0 \mid 123 \mid 456 \mid 789 \qquad\qquad T_0$$

separating the nine points other than *0* into three triads each of which makes an *h*-set with *0*. The order in which the three points of a triad are written is immaterial here. Had we selected not *4* but, say, *5* the same trisection T_0 would have arisen; *6* and *4* are harmonic inverses in *0* and *5*, and those of *1, 2, 3* must be *7, 8, 9* in some order because if two of *1, 2, 3* are harmonic inverses in two points one of which is *0* the other must be the third member of *1, 2, 3* and cannot be *5*. The three *h*-sets of a trisection have the same sign because any two of them can be transformed into one another by a harmonic inversion; we therefore call the trisection positive or negative with the *h*-sets that compose it. Once the point on which a trisection is based has been selected the trisection is fixed by the choice of an *h*-set that is to belong to it, so that there are $6 \div 3 = 2$ trisections of either sign based on each point of *L*.

By way of example let us display the trisections based, in the original labelling, on ∞. Three points form an *h*-set with ∞ when the sum of their parameters is zero; the trisections are two positive ones

$$\infty \mid 0, \; j, \; -j \mid 1, \quad j^2, \; j^3 \mid -1, \; -j^2, \; -j^3$$

$$\infty \mid 0, \; j^3, \; -j^3 \mid 1, \; -j^2, \; j \mid -1, \quad j^2, \; -j$$

and two negative ones

$$\infty \mid 0, \; 1, \; -1 \mid j, \quad j^2, \; -j^3 \mid \quad j^3, \; -j^2, \; -j$$

$$\infty \mid 0, \; j^2, \; -j^2 \mid 1, \; -j, \; -j^3 \mid -1, \quad j, \quad j^3.$$

9. Call those trisections which include the *h*-set *0123* and are based on *0, 1, 2, 3* respectively T_0, T_1, T_2, T_3; those which include *0456* and are based on *0, 4, 5, 6* respectively T_0, T_4, T_5, T_6.

Since the *h*-set which includes *4, 5, 6* is completed by *0* the points *4, 5, 6* do not form an *h*-set with *1*; they therefore belong two to one and one to the other of the two *h*-sets which, together with *0123*, compose T_1. Select this latter *h*-set, which includes only one of *4, 5, 6*; we may label this one point *4*. This *h*-set, as containing *1* but neither *2* nor *3*, is also the *h*-set of T_4 which contains only one of *1, 2, 3*. Using the other trisections similarly we obtain a *quintuple* of *h*-sets labelled, as yet incompletely,

$$0123$$
$$0456$$
$$14$$
$$25$$
$$36$$

where the blanks, two in each of the last three sets, are occupied only by *7, 8, 9*.

Now there is on L a projectivity $\wp$ wherein

$$0123 \barwedge 0231$$

and which, being of period three, belongs to $LF(2, 3^2)$. Since both the point 0 and the h-set 0123, although admitting a permutation of its members, are invariant for $\wp$ so is T_0; and since $\wp$ cannot, in virtue of its odd period, transpose the other two h-sets of T_0 these, 0456 and 0789, must both be invariant. Since $\wp$ is genuinely of period 3 and not identity it must permute $4, 5, 6$ cyclically, as also $7, 8, 9$. The above quintuple, as its construction shows, is uniquely determined by 0123 and 0456 and so cannot be altered by $\wp$; the remaining three h-sets in it must be permuted cyclically with $1, 2, 3$ and so, with the labelling now used, $\wp$ transforms $4, 5, 6$ into $5, 6, 4$. Of the two digits among $7, 8, 9$ that complete the third h-set of the quintuple one, which we now label 7, has to be transformed by $\wp$ into the other, which we now label 8. And so the quintuple is

$$0123$$
$$0456$$
$$1478$$
$$2589$$
$$3697$$

and includes each point of L exactly twice. Indeed the points are determined as being common one to each of the ${}^5C_2$ pairs of h-sets in the quintuple. Since two h-sets with a single common point are members of a trisection based on this point all the h-sets of a quintuple have the same sign, and we describe a quintuple as positive or negative with the h-sets that compose it.

10. Two h-sets of a trisection determine the whole quintuple to which they belong, and this quintuple is equally well determined by any two of its five members. Now there are twenty positive trisections, two based on each point of L, each providing three pairs of h-sets; hence the number of positive quintuples is

$$20 \times 3 \div {}^5C_2 = 6.$$

Every positive h-set, moreover, belongs to two of these six quintuples. For it belongs to four trisections (one based on each of its four points) and determines a quintuple when taken with either of the other two h-sets of any of these four trisections; but, since it determines the same quintuple when taken with any of the other four members of any quintuple to which it does belong, it occurs in $4 \times 2 \div 4 = 2$ quintuples. Indeed the 15 positive h-sets prove to be the " intersections by pairs " of the 6 positive quintuples.

All this is exactly paralleled in the negative quintuples, and it only remains to exhibit the quintuples in their entirety.

The twelve quintuples of harmonic sets

NEGATIVE

∞	0	1	-1	∞	0	1	-1	∞	j	j^2	$-j^3$
∞	j	j^2	$-j^3$	∞	$-j$	$-j^2$	j^3	∞	$-j$	$-j^2$	j^3
0	j^3	$-j$	j^2	0	$-j^3$	j	$-j^2$	j	0	1	j^3
1	$-j^2$	j^3	$-j^3$	1	j^2	$-j$	j	j^2	-1	1	$-j^2$
-1	$-j^2$	$-j$	j	-1	j^2	$-j^3$	j^3	$-j^3$	-1	0	$-j$

0	j^3	$-j$	j^2	1	$-j^2$	j^3	$-j^3$	-1	$-j^2$	$-j$	j
0	$-j^3$	j	$-j^2$	1	j^2	$-j$	j	-1	j^2	$-j^3$	j^3
j^3	∞	-1	j	$-j^2$	∞	0	j^2	$-j^2$	0	∞	j^2
$-j$	1	∞	$-j^3$	j^3	-1	∞	j	$-j$	∞	1	$-j^3$
j^2	-1	1	$-j^2$	$-j^3$	0	-1	$-j$	j	1	0	j^3

POSITIVE

∞	0	j	$-j$	∞	0	j	$-j$	∞	j^2	j^3	1
∞	j^2	j^3	1	∞	$-j^2$	$-j^3$	-1	∞	$-j^2$	$-j^3$	-1
0	-1	$-j^2$	j^3	0	1	j^2	$-j^3$	j^2	0	j	-1
j	$-j^3$	-1	1	j	j^3	$-j^2$	j^2	j^3	$-j$	j	$-j^3$
$-j$	$-j^3$	$-j^2$	j^2	$-j$	j^3	1	-1	1	$-j$	0	$-j^2$

0	-1	$-j^2$	j^3	j	$-j^3$	-1	1	$-j$	$-j^3$	$-j^2$	j^2
0	1	j^2	$-j^3$	j	j^3	$-j^2$	j^2	$-j$	j^3	1	-1
-1	∞	$-j$	j^2	$-j^3$	∞	0	j^3	$-j^3$	0	∞	j^3
$-j^2$	j	∞	1	-1	$-j$	∞	j^2	$-j^2$	∞	j	1
j^3	$-j$	j	$-j^3$	1	0	$-j$	$-j^2$	j^2	j	0	-1

11. Any operation of $LF(2, 3^2)$ has to permute the positive quintuples. If it imposes the identity permutation on them then every positive h-set, being common to two quintuples, is also invariant, as is every point of L, common to six positive h-sets. The operation is thus identity. Wherefore the 360 operations of $LF(2, 3^2)$ impose a group of 360 distinct permutations on the positive quintuples; this can only be the alternating group $\mathcal{A}_6$. The same discussion applies to the negative quintuples. This conclusion establishes the known isomorphism between $LF(2, 3^2)$ and $\mathcal{A}_6$ by reasoning which affords the geometrical explanation of why it has to occur. And, almost incidentally, the geometry shows at a glance the subgroups $\mathcal{A}_5$ and $\mathcal{S}_4$, there being, because of the fundamental opposition between oppositely signed constructs, two conjugate sets of each of these two types of subgroups; there are two conjugate sets each of six $\mathcal{A}_5$, each subgroup consisting of operations for which a quintuple is invariant, and two conjugate sets each of fifteen $\mathcal{S}_4$, each subgroup consisting of operations for which an h-set is invariant.

12. The foregoing description of the representation of $\mathcal{A}_6$ as $LF(2, 3^2)$ is complete in itself, but it is allied to the representation of $\mathcal{A}_6$ as the second projective orthogonal group $PO_2(4, 3)$ in four variables over $GF(3)$ that was described in [2]; we therefore conclude by giving the direct relation between the two representations. The essential bridge is the (1, 1) correspondence between the ten points of L and the ten points of the ellipsoid F used in [2].

When $GF(3)$ is extended to J two complementary reguli, of ten lines each, appear on what was F. Parameters u, v, one for each regulus, can be chosen so that the ten points of F occur when $v = \bar{u}$. Indeed the quadric

$$x^2 + y^2 + z^2 = t^2$$

admits, when marks of J are available, the parametric representation

$$x = u+v, \quad y = -j^2(u-v), \quad z = uv-1, \quad t = uv+1.$$

The points

$$x = u+\bar{u}, \quad y = -j^2(u-\bar{u}), \quad z = u\bar{u}-1, \quad t = u\bar{u}+1$$

are as follows:

u	0	1	j	j^2	j^3	-1	$-j$	$-j^2$	$-j^3$	∞
x	.	-1	1	.	1	1	-1	.	-1	.
y	.	.	-1	-1	1	.	1	1	-1	.
z	-1	.	1	.	1	.	1	.	1	1
t	1	-1	.	-1	.	-1	.	-1	.	1

Since the points arise on assigning the same marks to u that label the points of L there is a (1, 1) correspondence between the points of L [over J] and the points of the ellipsoid [over $GF(3)$].

13. The points of the three-dimensional space in which F lies all have coordinates in $GF(3)$, and those points of F that lie in the polar plane of (ξ, η, ζ, τ) are given by

$$\xi(u+\bar{u})+\eta j^2(\bar{u}-u)+\zeta(u\bar{u}-1) = \tau(u\bar{u}+1),$$

$$(\zeta-\tau)u\bar{u}+(\xi+\eta j^2)\bar{u}+(\xi-\eta j^2)u-(\zeta+\tau) = 0.$$

The expression on the left here is, if u is replaced by u_1/u_2, an H-form; its discriminant is

$$-(\zeta-\tau)(\zeta+\tau)-(\xi+\eta j^2)(\xi-\eta j^2) \equiv \tau^2-\xi^2-\eta^2-\zeta^2,$$

so that the H-form is non-singular provided that (ξ, η, ζ, τ) does not lie on F. Furthermore this discriminant is of one sign or the other according to which of the two categories ([2], 271) of points not on F includes (ξ, η, ζ, τ). This result shows that the h-sets on L correspond one to each non-singular plane section of F, and conversely, and that h-sets are similarly or oppositely signed according as the poles of the corresponding plane sections of F are similarly or oppositely signed. This is the cardinal feature of the correspondence; it secures that the projectivities of $PO_2(4, 3)$ in the three-dimensional space answer one to each direct projectivity on L.

We may note a few other features without delaying to substantiate them in every detail.

That the complement of an h-set on L is a sextuple answers to the fact that the six points of F not on a given non-singular plane section lie by pairs on three concurrent chords ([2], 270).

The three h-sets of a trisection based on θ answer to sections of F by the three planes, other than the tangent plane, through one of the four tangent lines whose contact with F is the point corresponding to θ. There is a (1, 1) correspondence between trisections on L and tangent lines of F; when trisections are similarly signed so are the corresponding tangent lines, and conversely.

Just as the plane of a non-singular section contains four tangent lines of F, all signed alike, so any h-set belongs to four trisections, all signed alike. The five h-sets of a quintuple are such that every pair belongs to some trisection; they correspond to the sections of F by the faces of a pentahedron whose edges are all tangent lines.

An anharmonic pentad on L corresponds to a cycle ([2], 281) on F; just as there are 36 complementary pairs of anharmonic pentads on L so there are 36 complementary pairs of cycles on F.

An involution whose foci, like α and α' in §4, correspond in the (1, 1) correspondence between complementary anharmonic pentads q, q', leaves both q and q' invariant; the five such involutions that arise from q and q' belong to a dihedral group of order ten which answers to the D_{10} at the foot of p. 283 of [2]. Each of the 36 pairs of complementary anharmonic pentads affords such a group, and the 36 such subgroups of $\mathscr{A}_6$ are all thereby accounted for.

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