Blown-up toric surfaces with non-polyhedral effective cone

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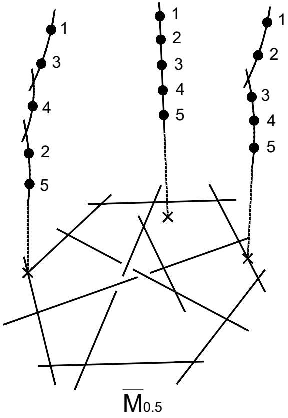
Moduli space of stable rational curves



- $M_{0,n} = \left\{ {p_1,\ldots,p_n \in \mathbb{P}^1 \atop p_i \neq p_j} \right\} / \mathrm{PGL}_2$
- $M_{0,3} = \mathrm{pt}$ (send $p_1, p_2, p_3 \to 0, 1, \infty$)
- $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ via cross-ratio
- $\overline{M}_{0,4} = \mathbb{P}^1$
- $\overline{M}_{0,n}$ functorial compactification
- $\overline{M}_{0,5} = dP_5$ (del Pezzo of degree 5)
- $\overline{M}_{0,6}$ = blow-up of the Segre cubic at the 10 nodes ($-K$ is big and nef)
- $\overline{M}_{0,n}$, $n \geq 8$: $-K$ not pseudo-effective

The effective cone of $\overline{M}_{0,n}$

- (Kapranov models) $\overline{M}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-1}{3}} \mathrm{Bl}_{\binom{n-1}{2}} \mathrm{Bl}_{n-1} \mathbb{P}^{n-3}$ (blow-up $n-1$ points, all lines, planes,... spanned by them)
- Every boundary divisor is contracted by a Kapranov map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ and generates an extremal ray of $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$
- $\overline{\mathrm{Eff}}(\overline{M}_{0,5})$ is generated by the 10 boundary divisors (-1 curves)
- $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$ is generated by boundary and Keel–Vermeire divisors (Hassett–Tschinkel 2002)

The effective cone of $\overline{\mathrm{M}}_{0,n}$

- $\overline{\mathrm{Eff}}(\overline{\mathrm{M}}_{0,n})$ has many extremal rays, generated by hypertree divisors, contractible by birational contractions (C.–Tevelev 2013)
- More extremal divisors for $n \geq 7$ (Opie 2016, based on Chen–Coskun 2014, Doran–Giansiracusa–Jensen 2017, Gonzàlez 2020)

Theorem (C.–Laface–Tevelev–Ugaglia 2020)

The cone $\overline{Eff}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic p, for an infinite set of primes p of positive density (including all primes up to 2000).

Rational contractions

Definition

A rational contraction $X \dashrightarrow Y$ between $\mathbb{Q}$-factorial, normal projective varieties, is a rational map that can be decomposed into a sequence of

- small $\mathbb{Q}$-factorial modifications,
- surjective morphisms between $\mathbb{Q}$-factorial varieties.

Theorem

Let $X \dashrightarrow Y$ be a rational contraction. If X has any of these properties then Y does as well:

- *Mori Dream Space (Keel–Hu 2000, Okawa 2016)*
- *(rational) polyhedral effective cone (BDPP 2013)*

$\overline{\mathrm{M}}_{0,n}$ and blow-ups of toric varieties

Philosophy (Fulton)

$\overline{M}_{0,n}$ is similar to a toric variety.

Not quite true. Instead, $\overline{\mathrm{M}}_{0,n}$ is similar to a blown up toric variety:

Theorem (C.–Tevelev 2015)

There are rational contractions

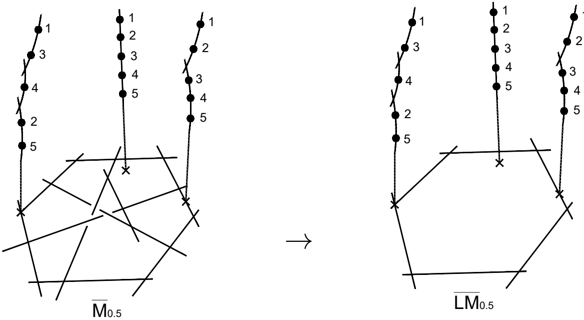
$$Bl_e\,\overline{LM}_{0,n+1} \dashrightarrow \overline{M}_{0,n} \to Bl_e\,\overline{LM}_{0,n},$$

where $\overline{LM}_{0,n}$ is the Losev-Manin moduli space of dimension $n-3$, $e =$ identity point of the open torus $\mathbb{G}_m^{n-3} \subseteq \overline{LM}_{0,n}$.

Kapranov description: $\overline{\mathrm{LM}}_{0,n} = \ldots \mathrm{Bl}_{\binom{n-2}{3}}\,\mathrm{Bl}_{\binom{n-2}{2}}\,\mathrm{Bl}_{n-2}\,\mathbb{P}^{n-3}$
(blow-up $n-2$ points, all lines, planes,... spanned by them)

The Losev-Manin moduli space $\overline{\mathrm{LM}}_{0,n}$

The Losev-Manin moduli space $\overline{\mathrm{LM}}_{0,n}$ is the Hassett moduli space of stable rational curves with n markings and weights $1, 1, \epsilon, \ldots, \epsilon$.



trees of $\mathbb{P}^1$'s

chains of $\mathbb{P}^1$'s

Universal blown up toric variety

Theorem

X projective $\mathbb{Q}$-factorial toric variety. For $n \gg 0$

- *there exists a toric rational contraction $\overline{LM}_{0,n} \dashrightarrow X$*
- *there exists a rational contraction $Bl_e \overline{LM}_{0,n} \dashrightarrow Bl_e X$*

Corollary (C.–Tevelev, 2015)

$\overline{M}_{0,n}$ is not a MDS in characteristic 0 for $n \gg 0$. There exists a rational contraction

$$\overline{M}_{0,n} \dashrightarrow Bl_e \mathbb{P}(a, b, c)$$

for some a, b, c such that $Bl_e \mathbb{P}(a, b, c)$ has a nef but not semi-ample divisor (Goto–Nishida–Watanabe 1994).

Remark

This argument cannot work in characteristic p, where, by Artin's contractibility criterion, a nef divisor on $Bl_e \mathbb{P}(a, b, c)$ is semi-ample.

Blown up toric surfaces

Theorem (C.-Laface-Tevelev-Ugaglia 2020)

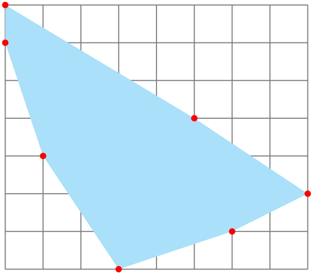
There exist projective toric surfaces $\mathbb{P}_\Delta$, given by good polygons Δ, such that $\overline{Eff}(Bl_e \mathbb{P}_\Delta)$ is not polyhedral in characteristic 0.

For some of these toric surfaces, $\overline{Eff}(Bl_e \mathbb{P}_\Delta)$ is not polyhedral in characteristic p for an infinite set of primes p of positive density.

Corollary

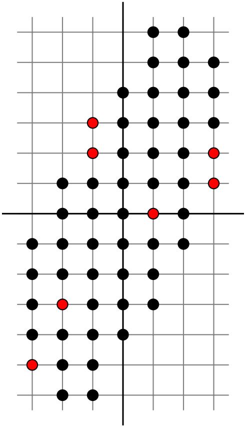
For $n \geq 10$, the space $\overline{M}_{0,n}$ is not a MDS both in characteristic 0 and in characteristic p for an infinite set of primes of positive density, including all primes up to 2000.

Example of a good polygon



Example of a good polygon

There is a rational contraction $\overline{\mathrm{M}}_{0,10} \to \mathrm{Bl}_e\, \overline{\mathrm{LM}}_{0,10} \dashrightarrow \mathrm{Bl}_e\, \mathbb{P}_\Delta$:



Red $\to$ normal fan of Δ Black $\to$ projection of fan of $\overline{\mathrm{LM}}_{0,10}$

Elliptic Pairs

A good polygon will correspond to an elliptic pair $(\mathrm{Bl}_e \mathbb{P}_\Delta, C)$.

Definition

An elliptic pair (C, X) consists of

- *a projective rational surface X with log terminal singularities,*
- *an arithmetic genus 1 curve $C \subseteq X$ such that $C^2 = 0$,*
- *C disjoint from singularities of X.*

Restriction map $\mathrm{res} : C^\perp \to \mathrm{Pic}^0(C), \quad D \mapsto \mathcal{O}(D)|_C$

$C^\perp \subseteq \mathrm{Cl}(X)$ orthogonal complement of C, $C^\perp$ contains C

Definition

The order $e(C, X)$ of the pair (C, X) is the order of $\mathrm{res}(C)$ in $\mathrm{Pic}^0(C)$.

In characteristic p, we have $e(C, X) < \infty$.

Order of an elliptic pair

The order $e(C,X)$ is the smallest integer $e > 0$ such $h^0(eC) > 1$.

Lemma

- *If $e = e(C,X) < \infty$, then $h^0(eC) = 2$ and $|eC| : X \to \mathbb{P}^1$ is an elliptic fibration with C a multiple fiber.*
- *If $e(C,X) = \infty$, then C is rigid :*

$$h^0(nC) = 1 \quad \text{for all} \quad n \geq 1.$$

In this case, $\overline{\mathit{Eff}}(X)$ is not polyhedral if $\rho(X) \geq 3$.

Proof.

Observation (Nikulin): If $\rho(X) \geq 3$ and $\overline{\mathrm{Eff}}(X)$ is polyhedral, then

- $\overline{\mathrm{Eff}}(X)$ is generated by negative curves,
- every irreducible curve with $C^2 = 0$ is contained in the interior of a facet; in particular, a multiple moves. □

Minimal elliptic pairs

Polyhedrality when $e(C, X) < \infty$? In general, for any $e(C, X)$:

Definition

An elliptic pair (C, X) is called minimal if there are no smooth rational curves $E \subseteq X$ such that $K \cdot E < 0$ and $C \cdot E = 0$.

Theorem

For an elliptic pair (C, X), there exists a minimal elliptic pair (C, Y) and a morphism $\pi : X \to Y$, which is an isomorphism in a neighborhood of C.

In particular, $e(C, X) = e(C, Y)$.

Proof.

$\mathcal{O}(K + C)|_C \simeq \mathcal{O}_C \Rightarrow K \cdot C = 0$

(C, X) is minimal $\Leftrightarrow K + C$ is nef $\Leftrightarrow K + C \sim \alpha C$, $\alpha \in \mathbb{Q}$

Run $(K + C)$-MMP: contract all curves $E \subseteq X$ with $K \cdot E < 0$, $C \cdot E = 0$.

□

Minimal + Du Val singularities

Definition

Since $K \cdot C = 0$, define on $\mathrm{Cl}_0(X) = C^{\perp}/\langle K\rangle$ the reduced restriction map

$$\overline{\mathrm{res}}: \mathrm{Cl}_0(X) \to \mathrm{Pic}^0(C)/\langle \mathrm{res}(K)\rangle$$

Theorem

Let (C, Y) be an elliptic pair such that Y has Du Val singularities. Let Z be the minimal resolution of Y. Then

$$(C, Y) \quad \textit{minimal} \quad \Leftrightarrow \quad (C, Z) \quad \textit{minimal} \quad \Leftrightarrow \quad \rho(Z) = 10.$$

In this case $\mathrm{Cl}_0(Z) \simeq \mathbb{E}_8$.

Assume (C, Y) minimal elliptic pair with $\rho(Y) \geq 3$ and $e(C, Y) < \infty$:

$$\overline{\mathit{Eff}}(Y) \quad \textit{polyhedral} \quad \Leftrightarrow \quad \overline{\mathit{Eff}}(Z) \quad \textit{polyhedral} \quad \Leftrightarrow$$

$$\mathrm{Ker}(\overline{\mathrm{res}}) \quad \textit{contains 8 linearly independent roots of} \quad \mathbb{E}_8.$$

Upshot

(C, Y) = minimal model of elliptic pair (C, X)

- $e(C,X) = \infty \Rightarrow \overline{\mathrm{Eff}}(X)$, $\overline{\mathrm{Eff}}(Y)$ not polyhedral (if $\rho \geq 3$)

  In this case, Y *is* Du Val: $\mathcal{O}(C)|_C$ not torsion implies $-K_Y \sim C$
- $e(C,X) < \infty$ *and* Y is Du Val $\Rightarrow$ polyhedrality criterion for $\overline{\mathrm{Eff}}(Y)$

Problem

- *Suppose* $C, X, \mathrm{Cl}(X)$ *are defined over* $\mathbb{Q}$, $e(C,X) = \infty$
- $X \to Y$ *extends to the morphism of integral models* $\mathcal{X} \to \mathcal{Y}$ *over* $\operatorname{Spec}\mathbb{Z}$ *(outside of finitely many primes of bad reduction)*
- (C_p, Y_p) *is still the minimal elliptic pair associated to* (C_p, X_p)
- $e(C_p, X_p) < \infty$. *Study distribution of "polyhedral" primes*

Blown up toric surfaces

Lattice polygon $\Delta \subseteq \mathbb{R}^2 \Longrightarrow (\mathbb{P}_\Delta, \mathcal{L}_\Delta)$ associated polarized toric surface

Set $X = \mathrm{Bl}_e \mathbb{P}_\Delta$ and let $m > 0$ integer. Then $X, \mathrm{Cl}(X)$ are defined over $\mathbb{Q}$.

Definition

A lattice polygon Δ with at least 4 vertices is good if there exists

$$C \in |\mathcal{L}_\Delta - mE|$$

irreducible such that (C, X) is an elliptic pair with $e(C, X) = \infty$:

(I) *The Newton polygon of C coincides with Δ ($\Leftrightarrow C \subseteq X^{smooth}$),*

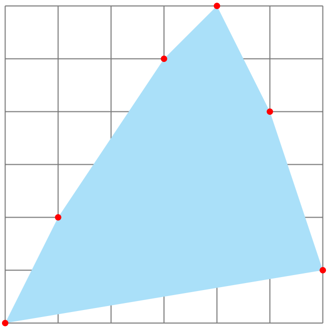
(II) *$Vol(\Delta) = m^2$ and $|\partial\Delta \cap \mathbb{Z}^2| = m$ ($\Leftrightarrow C^2 = 0$, $p_a(C) = 1$),*

(III) *The restriction $\mathrm{res}(C) = \mathcal{O}_X(C)|_C$ is not torsion in $\mathrm{Pic}^0(C)$ over $\mathbb{Q}$.*

Theorem

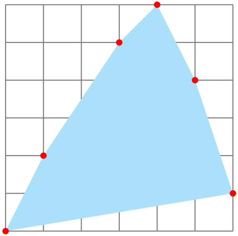
If Δ is a good polygon, then $\overline{\mathit{Eff}}(X)$ is not polyhedral in characteristic 0.

Example



$$\mathrm{Vol}(\Delta) = 36, \quad |\partial\Delta \cap \mathbb{Z}^2| = 6$$

Example of a good polygon

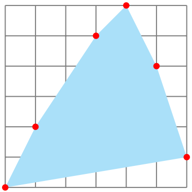


$$\mathrm{Vol}(\Delta) = 36, \quad |\partial\Delta \cap \mathbb{Z}^2| = 6$$

The linear system $|\mathcal{L}_\Delta - 6E|$ contains a unique curve C with equation

$$x^4y^6 + 6x^5y^4 - 2x^4y^5 - 14x^5y^3 - 17x^4y^4 - 4x^3y^5 +$$
$$+ x^6y + 11x^5y^2 + 38x^4y^3 + 26x^3y^4 - 9x^5y - 27x^4y^2 -$$
$$-34x^3y^3 + 22x^4y + 16x^3y^2 - 10x^2y^3 - 24x^3y +$$
$$+10x^2y^2 + 15x^2y + 5xy^2 - 11xy + 1 = 0.$$

Example of a good polygon



The curve C is a smooth elliptic curve labelled 997.a1 in the LMFDB database. It has the minimal equation

$$y^2 + y = x^3 - x^2 - 24x + 54$$

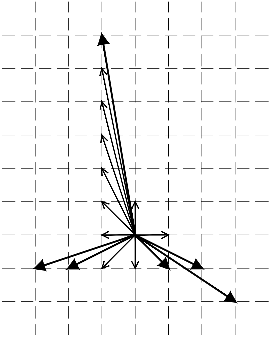
The Mordell-Weil group $C(\mathbb{Q})$ is $\mathbb{Z} \times \mathbb{Z}$, with generators

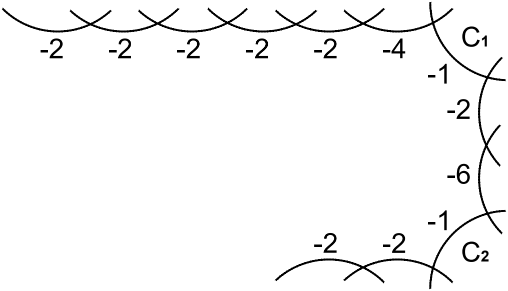
$$Q = (1, 5), \quad P = (6, -10)$$

Computation : $\mathrm{res}(C) = -Q$ (not torsion, so Δ is good)

Example - Minimal resolution

Fan of the minimal resolution $\tilde{\mathbb{P}}_\Delta$ of $\mathbb{P}_\Delta$:





The proper transforms C_1, C_2 of 1-parameter subgroups $\{v = 1\}, \{u = 1\}$

- have self-intersection -1 on $\mathrm{Bl}_e \tilde{\mathbb{P}}_\Delta$, and also on $X = \mathrm{Bl}_e \mathbb{P}_\Delta$
- have $C \cdot C_1 = C \cdot C_2 = 0$

Example - Minimal elliptic pair

(C, X) elliptic pair, $X = \mathrm{Bl}_e \mathbb{P}_\Delta$

Zariski decomposition $K_X + C = N + P$, $N = 3C_1 + 2C_2$, $P = 0$

To get minimal elliptic pair (C, Y), contract C_1, C_2.

$$\begin{array}{ccc} \mathrm{Bl}_e \tilde{\mathbb{P}}_\Delta & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

$Z \to Y$ minimal resolution, $\rho(X) = 5$, $\rho(Y) = 3$, $\rho(Z) = 10$

T = sublattice spanned by classes of (-2) curves above singularities of Y

Computation : $T = \mathbb{A}^7$

Example - Minimal resolution

$Z \to Y$ minimal resolution of Y, $\mathrm{Cl}(Z) = \mathrm{Cl}(Y) \oplus T$

T = sublattice spanned by classes of (-2) curves above singularities of Y

$\mathrm{Cl}_0(Y) = \mathrm{Cl}_0(Z)/T = \mathbb{E}_8/\mathbb{A}^7 \cong \mathbb{Z}$

Reduced restriction map $\overline{\mathrm{res}} : \mathrm{Cl}_0(Y) \to \mathrm{Pic}^0(C)/\langle Q \rangle$, $Q = (1, 5)$

$\overline{\mathrm{Eff}}(Y)$ is not polyhedral in characteristic $p \Leftrightarrow$

$\Leftrightarrow \overline{\mathrm{res}}(\beta) \neq 0$ for all β = image in $\mathrm{Cl}_0(Y)$ of a root in $\mathbb{E}_8 \setminus T$

If $\alpha \in \mathrm{Cl}_0(Y)$ generator $\Longrightarrow$ Images of roots of $\mathbb{E}_8$ are $\pm k\alpha$, for $0 \leq k \leq 3$

Computation : $\mathrm{res}(\alpha) = P - Q$, where $P = (6, -10)$

$\overline{\mathrm{Eff}}(Y)$ not polyhedral in characteristic $p \Leftrightarrow k\overline{P} \notin \langle \overline{Q} \rangle$ for $k = 1, 2, 3$

Example - Non-polyhedral primes

Prove that the set of primes p such that

$$\overline{P}, 2\overline{P}, 3\overline{P} \notin \langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$$

has positive density.

Fix q prime. It suffices to prove that the set of primes p such that

- q divides the index of $\langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$
- q does not divide the index of $\langle 6\overline{P} \rangle \subseteq C(\mathbb{F}_p)$

has positive density.

Apply Chebotarev's Density theorem + a theorem of Lang-Trotter

Lang-Trotter Criterion

C elliptic curve defined over $\mathbb{Q}$, without complex multiplication over $\overline{\mathbb{Q}}$.

Fix q prime and let $C[q] \subset C(\overline{\mathbb{Q}})$ be the q-torsion points of C.

For $x \in C(\mathbb{Q})$, choose $x/q \in C(\overline{\mathbb{Q}})$ and consider the Galois extension of $\mathbb{Q}$

$$K_x = \mathbb{Q}(C[q], x/q)$$

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$$K_x = \mathbb{Q}(C[q], x/q)$$

For almost all primes q, we have $\mathrm{Gal}(K_x/\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2$

For any $L/\mathbb{Q}$ Galois, for almost all primes p, there is a Frobenius element $\sigma_p \in \mathrm{Gal}(L/\mathbb{Q})$ of p in $L/\mathbb{Q}$ (well-defined up to conjugacy).

Lang-Trotter (1976): q divides the index of $\langle \overline{x} \rangle \subseteq C(\mathbb{F}_p) \Leftrightarrow$

$$\Leftrightarrow \text{the Frobenius element } \sigma_p = (\gamma_p, \tau_p) \in \mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2$$

with γ_p with 1 as an eigenvalue, and either $\gamma_p = 1$, or $\tau_p \in Im(\gamma_p - 1)$.

Non-polyhedral primes

C elliptic curve defined over $\mathbb{Q}$, without complex multiplication over $\overline{\mathbb{Q}}$.

For $x, y \in C(\mathbb{Q})$, let $K_{x,y} = \mathbb{Q}(C[q], x/q, y/q)$ (Galois extension of $\mathbb{Q}$).

The Frobenius element σ_p of p in $K_{x,y}/\mathbb{Q}$ is

$$\sigma_p = (\gamma_p, \tau_p, \tau'_p) \in \mathrm{Gal}(K_{x,y}/\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes ((\mathbb{Z}/q\mathbb{Z})^2)^2$$

where $(\gamma_p, \tau_p) \in \mathrm{Gal}(K_x/\mathbb{Q})$, $(\gamma_p, \tau'_p) \in \mathrm{Gal}(K_y/\mathbb{Q})$ (Frobenius elements).

By Lang-Trotter, the set of primes p such that

- q divides the index of $\langle \overline{x} \rangle \subseteq C(\mathbb{F}_p)$
- q does not divide the index of $\langle \overline{y} \rangle \subseteq C(\mathbb{F}_p)$

is the set of primes p such that:

γ_p has 1 as an eigenvalue, $\tau_p \in Im(\gamma_p - 1)$, $\tau'_p \notin Im(\gamma_p - 1)$

This condition is closed under conjugacy (and such elements exist).

Non-polyhedral primes

The set of non-polyhedral primes $p < 2000$ for our running example of a good polygon:

$$7, 11, 41, 67, 173, 307, 317, 347, 467, 503, 523, 571, 593, 631, 677, 733,$$

$$797, 809, 811, 827, 907, 937, 1019, 1021, 1087, 1097, 1109, 1213, 1231,$$

$$1237, 1259, 1409, 1433, 1439, 1471, 1483, 1493, 1567, 1601, 1619, 1669,$$

$$1709, 1801, 1811, 1823, 1867, 1877, 1933, 1951, 1993$$

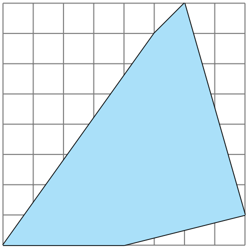
This gives 18% of the primes under 2000.

Further Examples

There are:

- 135 toric surfaces corresponding to good polygons with volume ≤ 49;
- Infinite sequences of good pentagons with all primes polyhedral;
- Infinite sequences of good heptagons. For all but finitely many, the set of non-polyhedral primes has positive density.

An infinite sequence of pentagons



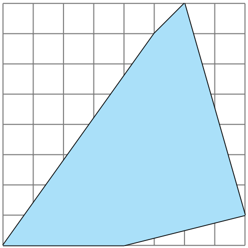
Polygon Δ is a pentagon with vertices

$$(0,0), \quad (2k,0), \quad (2k+4,1), \quad (2k+2,2k+4), \quad (2k+1,2k+3)$$

$$\mathrm{Vol}(\Delta) = (2k+4)^2, \quad |\partial\Delta \cap \mathbb{Z}^2| = 2k+4$$

Then Δ is good for every $k \geq 1$.

An infinite sequence of pentagons



Equation of C is:

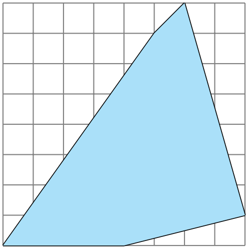
$$(uv + 2x_0^{k+2})\left(u - 2x_0^{k+1}\right)^{2k+3} - 2u^{k+1}(v + x_0)^{k+2}\left(u - 2x_0^{k+1}\right)^{k+2} -$$

$$-u^{2k+1}(v + x_0)^{2k+3}\left(uv + u(x_0 - x_1) + 2x_1 x_0^{k+1}\right) = 0,$$

where

$$x_0 = 2(k+1)(3k+2), \quad x_1 = 2(k+1)(3k+4).$$

An infinite sequence of pentagons



The curve C has Weierstrass equation

$$y^2 = x(x^2 + ax + b), \quad \text{where}$$

$$a = -(12k^2 + 24k + 11), \quad b = 4(k+1)^2(3k+2)(3k+4).$$