TRITANGENT PLANES OF BRING'S CURVE

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ABSTRACT

Bring's curve, being canonical of genus 4, has 120 tritangent planes. Their equations, and the coordinates of their contacts, are all found.

A canonical curve of genus 3 is a non-singular plane quartic whose 28 bitangents compose one of the most widely known figures in plane geometry. The canonical model of a non-hyperelliptic curve of genus p is of order $2p-2$ and lies in $[p-1]$, projective space of $p-1$ dimensions; it possesses $2^{p-1}(2^p-1)$ contact primes: 28 bitangent lines for $p = 3$, 120 tritangent planes for $p = 4$. While certain contributions have been made to the geometry when $p = 4$—indeed Coble's book [**2**] grew out of lectures on this very curve and Coxeter encountered in his early work [**3**; p. 169] a specialisation of the curve whose tritangent planes corresponded to the 120 diagonals of a polytope in 7-dimensional Euclidean space—much more remains to be discovered.

It may, therefore, be appropriate to record information recently acquired about the curve, Bring's curve, of genus 4 specialised to admit a group of 120 self-projectivities isomorphic to the symmetric group S_5 of degree 5. Two earlier appearances of B were recorded in [**4**]; as tritangent planes are now to be examined a third appearance [**5**] has to be acknowledged.

1.

Bring's curve B is the intersection of the quadric

$$Q : x^2+y^2+z^2+t^2+u^2 = 0 \tag{1.1}$$

with the (diagonal) cubic surface

$$D : x^3+y^3+z^3+t^3+u^3 = 0, \tag{1.2}$$

the supernumerary homogeneous coordinates being subject to the identity

$$x+y+z+t+u \equiv 0. \tag{1.3}$$

It is invariant under the 120 permutations of the five coordinates. It has [**4**; p. 544] 60 stalls, points where the osculating plane has 4-point intersection; their coordinates are permutations of $(1, 1, \alpha, \beta, \gamma)$ where α, β, γ are, and will be throughout this note, the roots of

$$\theta^3+2\theta^2+3\theta+4 = 0. \tag{1.4}$$

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The stalls lie [4; p. 541] six in each of ten planes such as $x = y$, and the tangents of B at such a coplanar hexad of stalls concur at a vertex V_{ij} of the pentahedron P of coordinate planes; the tangents at the hexad in $x = y$ concur at $V_{12}(1, -1, 0, 0, 0)$. So the tangent at $(1, 1, \alpha, \beta, \gamma)$, its join to V_{12}, is

$$z/\alpha = t/\beta = u/\gamma\,. \tag{1.5}$$

But then the plane $t/\beta = u/\gamma$ contains not only this line but also

$$x/\alpha = t/\beta = u/\gamma \quad \text{and} \quad y/\alpha = t/\beta = u/\gamma$$

which are the tangents of B at $(\alpha, 1, 1, \beta, \gamma)$ and $(1, \alpha, 1, \beta, \gamma)$; $t/\beta = u/\gamma$ is therefore a *tritangent plane* of B. Each tangent at a stall thus lies in three tritangent planes; for example, (1.5) lies in

$$t/\beta = u/\gamma, \quad u/\gamma = z/\alpha, \quad z/\alpha = t/\beta \tag{1.6}$$

so that $60 \times 3/3 = 60$ of the 120 tritangent planes of B are accounted for. This substantiates part of a twice discovered theorem, the two enunciations running as follows.

THEOREM (Emch, 1934). *If a sextic of genus* 4 *is on ten cubic cones whose vertices are the corners of a pentahedron it has six tritangent planes through each of the ten edges of the pentahedron. Through each of the sixty lines of contact of these with the corresponding cones passes a fourth tritangent plane. The whole figure is invariant under a group of* 120 *collineations* [5; p. 13].

THEOREM (Du Val, 1979). *The curve B has two distinct types of tritangent planes*, 60 *of each. Of type* (i) *all three contacts are stalls; of type* (ii) *only one is a stall. Each stall is the contact of three planes of type* (i) *and one of type* (ii).

Du Val remarks in a letter (dated 31 July, 1979) "For the tritangent planes to share their points of contact in this way is unexpected but not, as far as I can see, impossible, though I don't quite know how it will fit in with the theta function theory".

Immediately on seeing [4] Du Val, interested in any visible shape of B, took P to be bounded by three real and one pair of conjugate complex planes, and so produced a sphere on which the real circuit of B was a closed sinusoidal curve confined to an equatorial zone and cutting the equator at the vertices of a regular hexagon. It was thereupon suggested to him that B would have a pair of tritangent planes parallel to the equatorial plane, and he then quickly found the theorem.

2.

The identity

$$2(x^3+y^3) \equiv (x+y)\{3(x^2+y^2)-(x+y)^2\} \tag{2.1}$$

indicates the procedure for eliminating x and y between (1.1), (1.2) and (1.3); the outcome [4; p. 542] is

$$ztu+(z+t+u)(z^2+t^2+u^2) = 0\,, \tag{2.2}$$

the cubic cone K_{12} of chords, including six tangents, of B through V_{12}.

The join of the two contacts other than $(1, 1, \alpha, \beta, \gamma)$ of any of the three planes (1.6) contains V_{12}: for instance $u/\gamma = z/\alpha$ touches B at $(\beta, 1, \alpha, 1, \gamma)$ and $(1, \beta, \alpha, 1, \gamma)$ whose join is $t = u/\gamma = z/\alpha$; the tritangent plane touches K_{12} along this line. So each of the three planes (1.6) through the generator (1.5) of K_{12} touches K_{12} along a second generator. There are however four tangent planes of K_{12} through (1.5) in addition to its tangent plane along this line; it is the fourth of these that is of Du Val's type (ii).

Regard (2.2) for the present as a plane cubic. The polar conic of (α, β, γ) is

$$\alpha tu+\beta uz+\gamma zt+(\alpha+\beta+\gamma)(z^2+t^2+u^2)+2(z+t+u)(\alpha z+\beta t+\gamma u) = 0 . \qquad (2.3)$$

The six intersections of the cubic and the conic consist of a contact at (α, β, γ) and of the contacts of the four tangents from (α, β, γ) to the cubic. Three of these four contacts are $(1, \beta, \gamma)$, $(\alpha, 1, \gamma)$, $(\alpha, \beta, 1)$; where is the fourth? Wherever it is it will be fixed under simultaneous cyclic permutations (ztu) and $(\alpha\beta\gamma)$.

3.

Dialytic elimination of z between (2.2) and (2.3) provides a binary sextic S in t and u with coefficients of degree three in α, β, γ; $S = 0$ is the set of six lines joining $t = u = 0$ to the intersections of the cubic and conic. Among these six lines $t/\beta = u/\gamma$ occurs thrice, $\gamma t = u$ once, $t = \beta u$ once. In order to identify the sixth line it is only necessary, the other five all being known, to find the coefficients of t^6 and u^6 in S. Each of these coefficients is the outcome of transposing β and γ in the other.

So put $u = 0$ before eliminating z; the resulting eliminant is the coefficient of t^6 in S. As (2.2) and (2.3) then become

$$(z+t)(z^2+t^2) = 0$$

and

$$\gamma zt+(\alpha+\beta+\gamma)(z^2+t^2)+2(z+t)(\alpha z+\beta t) = 0 ,$$

or

$$z^3+tz^2+t^2z+t^3 = 0$$

and

$$(3\alpha+\beta+\gamma)z^2+(2\alpha+2\beta+\gamma)zt+(\alpha+3\beta+\gamma)t^2 = 0 ,$$

the eliminant is

$$\begin{vmatrix} 1 & t & t^2 & t^3 & . \\ . & 1 & t & t^2 & t^3 \\ . & . & 3\alpha+\beta+\gamma & (2\alpha+2\beta+\gamma)t & (\alpha+3\beta+\gamma)t^2 \\ . & 3\alpha+\beta+\gamma & (2\alpha+2\beta+\gamma)t & (\alpha+3\beta+\gamma)t^2 & . \\ 3\alpha+\beta+\gamma & (2\alpha+2\beta+\gamma)t & (\alpha+3\beta+\gamma)t^2 & . & . \end{vmatrix} .$$

The combination

$$C_3' \equiv t^2C_1-tC_2+C_3-t^{-1}C_4+t^{-2}C_5$$

of the columns shows the determinant to have the factor $2\alpha+2\beta+\gamma$; remove this, and then use routine operations on the rows and columns of the determinant that do not alter its value (a start could be $R_1' \equiv R_1 - tR_2$ followed by $C_5' \equiv C_5 + t^4 C_1$). The final result is

$$(2\alpha+2\beta+\gamma)\{(2\alpha+2\beta+\gamma)^2+4(\alpha-\beta)^2\}t^6 \tag{3.1}$$

which, being symmetric in α and β, can be expressed in terms of γ only.

4.

At this juncture a momentary digression concerning calculations that involve α, β, γ is in order. As roots of (1.4) they satisfy

$$\alpha+\beta+\gamma = -2, \quad \beta\gamma+\gamma\alpha+\alpha\beta = 3, \quad \alpha\beta\gamma = -4, \tag{4.1}$$

while their power sums $\sigma_k \equiv \alpha^k+\beta^k+\gamma^k$ satisfy the recurrence relation

$$\sigma_{k+3}+2\sigma_{k+2}+3\sigma_{k+1}+4\sigma_k = 0 \tag{4.2}$$

with initial conditions [4; p. 544]

$$\sigma_0 = 3, \quad \sigma_1 = \sigma_2 = \sigma_3 = -2.$$

Any symmetric function of two of α, β, γ is expressible as a function of the third only. For example,

$$\begin{aligned}(2\alpha+2\beta+\gamma)^2+4(\alpha-\beta)^2 &= 8(\alpha^2+\beta^2)+4\gamma(\alpha+\beta)+\gamma^2\\ &= 8(-2-\gamma^2)+4\gamma(-2-\gamma)+\gamma^2\\ &= -11\gamma^2-8\gamma-16,\end{aligned}$$

so that the multiplier of t^6 in (3.1) is

$$\begin{aligned}(\gamma+4)(11\gamma^2+8\gamma+16) &= 11\gamma^3+52\gamma^2+48\gamma+64\\ &= 11\gamma^3+52\gamma^2-16(\gamma^3+2\gamma^2) \qquad \text{by (1.4)}\\ &= -5\gamma^2(\gamma-4).\end{aligned}$$

But, since

$$0 = (\gamma^3+2\gamma^2+3\gamma+4)(\gamma-1) = \gamma^4+\gamma^3+\gamma^2+\gamma-4$$

an alternative and, it will soon be seen, more convenient form for this multiplier is

$$5\gamma^4(\gamma^2+\gamma+1).$$

5.

It is now apparent that the equation of the six lines $t = \lambda u$ joining $t = u = 0$ to the six intersections of (2.2) and (2.3) includes the pair of terms

$$5\gamma^4(\gamma^2+\gamma+1)t^6+5\beta^4(\beta^2+\beta+1)u^6\,,$$

so that the product of the six values of λ is

$$\frac{\beta^4(\beta^2+\beta+1)}{\gamma^4(\gamma^2+\gamma+1)}\,.$$

But the product of five of them is, as explained at the start of §3, β^4/γ^4; the sixth is, therefore,

$$(\beta^2+\beta+1)/(\gamma^2+\gamma+1)\,,$$

which indicates that the sought intersection has

$$z = \alpha^2+\alpha+1\,,\quad t = \beta^2+\beta+1\,,\quad u = \gamma^2+\gamma+1\,.$$

It only remains to check that this point (z, t, u) does satisfy (2.2) and (2.3). It makes

$$ztu = \frac{(\alpha^3-1)(\beta^3-1)(\gamma^3-1)}{(\alpha-1)(\beta-1)(\gamma-1)} = \frac{\alpha^3\beta^3\gamma^3-(\beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3)+\sigma_3-1}{\alpha\beta\gamma-(\beta\gamma+\gamma\alpha+\alpha\beta)+\sigma_1-1} = 7\,, \tag{5.1}$$

where one may use

$$\beta^3\gamma^3+\gamma^3\alpha^3+\alpha^3\beta^3-3\alpha^2\beta^2\gamma^2 \equiv (\beta\gamma+\gamma\alpha+\alpha\beta)\{(\beta\gamma+\gamma\alpha+\alpha\beta)^2-3\alpha\beta\gamma(\alpha+\beta+\gamma)\}\,.$$

Also

$$z+t+u = \sigma_2+\sigma_1+\sigma_0 = -1\,,$$

$$z^2+t^2+u^2 = \sigma_4+2\sigma_3+3\sigma_2+2\sigma_1+\sigma_0 = -2\sigma_1+\sigma_0 = 7\,,$$

and (2.2) is satisfied. Similar routine work verifies (2.3).

The fourth tritangent plane of B that contains (1.5) therefore meets $x = y$ in a line through

$$(\tfrac{1}{2}, \tfrac{1}{2}, \alpha^2+\alpha+1, \beta^2+\beta+1, \gamma^2+\gamma+1)$$

and touches B at its two intersections with the join of this point to V_{12}. These two contacts are obtainable from each other by transposing their first two coordinates ξ, η. But

$$\xi+\eta = -z-t-u = 1\,,\quad \xi^2+\eta^2 = -z^2-t^2-u^2 = -7\,,$$

so that $\xi\eta = 4$ and ξ, η are the roots of $X^2-X+4 = 0$. The contacts of that tritangent plane of Du Val's type (ii) of which one contact is the stall $(1, 1, \alpha, \beta, \gamma)$ are, therefore,

$$\left(\frac{1+i\sqrt{15}}{2}, \frac{1-i\sqrt{15}}{2}, \alpha^2+\alpha+1, \beta^2+\beta+1, \gamma^2+\gamma+1\right)$$

and (5.2)

$$\left(\frac{1-i\sqrt{15}}{2}, \frac{1+i\sqrt{15}}{2}, \alpha^2+\alpha+1, \beta^2+\beta+1, \gamma^2+\gamma+1\right).$$

These coordinates must, of course, also satisfy $\xi^3+\eta^3 = -z^3-t^3-u^3$ and can be verified to do so; (2.1) gives $\xi^3+\eta^3 = -11$.

As no two of ξ, η, z, t, u are equal the pair of contacts is one of 60 pairs and the 120 contacts compose, as Du Val expected, a single orbit under the symmetric group of permutations of the five coordinates. It seems unfitting not to display the equation of a tritangent plane of type (ii); a reader will be content with the equation itself and allow the details of finding it (they use §4) to be omitted. The plane with contacts (5.2) is

$$(\alpha-1)(\alpha+4)z+(\beta-1)(\beta+4)t+(\gamma-1)(\gamma+4)u = 0$$

and the others of type (ii) are obtained from this by permuting α, β, γ and imposing S_5.

6.

Clebsch [**1**; p. 238] remarked that the 360 contacts of the 120 tritangent planes of any canonical curve of genus 4 are its intersection with a surface of order 60 composed of 30 quadrics. For B [**4**; p. 545] the contacts are also its intersections with three surfaces of order 20 belonging to a pencil $\lambda S_4^5+\mu S_5^4 = 0$. It has now appeared that the 60 stalls are to be reckoned four times among the 360 contacts; as they are [**4**; p. 541] cut twice on B by $S_4^5 = 20S_5^4$, which is the square of ten planes, this surface must be included twice among the three members of the pencil. The third member cuts B at the 120 contacts, other than stalls, of the 60 tritangent planes of type (ii), and so a question asked in §9 of [**4**] is answered.

At points (5.2), in the notation of §2 of [**4**],

$$S_5 = -5f = 5\left(\frac{1}{4}+\frac{15}{4}\right)ztu = 140,$$

by (5.1), while

$$S_4 = -4e = -4\xi\eta(tu+uz+zt)-4(\xi+\eta)ztu$$

$$= -8\{(z+t+u)^2-(z^2+t^2+u^2)\}-28 = 20,$$

and the member of the pencil has $\lambda(20)^5+\mu(140)^4 = 0$. The contacts, other than stalls, of Du Val's second type of tritangent plane all satisfy $2401S_4^5 = 20S_5^4$.

7. *Principal chords of B*

A non-singular curve of order n, class n' and genus p has [7; p. 86]

$$\tfrac{1}{2}(n-3)(n-4)+\tfrac{1}{2}(n'-3)(n'-4)-12p$$

principal chords; chords, that is, which lie in the osculating planes at both their intersections with the curve. Equation (7.1) of [4] shows that the contacts of those osculating planes of B through a given point (X) are on a sextic surface so that, for B, $n' = 36$; as $n = 6$ and $p = 4$ there are 483 principal chords. These must be permuted among themselves by S_5, and fall into four classes. Indeed several of these chords, though not there remarked as principal, have appeared in [4].

Each cubic cone K_{ij} has nine inflectional generators δ; the tangent plane along δ osculates B at both its intersections with δ, which is then a principal chord. These δ include [4; p. 542] the 15 diagonals of P; each diagonal is common to two K_{ij}, each K_{ij} includes three coplanar diagonals and so has six other δ lying on a quadric cone. These latter δ, each belonging to only one K_{ij}, provide 60 principal chords.

It was also remarked [4; p. 543] that Q contains six quadrilaterals of tangents of B. If $aba'b'$ is one of these the osculating planes at a, b, a', b' are, respectively [4; p. 543],

$$b'ab, \quad aba', \quad ba'b', \quad a'b'a,$$

so that ab, ba', $a'b'$, $b'a$ are all principal chords and 24 are thus recognised. But these must be counted twice. Without using more space by giving a full justification let it suffice to say that ab is not merely a chord: it is a tangent as well as a secant, touching B at a and intersecting it at b. It thus lies in two "consecutive" osculating planes at a as well as in that at b. So the quadrilaterals on Q account for 48 principal chords.

Now $483-15-60-48 = 360$ and these residual chords will presumably compose three orbits under S_5.

8.

It is fitting to close on a note of interrogation.

Bring's curve B has 255 systems of contact quadrics [1; p. 238], each system associated with a 4-nodal cubic surface [5; p. 376] containing B; these 255 systems, with the 255 cubic surfaces, are permuted among themselves by S_5. But S_5, of order 120, cannot permute 255 objects transitively: how are they distributed in orbits? Is there an orbit of 15? If so, what geometrical features distinguish its members from the other 240?

Among the contact quadrics of any one system are 28 consisting of a pair of tritangent planes [1; p. 238]. How many of these 28 pairs are (*a*) both of type (i), (*b*) both of type (ii), (*c*) one of each type? The answers will be the same for systems in the same orbit but need not be for systems in different orbits. If, moreover, two tritangent planes are not both of type (ii) they may share a contact, or they may not.

Lastly, what are, for genus 4, the special theta relations associated with B? Do they suggest extensions to higher genera and, if so, what are the consequent properties of the specialised canonical curves?

References

1. A. Clebsch, "Ueber die Anwendung der Abelschen Functionen in der Geometrie", *Journal für Mathematik*, 63 (1864), 189–243.
2. A. B. Coble, *Algebraic geometry and theta functions* (American Mathematical Society, New York, 1929).
3. H. S. M. Coxeter, "The polytopes with regular-prismatic vertex figures (Part 2)", *Proc. London Math. Soc.* (2), 34 (1932), 126–189.
4. W. L. Edge, "Bring's curve", *J. London Math. Soc.* (2), 18 (1978), 539–545.
5. A. Emch, "Über eine bemerkenswerte Klasse von Raumkurven sechster Ordnung vom Geschlecht 4", *Comment. Math. Helv.*, 7 (1934–5), 1–13.
6. W. P. Milne, "Sextactic cones and tritangent planes of the same system of a quadricubic curve", *Proc. London Math. Soc.* (2), 21 (1923), 373–380.
7. F. Severi, "Sopra alcune singolarità delle curve di un iperspazio", Mem. Accad. Sci. Torino (2), 51 (1902), 81–114.

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