The Common Curve of Quadrics Sharing a Self-Polar Simplex (*).

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Summary. – *When $n-1$ quadrics in projective space $[n]$ of n dimensions have a common self-polar simplex their common curve Γ admits a group of 2^n self-projectivities. The consequent properties of Γ are investigated, and further specialisations are imposed which amplify the the group and endow Γ with further properties. There is some reference to the osculating spaces and principal chords of Γ, and some properties of particular curves in four and five dimensions are described.*

I. – Principal properties of the curve.

1. – A simplex S in a projective space $[n]$ of n dimensions provides $n+1$ involutory projectivities that are mutually commutative and have for product the identity projectivity: namely the harmonic inversions in the vertices and opposite bounding primes of S. When S is taken as simplex of reference for a system of homogeneous coordinates x_j then h_j, associated with the vertex X_j and the opposite bounding prime $x_j=0$, multiplies the single coordinate x_j by -1 and leaves the other n coordinates unchanged. That h_j is an involution, that the different h_j commute, and that

$$h_0\, h_1 \ldots h_n = I\ ,$$

are clear from the representation of the h_j as diagonal matrices. These h_j generate an elementary abelian group E of 2^n projectivities. The $[s-1]$ spanned by any s vertices of S and the $[n-s]$ spanned by the other $n+1-s$ vertices are a pair of fundamental spaces, or axes, of one of the 2^n-1 involutions in E.

Any quadric Ω for which S is self-polar is invariant under E; each point of Ω that is not in any face of S, i.e. for which no x_j is zero, is one of a batch B of 2^n points of Ω that is invariant under E. The same is true of the manifold common to any set of such quadrics. If there are $n-1$ quadrics in the set this manifold is, in general, an irreducible curve Γ_n of order 2^{n-1}. Since it is the complete intersection of $n-1$ primals its genus π_n is deducible from the known fact that the canonical series is cut on Γ_n by primals of order $2(n-1)-(n+1)=n-3$, so that

$$2\pi_n-2=2^{n-1}(n-3)$$

$$\pi_n=2^{n-2}(n-3)+1\ .$$

as stated on p. 185 of [1].

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2. – Γ_n is determined by $n-1$ simultaneous equations

$$\Omega_k \equiv \sum \alpha_j^{(k)} x_j^2 = 0 \qquad k = 0, 1, \ldots, n-2 \tag{2.1}$$

where summation is over j and runs, unless otherwise stated, from 0 to n. It is presumed that (2.1), regarded as linear equations in x_j^2, are linearly independent; the matrix $[\alpha_j^{(k)}]$ has full rank $n-1$. This implies, since all $(n-1)$ rowed determinants are non-zero, that Γ_n is skew to every bounding $[n-2]$ of S: no two of the $n+1$ coordinates can be zero simultaneously on Γ_n. A point of Γ_n in $x_j = 0$ is invariant under h_j and the batch determined by it consists not of 2^n but of only 2^{n-1} points; they compose the whole intersection of Γ_n and $x_j = 0$ and may be called a critical batch B_j^*.

There is a linear combination of the $n-1$ equations (2.1) from which any $n-2$ of the squares x_j^2 are absent, but it is not possible so to eliminate $n-1$ of the $n+1$ squares. Among the quadrics through Γ_n are cones whose vertices are the bounding $[n-3]$'s opposite to the plane faces of S, but Γ_n does not lie on any cone with an $[n-2]$ for vertex.

The invariance of Γ_n under the h_j discloses the special character of the points in the $n+1$ critical batches.

The tangents of Γ_n at all the points in B_j^* concur at X_j. If PP_j is a chord of Γ_n through X_j and P tends along Γ_n to a point in B_j^* then P_j, on the line X_jP, tends simultaneously to this same point; the osculating plane there has 4-point intersection with Γ_n. Repetition of the argument proves that the osculating $[s]$ has $2s$-point intersection for $s = 1, 2, \ldots, n-1$. These points, 2^{n-1} in each of the $n+1$ bounding primes of S, may be appropriately called *stalls* of Γ_n. The branch centred at a stall is characterized by the integers

$$\alpha_0 = \alpha_1 = 1, \qquad \alpha_2 = \alpha_3 = \ldots = \alpha_{n-1} = 2 .$$

For a definition of such integers see [13], p. 28 where they are designated as $\nu, \nu', \nu'', \ldots$; or [8], p. 246 where Segre's i_s is $\alpha_0 + \alpha_1 + \ldots + \alpha_s$.

The 2^n points of a batch are in perspective from each of the $n+1$ vertices of S. The chords of Γ_n through X_j generate a two-dimensional cone of order 2^{n-2}; all points on these chords satisfy $n-3$ linearly independent equations obtained by eliminating x_j^2 from (2.1). But these, x_j^2 being absent, are equations, one less in number, of the same form and involving one coordinate less: the chords of Γ_n through X_j meet $x_j = 0$ in a curve Γ_{n-1}. There is a $(2, 1)$ correspondence between Γ_n and this projection, branching at the 2^{n-1} points of B_j^*. An application of Zeuthen's formula ([12], p. 152, [13], p. 223, [8] p. 83, [10] p. 211) allows π_n to be calculated by recurrence, the initial condition being $\pi_3 = 1$ for the elliptic quartic, or even $\pi_2 = 0$ for a conic. The calculation yields the same value for π_n as in § 1, and has already been used on p. 338 of [6].

3. – There is a representation of Γ_n that proves to be convenient when investigating the geometry of the curve.

Regard the coefficients $\alpha^{(k)}$ as the homogeneous coordinates $y^{(k)}$ of $n+1$ points A_j in a projective space $[n-2]$. Through these points there passes a unique rational normal curve C. Choose a coordinate system in $[n-2]$ so that C has the standard parametric form $y^{(k)}=t^k$ $(k=0,1,\ldots,n-2)$; the points A_j will have for their parameters $n+1$ unequal numbers a_j. In order to avoid an infinite value, and so ensure that every A_j has its first coordinate $y^{(0)}$ non-zero, one has only to arrange that the contact of C with its osculating $[n-3]$ $y^{(0)}=0$ is not at any of the A_j. Then, A_j being $(1, a_j, a_j^2, \ldots, a_j^{n-2})$, Γ_n is given by

$$\Omega_k \equiv \sum a_j^k x_j^2 = 0 \qquad (k=0,1,\ldots,n-2) \tag{3.1}$$

where k is now an actual power and not a mere superscript. There is a practical illustration of this circumstance when $n=4$ in [3].

The equations (3.1) show that Γ_n depends only on the $n-2$ projective invariants of the $n+1$ numbers a_j; Γ_n has only $n-2$ moduli whereas a general non-hyperelliptic curve of genus π_n has $3\pi_n-3=2^{n-2}(3n-9)$. The specialisation is due to the invariance under E. Since Γ_n is projected from a bounding $[s]$ of S onto the opposite $[n-s-1]$ into a Γ_{n-s-1} covered 2^{s+1} times it is in multiple correspondence with curves of lower genera. If $n=4$, say, Γ_4 is a special canonical curve of genus 5 with only 2 moduli whereas the general curve of genus 5 has 12 moduli. Γ_4 is projected from each of the 5 vertices of S into an elliptic quartic, covered twice, with eight branch points; it is projected from each of the ten edges of S into a plane conic covered four times.

4. – Write, now

$$f(\theta) \equiv \prod_{j=0}^{n} (\theta - a_j) \equiv \sum_{p=0}^{n+1} (-1)^p e_p \theta^{n-p+1}$$

and

$$s_k = \sum_{=0}^{n} a_j^k / f'(a_j) \, .$$

Then, as is seen from the decomposition into partial fractions of $\theta^k/f(\theta)$, or otherwise,

$$s_0 = s_1 = \ldots = s_{n-1} = 0 \, , \qquad s_n = 1 \, . \tag{4.1}$$

So two solutions, and therefore the only two so far as linear dependence is concerned, of (3.1) are

$$x_j^2 = 1/f'(a_j) \, , \qquad x_j^2 = a_j/f'(a_j) \tag{4.2}$$

and the general solution is

$$x_j^2 = (\theta + a_j)/f'(a_j) \qquad j = 0, 1, \ldots, n. \tag{4.3}$$

This is a parametric form not, of course, for Γ_n but for a batch $B(\theta)$ on Γ_n; the 2^n different points of $B(\theta)$ answer to the different signing of the $n+1$ square roots of the x_j^2. The critical batch B_j^* of stalls is $B(-a_j)$.

The value of s_N for $N > n$ can be found by using initial conditions (4.1) and the recurrence relation

$$s_N - e_1 s_{N-1} + \ldots + (-1)^{n+1} e_{n+1} s_{N-n-1} = 0$$

so that, to give the simplest example, $s_{n+1} = e_1 s_n = e_1$. The x_j^2, as parametrised in (4.3), thus satisfy

$$\Omega_{n-1} = 1\,, \qquad \Omega_n = \theta + e_1\,,$$

and $B(\theta)$ is the intersection of Γ_n with the quadric $\Omega_n = (\theta + e_1)\Omega_{n-1}$. It will be found convenient to refer occasionally to the quadrics $\Omega_n = 0$ and $\Omega_{n-1} = 0$ which with the $n-1$ quadrics (3.1), compose a linearly independent set of $n+1$ linear combinations of the $n+1$ squares x_j^2.

This parametrisation of batches on Γ_n really goes as far back, for $n = 3$, as SALMON ([7], p. 195). The explicit form (4.3) in $[n]$ is given by BAKER ([1], p. 185) who exploits it to advantage for $n = 4$ in his account of Segre's cyclide, and for $n = 5$ in his geometrical treatment of the line geometry of Kummer's surface ([1], pp. 218 *et seq.*).

5. – If an algebraic curve Γ in $[n]$ has genus p, order m—the number of its intersections with a prime—and class m'—the number of its osculating primes through a point—then it is known ([9], p. 86) to have

$$\tfrac{1}{2}(m-n)(m-n-1) + \tfrac{1}{2}(m'-n)(m'-n-1) - n(n+1)p$$

principal chords, a chord being principal when it lies in the osculating primes at *both* its intersections with Γ.

As with so much successful enumerative work the procedure by which the number is derived gives no information about how these chords may be grouped, or about the geometry within such a group or the relations of such groups to one another. These matters are likely to be relevant when Γ has special properties: for example, any projectivity under which Γ is invariant must permute the principal chords among themselves.

The order 2^{n-1} and genus π_n of Γ_n have been given already; its class $2^{n-1}(2n-3)$ was found by elementary methods in [5] and can also be found in other ways. It follows that the number of principal chords of Γ_n is

$$2^{2n-2}(2n^2 - 6n + 5) - 2^{n-2}(n^3 + 2n^2 - 5n - 2)\,. \tag{5.1}$$

This accords with the normal elliptic quartic having 24 principal chords; this fact is well known as a special case of the number for the normal elliptic curve of any order, and has also been established independently in [4] by reasoning allied to that to be used now.

There will be two kinds of principal chords of Γ_n: such a chord may join points in the same batch, or it may not. If it does it is invariant under that involution in E which transposes these two points in the batch, and so meets a pair of opposite bounding spaces of S. For example: if $n=5$ such a chord may

(i) pass through a vertex X_j and meet $x_j=0$, or

(ii) meet an edge and opposite solid of S, or

(iii) meet an opposite pair of plane faces of S.

In (i) the « chord » is the tangent at a stall. This, in view of the high multiplicity $2n-2$ of the intersection of Γ_n with its osculating prime at the stall, raises the question: what is the number μ_n of times that the tangent at a stall is to be reckoned among the principal chords?

There will also be principal chords joining points in different batches. Should the points be in batches B_1 and B_2 the osculating prime at any point of B_1 contains a point of B_2 at which the osculating prime, in turn, contains the point of B_1. Such « transversal » principal chords occur in sets of 2^n.

6. – The osculating prime of Γ_n at $x=\xi$ is, by equation (3.1) on p. 40 of [5],

$$\sum \{f'(a_j)\}^{n-2}\xi_j^{2n-3}x_j=0$$

or, alternatively, if ξ belongs to $B(\theta)$,

$$\sum (\theta+a_j)^{n-2}\xi_j x_j=0\,.$$

If, then, ξ belongs to $B(\theta)$ and η to $B(\varphi)$ the chord $\xi\eta$ is principal if, and only if,

$$\sum (\theta+a_j)^{n-\frac{3}{2}}(\varphi+a_j)^{\frac{1}{2}}/f'(a_j)=0=\sum (\theta+a_j)^{\frac{1}{2}}(\varphi+a_j)^{n-\frac{3}{2}}/f'(a_j)\,.$$

Both these equations are unchanged if the signs of $\sqrt{(\theta+a_j)}$ and $\sqrt{(\varphi+a_j)}$ are both changed; hence a chord $\xi\eta$ that is principal is accompanied by other principal chords, one through each point of the batch to which ξ belongs.

If the intersections of Γ_n with a principal chord both belong to $B(\theta)$ then, for some selection of signs,

$$\sum \pm\,(\theta+a_j)^{n-1}/f'(a_j)=0\,. \tag{6.1}$$

Now this sum would be zero identically in θ by (4.1) were all signs the same; hence *both* sums of terms that are signed positively and of terms that are signed negatively, are zero: say, for convenience,

$$\sum_{0}^{p}(\theta + a_j)^{n-1}/f'(a_j) = 0 = \sum_{p+1}^{n}(\theta + a_j)^{n-1}/f'(a_j)\,. \tag{6.2}$$

These two sums are negatives of each other, and the two equations have the same roots $\theta_1, \theta_2, \dots, \theta_{n-1}$. Now

$$\sum_{0}^{p}(\theta + a_j)^{n-1}/f'(a_j) = \sum_{0}^{p}\{f'(a_j)\}^{n-2}\xi_j^{2n-2}.$$

When θ is assigned each ξ_j has either of two values, so that the sum on the right is, for each root θ_j, zero for 2^p points ξ in $x_{p+1} = x_{p+2} = \dots = x_n = 0$. Likewise there are 2^{n-p-1} in the opposite $[n-p-1]$ spanned by the vertices $X_{p+1}, X_{p+2}, \dots, X_n$ of S. Thus each of the $n-1$ roots θ_j provides 2^{n-1} principal chords joining pairs of points in $B(\theta_j)$. Take p over the range $1 \leqslant p \leqslant n-2$ so that each such chord is included twice. The number of principal chords so accounted for is

$$\frac{1}{2}(n-1)\cdot 2^{n-1}\left\{\binom{n+1}{2} + \binom{n+1}{3} + \dots + \binom{n+1}{n-1}\right\} = 2^{n-1}(n-1)(2^n - n - 2)\,. \tag{6.3}$$

If, in particular, $n = 3$ one obtains 24 principal chords; each of the three pairs of opposite edges of the tetrahedron S is associated with two batches on Γ_3, and each of these batches has its eight points joined in pairs by four principal chords all transversal to the pair of opposite edges. These facts agree with the findings in [4].

Incidentally, as these are all the principal chords of Γ_3, $\mu_3 = 0$.

The preceding discussion provides 240 principal chords of Γ_4, 24 transversal to any edge and opposite plane face of the simplex S. This edge and face are thus associated with three batches $B(\theta_1)$, $B(\theta_2)$, $B(\theta_3)$. Each θ_i yields a pair of points on the edge that are harmonic to the vertices of S as well as four points in the face forming a quadrangle with the vertices of S for its triangle of diagonal points; the eight joins of the two points on the edge to the four points in the face are all principal chords of Γ_4, and join points of $B(\theta_i)$ in pairs.

Γ_4 has, by (5.1), 536 principal chords in all so that 296 remain to be accounted for. But 296 is not divisible by 2^4 so that the tangents at the 40 stalls of Γ_4 must be included, and that an odd number of times; μ_4 is not zero, and is odd.

The number in (5.1) exceeds that in (6.3) by

$$2^{2n-2}\,(2n^2 - 8n + 7) - 2^{n-2}(n^3 - 7n + 2)\,.$$

If one subtracts from this the number of stalls counted μ_n times the residue is the number of principal chords joining points of different batches; this has to be a multiple of 2^n. But the residue is

$$2^{2n-2}(2n^2 - 8n + 7) - 2^{n-2}\{(n-1)^2(n+2) - 4n\} - (n+1)\,2^{n-1}\mu_n\,.$$

This, whatever the integer μ_n, is divisible by 2^n whenever n is odd. If, however, n is even then

$$\mu_n \text{ is even if } n \equiv 2 \pmod 4 ,$$

$$\mu_n \text{ is odd if } n \equiv 0 \pmod 4 .$$

Questions of multiplicity can present awkward problems, and the value of μ_n is not the only one. Suppose, as does happen upon occasion (see §§ 18, 21 below), that a principal chord $\xi\eta$ of Γ_n lies not only in the osculating primes at both ξ and η but also in the osculating spaces of dimensions $n-2$, $n-3$, ..., $n-s$ at both points. How often must such a chord be reckoned? For how many principal chords does it account?

7. – There is a special situation, instances of which are to be encountered later, in which principal chords can be detected joining points in $B(\infty)$ and joining points in $B(0)$. It occurs when $n+1$ is composite and Γ_n invariant under a regular permutation of the coordinates; say $n+1=gh$ with the coordinates permuted in g cycles of h:

$$(x_0\, x_g \ldots x_{g(h-1)}) \ldots (x_{g-1}\, x_{2g-1} \ldots x_{hg-1}) .$$

The reason why $B(\infty)$ and $B(0)$ obtrude is that they are both invariant under the imposed projectivity $\wp$ whereas all other batches are permuted in cycles of h.

This situation occurs if

$$\Omega_k = \sum_{j=0}^{g-1} a_j^k \{x_j^2 + \varepsilon^k x_{j+g}^2 + \varepsilon^{2k} x_{j+2g}^2 + \ldots + \varepsilon^{(h-1)k} x_{j+(h-1)g}^2\}$$

with ε a primitive h-th root of unity. The effect of $\wp$ is to multiply Ω_k by ε^{-k} and so leave each quadric $\Omega_k = 0$ invariant. Then

$$f(\theta) \equiv (\theta^h - a_0^h)(\theta^h - a_1^h) \ldots (\theta^h - a_{g-1}^h) \equiv \psi(\theta^h) ,$$

say so that $f'(\theta) = h\theta^{h-1}\psi'(\theta^h)$ whence it follows that

$$f'(a_j) : f'(\varepsilon a_j) : \ldots : f'(\varepsilon^{h-1} a_j) = 1 : \varepsilon^{-1} : \varepsilon^{-2} : \ldots : \varepsilon^{1-h} \tag{7.1}$$

This shows that the terms of either sum obtained by taking $\theta = \infty$ or 0 in (6.1) fall into g sets of h, the sum of any set being zero. For, λ being some constant,

$$\sum_{r=0}^{h-1} 1/f'(\varepsilon^r a_j) = \lambda \sum_0^{h-1} \varepsilon^r = 0 ,$$

and

$$\sum_{r=0}^{h-1} \frac{(\varepsilon^r a_j)^{gh-2}}{f'(\varepsilon^r a_j)} = (a_j)^{gh-2} \sum_0^{h-1} \frac{1}{\varepsilon^{2r} f'(\varepsilon^r a_j)} = \lambda a_j^{gh-2} \sum_0^{h-1} \varepsilon^{-r} = 0 .$$

II. – Curves invariant under additional self-projectivities of period 2 or 3.

8. – Γ_n can be specialised to admit ampler groups of self-projectivities than E and so have still fewer moduli.

If $n = 2p + 1$ is odd a specialisation suggests itself at once; choose the $2p + 2$ numbers a_j to be $p + 1$ pairs of an involution I (this is, of course, no specialisation if $p = 1$). There is no loss of generality in assigning parameters 0 and ∞ to the foci of I; then the parameters of any pair of I sum to zero: say $a_j + a_{j+p+1} = 0$, and Γ'_n is the common curve of the $2p$ quadrics

$$\Omega_k \equiv \sum_{j=0}^{p} a_j^k \{x_j^2 + (-1)^k x_{j+p+1}^2\} = 0 \qquad k = 0, 1, \ldots, 2p-1 . \tag{8.1}$$

It is invariant not only under E but also under the $(p+1)$-fold transposition.

$$(x_0 x_{p+1})\,(x_1 x_{p+2})\,\ldots\,(x_p x_{2p+1})$$

of the $2p + 2$ coordinates in pairs; for this leaves Ω_k unchanged when k is even, changes Ω_k into $-\Omega_k$ if k is odd. The corresponding projectivity is harmonic inversion J in the skew pair of $[p]$'s.

$$\begin{cases} x_0 - x_{p+1} = x_1 - x_{p+2} = \ldots = x_p - x_{2p+1} = 0 , \\ x_0 + x_{p+1} = x_1 + x_{p+2} = \ldots = x_p + x_{2p+1} = 0 . \end{cases} \tag{8.2}$$

Both these $[p]$'s lie on the p quadrics

$$\Omega_1 = 0 , \quad \Omega_3 = 0, \ldots, \Omega_{2p-1} = 0 ; \tag{8.3}$$

indeed (8.3) all hold whenever

$$x_0^2 - x_{p+1}^2 = x_1^2 - x_{p+2}^2 = \ldots = x_p^2 - x_{2p+1}^2 = 0 , \tag{8.4}$$

a set of $p + 1$ equations representing 2^{p+1} $[p]$'s composed of 2^p skew pairs of opposites of which (8.2) is but one. Each pair affords a harmonic inversion leaving Γ'_n invariant. These skew pairs are of course obtainable from (8.2) by using the h_j of § 1. Since the 2^{2p+1} operations of E permute the 2^p pairs among themselves one expects each pair to be invariant under a group of 2^{p+1} operations of E. For example: each member of (8.2) is unchanged by any of

$$h_0 h_{p+1},\ h_1 h_{p+2},\ \ldots,\ h_p h_{2p+1} \tag{8.5}$$

which, commuting and having product identity, generate an elementary abelian group of order 2^p; this is amplified to one of order 2^{p+1} by adjoining

$$h_0\, h_1 \dots h_p \equiv h_{p+1}\, h_{p+2} \dots h_{2p+1}$$

transposing the two $[p]$'s of (8.2).

It will not have escaped notice that just as (8.3) hold in consequence of (8.4), so

$$\Omega_0 = 0\,, \quad \Omega_2 = 0, \dots, \Omega_{2p-2} = 0$$

hold in consequence of

$$x_0^2 + x_{p+1}^2 = x_1^2 + x_{p+2}^2 = \dots = x_p^2 + x_{2p+1}^2 = 0\,. \tag{8.6}$$

This is a second set of 2^p skew pairs of opposite $[p]$'s, and each pair is invariant under the same group of order 2^{p+1}.

Each of the 2^p pairs in (8.4) is associated with a pair in (8.6). To explain this geometrically take (8.2). The equations (λ a parameter)

$$x_0/x_{p+1} = x_1/x_{p+2} = \dots = x_p/x_{2p+1} (= \lambda)$$

represent a singly-infinite set of $[p]$'s, all skew to one another, generating a locus V of dimension $p+1$, indeed the «Segre product» $[p]\times[1]$ mentioned on p. 174 of [8]. For $\lambda = \pm 1$ one has the pair (8.2); for $\lambda = \pm i$ one has the pair

$$\begin{cases} x_0 - ix_{p+1} = x_1 - ix_{p+2} = \dots = x_p - ix_{2p+1} = 0\,, \\ x_0 + ix_{p+1} = x_1 + ix_{p+2} = \dots = x_p + ix_{2p+1} = 0\,, \end{cases} \tag{8.7}$$

of (8.6). The harmonic inversions in (8.2) and (8.7) commute, their product being the inversion in those $[p]$ on V having $\lambda = 0,\ \infty$, i.e. in

$$x_0 = x_1 = \dots = x_p = 0 \quad \text{and} \quad x_{p+1} = x_{p+2} = \dots = x_{2p+1} = 0\,,$$

so that this product is just $h_0\, h_1\, h_2 \dots h_p$.

9. – After observing that any of the 2^{p+1} skew pairs of $[p]$'s can be transformed into any other by a projectivity leaving Γ_n' invariant it is sufficient to consider any one pair.

Since both $[p]$ lie on p of the $2p$ quadrics defining Γ_n' the 2^p common points of the quadric $(p-1)$-folds in which the remaining p quadrics meet either $[p]$ are on Γ_n'. Now every point of Γ_n' is paired with another in J, and the chords joining such pairs, all being transversal to both $[p]$'s, generate a scroll R, of order μ say, meeting the two $[p]$'s in directrix curves, of order m say (the curves have the same order since the $[p]$'s, say the pair (8.2), in which they lie are transformed into each other by harmonic inversion in the pair (8.7)). Among the generators of R those through the intersections of Γ_n' with the two $[p]$'s are tangents of Γ_n'.

The genus π of R can be found at once by applying Zeuthen's formula. For every directrix (say a prime section) of R is in $(1, 2)$ correspondence with Γ_n', the coincidences on Γ_n' being its 2^{p+1} contacts with generators of R; hence the formula gives

$$4(\pi-1)+2^{p+1}=2(\pi_n-1)=2^{2p}(2p-2)$$

$$\pi=1+(p-1)2^{2p-2}-2^{p-1}.$$

A prime meets Γ_n' in 2^{2p} points; if the prime contains either $[p]$ these consist of the 2^p points of Γ_n' in $[p]$ and two on each of those m generators of R that pass through the m intersections of the prime with the directrix in the opposite $[p]$; so

$$2^{2p}=2^p+2m,$$

$$m=2^{2p-1}-2^{p-1}.$$

And since the prime meets R in the directrix of order m and m generators

$$\mu=2m=2^{2p}-2^p.$$

There are 2^{p+1} such scrolls containing Γ_n'.

10. – The discussion in § 7 shows, with $h=2$ and $g=p+1$, that principal chords of Γ_n' join points in $B(\infty)$ as well as in $B(0)$. Since $f(\theta)$ is here an even, and so $f'(\theta)$ an odd, function, $f'(a_j)=-f'(-a_j)$; thus both sums

$$1/f'(a_j)+1/f'(-a_j) \qquad \text{and} \qquad a_j^{2p}/f'(a_j)+(-a_j)^{2p}/f'(-a_j)$$

are zero and the $2p+2$ terms in (6.1) fall now, should θ be either 0 or ∞, into $p+1$ pairs with each pair summing to zero. So one obtains, in either batch, 2^{2p} principal chords transversal to an edge X_jX_{j+p+1} and opposite $[2p-1]$ of S, 2^{2p} transversal to a solid $X_jX_{j+p+1}X_kX_{k+p+1}$ and opposite $[2p-3]$ of S, and so on. Thus, among the joins of either $B(\infty)$ or $B(0)$ there are

$$2^{2p}\cdot\frac{1}{2}\left\{\binom{p+1}{1}+\binom{p+1}{2}+\dots+\binom{p+1}{p}\right\}=2^{3p}-2^{2p}$$

principal chords. Since there are 2^{2p+1} points in a batch one expects there to be

$$2(2^{3p}-2^{2p})/2^{2p+1}=2^p-1$$

principal chords through each; indeed both $B(0)$ and $B(\infty)$ are partitioned into 2^{p+1} subsets of 2^p points, the join of any two points in the same subset being a principal chord. Each subset is an orbit under the group, of order 2^p, generated by the operations (8.5).

One now proceeds to illustrate these properties of Γ'_{2p+1}, by describing the figure in [5].

11. – Γ'_5, with $p=2$, has order 16 and genus 17; it lies on eight scrolls of order 12 each having two plane sextics of genus 7 for directrices.

Γ'_5 is the common curve of the four quadrics

$$\left\{\begin{aligned} x_0^2+x_3^2+\quad x_1^2+x_4^2+\quad x_2^2+x_5^2&=0\,,\\ a(x_0^2-x_3^2)+\ b(x_1^2-x_4^2)+\ c(x_2^2-x_5^2)&=0\,,\\ a^2(x_0^2+x_3^2)+b^2(x_1^2+x_4^2)+c^2(x_2^2+x_5^2)&=0\,,\\ a^3(x_0^2-x_3^2)+b^3(x_1^2-x_4^2)+c^3(x_2^2-x_5^2)&=0\,.\end{aligned}\right. \tag{11.1}$$

Any of the eight planes

$$x_0^2+x_3^2=x_1^2+x_4^2=x_2^2+x_5^2=0 \tag{11.2}$$

meets Γ'_5 in the four points for which

$$ax_0^2+bx_1^2+cx_2^2=a^3x_0^2+b^3x_1^2+c^3x_2^2=0\ ; \tag{11.3}$$

these 32 points will be shown to be $B(\infty)$. Any of the eight planes

$$x_0^2-x_3^2=x_1^2-x_4^2=x_2^2-x_5^2=0 \tag{11.4}$$

meets Γ'_5 in the four points for which

$$x_0^2+x_1^2+x_2^2=a^2x_0^2+b^2x_1^2+c^2x_2^2=0\ ; \tag{11.5}$$

these 32 points compose $B(0)$.

Since, here,

$$f(\theta)\equiv(\theta^2-a^2)(\theta^2-b^2)(\theta^2-c^2)\,,$$

it follows that

$$(b^2-c^2)f'(a)/a=(c^2-a^2)f'(b)/b=(a^2-b^2)f'(c)/c$$

and these relations, coupled with the fact of $f'(\theta)$ being an odd function, give the parametric form for $B(\theta)$, namely, by (4.3),

$$\xi_0^2=(b^2-c^2)(\theta+a)/a\,,\quad \xi_3^2=-(b^2-c^2)(\theta-a)/a$$

with the equations obtained from these by imposing simultaneously the cyclic permutations (abc), (012), (345). So $B(0)$ and $B(\infty)$ are as stated.

The osculating [4] of Γ'_5 at a point ξ of $B(\infty)$ is, by (3.1) of [5], $\sum \xi_j x_j = 0$; it clearly contains ξ itself by (11.2). But so do, for the same reason, the three [4]'s

$$\xi_0 x_0 + \xi_3 x_3 = 0\,, \qquad \xi_1 x_1 + \xi_4 x_4 = 0\,, \qquad \xi_2 x_2 + \xi_5 x_5 = 0\,.$$

Hence the osculating [4] at any of the points

$$\begin{array}{cccccc} \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ -\xi_0 & \xi_1 & \xi_2 & -\xi_3 & \xi_4 & \xi_5 \\ \xi_0 & -\xi_1 & \xi_2 & \xi_3 & -\xi_4 & \xi_5 \\ \xi_0 & \xi_1 & -\xi_2 & \xi_3 & \xi_4 & -\xi_5 \end{array}$$

of $B(\infty)$ contains all three others. The 32 points of $B(\infty)$ fall into 8 tetrahedra, the edges of the tetrahedra being principal chords. The vertices of a tetrahedron are obtained from any one of them by harmonic inversions in

the edge X_0X_3 and opposite bounding solid of S,
.... X_1X_4 ....,
.... X_2X_5 ....

The analogous situation holds with $B(0)$; at a point ξ of this batch the osculating [4], is, again by (3.1) of [5],

$$a^3(\xi_0 x_0 - \xi_3 x_3) + b^3(\xi_1 x_1 - \xi_4 x_4) + c^3(\xi_2 x_2 - \xi_5 x_5) = 0$$

and one now has the three [4]'s

$$\xi_0 x_0 = \xi_3 x_3\,, \qquad \xi_1 x_1 = \xi_4 x_4\,, \qquad \xi_2 x_2 = \xi_5 x_5$$

all containing ξ by (11.4).

12. – The chords of Γ'_5 which join pairs of points harmonic inverses of each other in the planes

(12.1) $x_0 - x_3 = x_1 - x_4 = x_2 - x_6 = 0$ and $x_0 + x_3 = x_1 + x_4 = x_2 + x_5 = 0$

generate a scroll R with a directrix in each plane. The pairs on Γ'_5 are

(12.2) $$(x_0, x_1, x_2, x_3, x_4, x_5) \qquad (x_3, x_4, x_5, x_0, x_1, x_2)$$

with the six coordinates subject to (11.1). The directrix in (12.1) is the locus of

$$(x_0+x_3,\, x_1+x_4,\, x_2+x_5,\, x_3+x_0,\, x_4+x_1,\, x_5+x_2)$$

when the points (12.2) trace Γ_5'. But then, by (11.1),

$$x_0^2+x_3^2 : x_1^2+x_4^2 : x_2^2+x_5^2 = b^2-c^2 : c^2-a^2 : a^2-b^2\,,$$
$$x_0^2-x_3^2 : x_1^2-x_4^2 : x_2^2-x_5^2 = bc(b^2-c^2) : ca(c^2-a^2) : ab(a^2-b^2)$$

so that there are constants μ, ν for which

$$x_0^2=(b^2-c^2)(\mu+\nu bc)\,, \qquad x_3^2=(b^2-c^2)(\mu-\nu bc)\,.$$

Thus the point $(\xi, \eta, \zeta, \xi, \eta, \zeta)$ on R in (12.1) is such that

$$\begin{aligned}\xi &= (b^2-c^2)^{\frac12}\{(\mu+\nu bc)^{\frac12}+(\mu-\nu bc)^{\frac12}\}\,,\\ \tfrac12\xi^2 &= (b^2-c^2)\{\mu+(\mu^2-\nu^2b^2c^2)^{\frac12}\}\\ \tfrac14\xi^4 &= (b^2-c^2)^2\{2\mu^2-\nu^2b^2c^2+2\mu(\mu^2-\nu^2b^2c^2)^{\frac12}\}\\ &= (b^2-c^2)\{\mu\xi^2-\nu^2b^2c^2(b^2-c^2)\}\,.\end{aligned}$$

Two similar relations, derived from this by simultaneous cyclic permutations $(\xi\eta\zeta)$ and (abc), also hold; elimination of μ and ν from the three relations gives

$$\begin{vmatrix} \xi^4 & (b^2-c^2)\xi^2 & b^2c^2(b^2-c^2)^2\\ \eta^4 & (c^2-a^2)\eta^2 & c^2a^2(c^2-a^2)^2\\ \zeta^4 & (a^2-b^2)\zeta^2 & a^2b^2(a^2-b^2)^2\end{vmatrix}=0\,,$$

a plane sextic with nodes, indeed biflecnodes, at the vertices of the triangle $\xi\eta\zeta=0$. Since the genus is known to be 7 three nodes on the plane curve are to be expected.

13. – The preceding paragraphs have been concerned with Γ_n' invariant, when $n=2p+1$, under a $(p+1)$-fold transposition of the $2p+2$ homogeneous coordinates. If one seeks by analogy a curve Γ_n'' invariant under a permutation, of the homogeneous coordinates, consisting wholly of 3-cycles then $n=3s-1$; if ω is either complex cube root of 1 take Γ_n'' to be defined by

$$\Omega_k \equiv \sum_{j=0}^{s-1} a_j^k(x_j^2+\omega^k x_{j+s}^2+\omega^{2k}x_{j+2s}^2)=0 \qquad k=0,1,2,\dots,3s-3.$$

It is invariant under the projectivity $\wp$ induced by the permutation

(13.1) $$(x_0x_sx_{2s})(x_1x_{1+s}x_{1+2s})\dots(x_{s-1}x_{2s-1}x_{3s-1})$$

since this merely multiplies Ω_k by ω^{2k}.

The points invariant under $\wp$ are those, and only those, of the three $[s-1]$'s

(13.2) $\alpha: x_j = x_{j+s} = x_{j+2s}$; $\beta: x_j = \omega x_{j+s} = \omega^2 x_{j+2s}$; $\bar{\beta}: x_j = \omega^2 x_{j+s} = \omega x_{j+2s}$;

where j runs over $0, 1, 2, \ldots, s-1$ and each written pair of equations is but one of s pairs of linear equations holding simultaneously. These three spaces are mutually skew and together span the whole $[3s-1]$; the pairs span the $[2s-1]$'s,

$$\beta\bar{\beta}: x_j + x_{j+s} + x_{j+2s} = 0 \; ; \quad \alpha\bar{\beta}: x_j + \omega x_{j+s} + \omega^2 x_{j+2s} = 0 \; ;$$
$$\alpha\beta: x_j + \omega^2 x_{j+s} + \omega x_{j+2s} = 0 \; .$$

Any point not in any of α, β, $\bar{\beta}$ is one of a trio cyclically permuted by (13.1); should the point be in any of the three $[2s-1]$'s the members of the trio are collinear; otherwise they span a plane meeting all of α, β, $\bar{\beta}$. One may note the s trios $X_j X_{j+s} X_{j+2s}$ spanned by vertices of S. Their s intersections with α, β or $\bar{\beta}$ form a simplex in this $[s-1]$; the harmonic inversions in its vertices and opposite bounding $[s-2]$'s being induced in α, β or $\bar{\beta}$ by

$$h_0 h_s h_{2s}, \ldots, h_j h_{j+s} h_{j+2s}, \ldots, h_{s-1} h_{2s-1} h_{3s-1} \, .$$

These, commuting and having identity for their product, generate an elementary abelian group e of order 2^{s-1}; it is clear from (13.2) that all of them leave α, β, $\bar{\beta}$ invariant. The three $[s-1]$'s are thus only one of $2^{3s-1}/2^{s-1} = 2^{2s}$ such sets of spaces associated with Γ''_n, those of a set being transformed into those of any other by the 2^{s-1} operations of a coset of e in E.

α lies on $2s-2$ of the $3s-2$ quadrics $\Omega_k = 0$, namely those for which $k \not\equiv 0 \pmod 3$. But each of β and $\bar{\beta}$ lies on $2s-1$ of the $3s-2$ quadrics: β on those having $k \not\equiv 2$, $\bar{\beta}$ on those having $k \not\equiv 1$. It follows that β and $\bar{\beta}$ both meet Γ''_n in 2^{s-1} points, namely those, in the case of β, common to the sections by β of the $s-1$ quadrics $\Omega_k = 0$ having $k \equiv 2$ and those, in the case of $\bar{\beta}$, common to the sections by $\bar{\beta}$ of the $s-1$ quadrics $\Omega_k = 0$ having $k \equiv 1$. The planes spanned by trios include the osculating planes of Γ''_n at these 2^s points in β and $\bar{\beta}$.

The 2^{s-1} points in, for example, β satisfy

$$\sum a_j^2 x_j^2 = \sum a_j^5 x_j^2 = \ldots = \sum a_j^{3s-4} x_j^2 = 0 \; ;$$

the obvious determinantal solution of these $s-1$ linear equations for the s « unknowns » x_j^2 shows that none of the x_j is zero.

14. – If ξ is a point on Γ''_n the tangent there is the line

$$\sum_{j=0}^{s-1} a_j^k (\xi_j x_j + \omega^k \xi_{j+s} x_{j+s} + \omega^{2k} \xi_{j+2s} x_{j+2s}) = 0 \qquad k = 0, 1, \ldots, 3s-3$$

Should ξ happen to be one of the points of Γ''_n in β these equations would be

$$\sum a_j^k(x_j + \omega^{k+2}x_{j+s} + \omega^{2k+1}x_{j+2s})\,\xi_j = 0\ .$$

Now a point x in α would satisfy all these equations if only, for each k,

$$\sum a_j^k(1 + \omega^{k+2} + \omega^{2k+1})\,x_j\xi_j = 0\ ;$$

but all these $3s-2$ equations other than those $s-1$ for which $k \equiv 1$ are nugatory so that there are only these $s-1$ linear equations to be satisfied by the s coordinates x_j, namely

$$\sum a_j^k x_j\xi_j = 0 \qquad k = 1, 4, \ldots, 3s-2\ .$$

Since no two a_j are equal nor, as just remarked, is any ξ_j zero, the matrix $[a_j^k\xi_j]$ has rank $s-1$ and the ratios of the x_j are uniquely determined. So the tangents to Γ''_n at its 2^{s-1} intersections with β all meet α as, by similar reasoning, do the tangents at its 2^{s-1} intersections with $\bar\beta$.

When Γ_n is specialised to Γ''_n the parametric form (4.3) implies, as is seen on using (7.1) with $h=3$, the relations

$$x_j^2 : x_{j+s}^2 : x_{j+2s}^2 = \theta + a_j : \omega(\theta + \omega a_j) : \omega^2(\theta + \omega^2 a_j) \qquad j = 0, 1, \ldots, s-1.$$

But, on β, $x_j : x_{j+s} : x_{j+2s} = 1 : \omega^2 : \omega$ so that all 2^{s-1} intersections of Γ''_n with β belong to $B(\infty)$. This batch of 2^{3s-1} points is completed on appropriating the points of Γ''_n in those $2^{2s}-1$ other spaces derived by applying E to β. Similar remarks apply to $\bar\beta$ and $B(0)$.

The osculating prime of Γ''_n at a point of $B(0)$ is, by (3.1) of [5] with $n = 3s-1$,

$$\sum_{j=0}^{s-1} a_j^{3s-3}(\xi_j x_j + \xi_{j+s}x_{j+s} + \xi_{j+2s}x_{j+2s}) = 0\ .$$

When ξ is in $\bar\beta$ this equation is, by (13.2).

$$\sum a_j^{3s-3}(x_j + \omega x_{j+s} + \omega^2 x_{j+2s})\,\xi_j = 0$$

and is satisfied by every point of $\alpha\bar\beta$; the osculating primes of Γ''_n at its 2^{s-1} points in $\bar\beta$ all contain the $[2s-1]\alpha\bar\beta$. All $2^{s-2}(2^{s-1}-1)$ joins of the 2^{s-1} points are therefore principal chords, and the operations of E applied to them produce $2^{3s-2}(2^{s-1}-1)$ of the principal chords joining pairs of points in $B(0)$. Analogous statements hold for $B(\infty)$.

15. – The planes of the trios generate a threefold W whose genus appears on using Zeuthen's formula. For there is a $(1, 3)$ correspondence between the planes of W and the points of Γ_n'' in which the only coincidences occur at the 2^s intersections of Γ_n'' with β and $\bar{\beta}$. But each of these is a coincidence of all three members of a trio and so ([8], p. 238) is to be counted twice; whence the formula gives

$$\begin{aligned} 6(\pi - 1) + 2.2^s &= 2(\pi_{3s-1} - 1) \\ &= 2.2^{3s-3}(3s-4) , \end{aligned}$$

π being the unknown genus of W. Thus

$$\begin{aligned} \pi &= 1 + s\cdot 2^{2s-3} - 2^s(2^{2s-1}+1)/3 \\ &= 1 + s\cdot 2^{3s-3} - 2^s(1 - 2 + 2^2 - \ldots + 2^{2s-2}) . \end{aligned} \tag{15.1}$$

α, β, $\bar{\beta}$ contain directrices of W; $\beta\bar{\beta}$, $\alpha\bar{\beta}$, $\alpha\beta$ meet W in scrolls. Among the 2^{3s-2} intersections of Γ_n'' with a prime through $\beta\bar{\beta}$ are 2^{s-1} points on β and 2^{s-1} on $\bar{\beta}$; the $2^{3s-2} - 2^s$ others consist of trios whose $M = \frac{1}{3}(2^{2s-2} - 1)2^s$ planes meet α on the directrix there, whose order is therefore M. The other two directrices have the same order because a prime through $\alpha\bar{\beta}$ or $\alpha\beta$ contains the tangents at 2^{s-1} points of Γ_n'' which account for 2^s intersections. The generators of each scroll put a pair of directrices in $(1, 1)$ correspondence; the scrolls have order $2M$, W has order $3M$.

16. – The simple instance when $s = 2$ furnishes a threefold of order 12 with three quadruple lines and merits some description. Γ_5'', of order 16 and genus 17, is defined by four simultaneous equations $(a \neq b)$

$$\begin{aligned} x_0^2 + x_2^2 + x_4^2 + x_1^2 + x_3^2 + x_5^2 &= 0 \\ a(x_0^2 + \omega x_2^2 + \omega^2 x_4^2) + b(x_1^2 + \omega x_3^2 + \omega^2 x_5^2) &= 0 \\ a^2(x_0^2 + \omega^2 x_2^2 + \omega x_4^2) + b^2(x_1^2 + \omega^2 x_3^2 + \omega x_5^2) &= 0 \\ a^3(x_0^2 + x_2^2 + x_4^2) + b^3(x_1^2 + x_3^2 + x_5^2) &= 0 \end{aligned} \tag{16.1}$$

Hence, on Γ_5'',

$$\begin{gathered} x_0^2 + x_2^2 + x_4^2 = x_1^2 + x_3^2 + x_5^2 = 0 , \\ x_0^2 + \omega x_2^2 + \omega^2 x_4^2 = \varrho b , \quad x_1^2 + \omega x_3^2 + \omega^2 x_5^2 = -\varrho a , \\ x_0^2 + \omega^2 x_2^2 + \omega x_4^2 = \sigma b^2 , \quad x_1^2 + \omega^2 x_3^2 + \omega x_5^2 = -\sigma a^2 ; \end{gathered}$$

$$\begin{aligned} 3x_0^2 &= \varrho b + \sigma b^2 , & 3x_2^2 &= \omega^2 \varrho b + \omega\sigma b^2 , & 3x_4^2 &= \omega\varrho b + \omega^2\sigma b^2 , \\ -3x_1^2 &= \varrho a + \sigma a^2 , & -3x_3^2 &= \omega^2\varrho a + \omega\sigma a^2 , & -3x_5^2 &= \omega\varrho a + \omega^2\sigma a^2 . \end{aligned} \tag{16.2}$$

Those points that are invariant when the x_j undergo the permutation $(x_0x_2x_4)(x_1x_3x_5)$ are on the lines

$$\alpha: x_0 = x_2 = x_4 , \quad x_1 = x_3 = x_5 ;$$

$$\beta: x_0 = \omega x_2 = \omega^2 x_4 , \quad x_1 = \omega x_3 = \omega^2 x_5 ;$$

$$\bar{\beta}: x_0 = \omega^2 x_2 = \omega x_4 , \quad x_1 = \omega^2 x_3 = \omega x_5 .$$

Should a point of Γ_5'' be on β then, from (16.2), $\varrho = 0$; there are two intersections

$$x_0^2 : x_2^2 : x_4^2 : x_1^2 : x_3^2 : x_5^2 = b^2 : \omega b^2 : \omega^2 b^2 : -a^2 : -\omega a^2 : -\omega^2 a^2 .$$

Likewise there are two intersections with $\bar{\beta}$; but none with α.

The solids containing collinear trios are

$$\beta\bar{\beta}: x_0 + x_2 + x_4 = x_1 + x_3 + x_5 = 0 ,$$

$$\alpha\bar{\beta}: x_0 + \omega x_2 + \omega^2 x_4 = x_1 + \omega x_3 + \omega^2 x_5 = 0 ,$$

$$\alpha\beta: x_0 + \omega^2 x_2 + \omega x_4 = x_1 + \omega^2 x_3 + \omega x_5 = 0 .$$

Take, now, three points $(x_0, x_2, x_4, x_1, x_3, x_5)$ one on each of α, β, $\bar{\beta}$: say

$$(1,\ 1,\ 1,\ k,\ k,\ k) \quad \text{on } \alpha ,$$

$$(1,\ \omega^2,\ \omega,\ m,\ \omega^2 m,\ \omega m) \quad \text{on } \beta ,$$

$$(1,\ \omega,\ \omega^2,\ n,\ \omega n,\ \omega^2 n) \quad \text{on } \bar{\beta} .$$

The object is to select k, m, n so that the plane spanned by these points contains a trio on Γ_5''; in order that it should do so it is sufficient for it to meet Γ_5'' once. But every point of the plane is obtained by varying λ, μ, ν in

(16.3) $$(\lambda + \mu + \nu,\ \lambda + \mu\omega^2 + \nu\omega,\ \lambda + \mu\omega + \nu\omega^2,$$
$$\lambda k + \mu m + \nu n,\ \lambda k + \mu\omega^2 m + \nu\omega n,\ \lambda k + \mu\omega m + \nu\omega^2 n)$$

and on substituting these coordinates for the x_j in (16.1) one has the conditions

$$\lambda^2 + 2\mu\nu + \lambda^2 k^2 + 2\mu\nu mn = 0 ,$$

$$a(\nu^2 + 2\lambda\mu) + b(\nu^2 n^2 + 2\lambda\mu km) = 0 ,$$

$$a^2(\mu^2 + 2\nu\lambda) + b^2(\mu^2 m^2 + 2\nu\lambda kn) = 0 ,$$

$$a^3(\lambda^2 + 2\mu\nu) + b^3(\lambda^2 k^2 + 2\mu\nu mn) = 0 .$$

The first and fourth conditions require

$$\lambda^2 + 2\mu\nu = \lambda^2 k^2 + 2\mu\nu mn = 0$$

so that *either* $\lambda^2 + 2\mu\nu = k^2 - 2mn = 0$ *or* $\lambda^2 = 2\mu\nu = 0$. And, in either alternative, one has to satisfy the second and third of the four conditions.

If, in the second alternative, $\lambda = \mu = 0$, the third condition is identically satisfied, the second if $a + bn^2 = 0$; one obtains the two points of Γ_5'' on $\bar{\beta}$ and their osculating planes. Similarly, $\lambda = \nu = 0$ yields the two points on β. So one considers the first alternative. The two conditions are

$$\nu^2(a + bn^2) = -2\lambda\mu(a + bkm)\,, \qquad \mu^2(a^2 + b^2m^2) = -2\nu\lambda(a^2 + b^2kn)\,,$$

which give, on multiplication and cancelling $\mu\nu$,

$$\mu\nu(a + bn^2)(a^2 + b^2m^2) = 4\lambda^2(a + bkm)(a^2 + b^2kn)$$

or, since $\lambda^2 = -2\mu\nu$,

$$(16.4) \qquad (a + bn^2)(a^2 + b^2m^2) + 8(a + bkm)(a^2 + b^2kn) = 0\,.$$

This is the relation sought; when it is satsfied, in addition to $k^2 = 2mn$, the plane belongs to W. If any one of k, m, n is now eliminated the outcome is a (4, 4) correspondence between the other two; this yields an octavic scroll with two of α, β, $\bar{\beta}$ for quadruple lines. For example: the elimination of k leads to

$$(16.5) \qquad (9a^3 + a^2bn^2 + ab^2m^2 + 17b^3m^2n^2)^2 = 128a^2b^2(am + bn)^2mn\,.$$

The plane (16.3) meets $\beta\bar{\beta}$ in the line

$$(\mu + \nu,\ \mu\omega^2 + \nu\omega,\ \mu\omega + \nu\omega^2,\ \mu m + \nu n,\ \mu\omega^2 m + \nu\omega n,\ \mu\omega m + \nu\omega^2 n)$$

so that one has only to take

$$m = (\omega^2 x_3 - \omega x_5)/(\omega^2 x_2 - \omega x_4)\,, \qquad n = (\omega x_3 - \omega^2 x_5)/(\omega x_2 - \omega^2 x_4)$$

and substitute in (16.5) to obtain an equation for the octavic scroll R_α in which $\beta\bar{\beta}$ meets W.

The generators of R_α are paired in the harmonic inversion $h_0h_2h_4 \equiv h_1h_3h_5$; this accords with (16.5) being unchanged when m, n are replaced by $-m$, $-n$. A plane section of R_α has quadruple points, one on β and one on $\bar{\beta}$, so that, being of genus 5 by (15.1), it is required to have 4 nodes; R_α has 4 double generators.

III. – Curves invariant when the coordinates are permuted in a single cycle.

17. – There is a specialisation of Γ_n having no surviving moduli; the properties of such a curve Δ_n include those of the less specialised curves already considered.

Take $a_j = \varepsilon^j$ where $\varepsilon = \exp[2\pi i/(n+1)]$, a primitive $(n+1)$-th root of unity. Then Δ_n admits, in addition to E, the cyclic self-projectivity $\wp$, of period $n+1$, that permutes the coordinates in a single cycle $(x_0 x_1 \dots x_n)$. For this permutation merely divides Ω_k by ε^k because, now,

$$\Omega_k \equiv \sum \varepsilon^{jk} x_j^2 .$$

Every quadric through Δ_n is invariant under a group of $2^n(n+1)$ projectivities; though Δ_n itself will be invariant for a larger group should there be projectivities permuting the quadrics through Δ_n among themselves.

Since, now,

$$f(\theta) = \theta^{n+1} - 1 , \quad f'(\theta) = (n+1)\theta^n , \quad f'(\varepsilon^j) = (n+1)\varepsilon^{-j}$$

the parametrisation (4.3) is

$$x_j^2 = (\theta + \varepsilon^j)\varepsilon^j . \tag{17.1}$$

When $\wp$ is imposed $B(\theta)$ becomes $B(\varphi)$ where the ratio

$$(\theta + \varepsilon^j)\varepsilon^j : (\varphi + \varepsilon^{j+1})\varepsilon^{j+1}$$

is independent of j, hence $\varphi = \varepsilon\theta$. The batches are permuted in cycles of $n+1$ save that $B(0)$ and $B(\infty)$ are unmoved. The $n+1$ critical batches are $B(-\varepsilon^j)$.

18. – $\wp$ has the $n+1$ distinct fixed points

$$A_k : x_j = \varepsilon^{jk} \qquad k = 0, 1, \dots, n,$$

vertices of a simplex $\mathfrak{S}$. Each A_k lies on all $n+1$ quadrics Ω_k except Ω_{n+1-2k} (the suffix *modulo* $n+1$) so that A_k is on Δ_n when Ω_{n+1-2k} is either Ω_{n-1} or Ω_n. Thus A_1, not being on Ω_{n-1}, is on Δ_n, indeed in $B(0)$.

If $n = 2p+1$ is odd, so that $\varepsilon^{p+1} = -1$, the inversion (18.1) in a pair of opposite bounding $[p]$'s of S transposes A_1 and A_{p+2} so that this last point is also on Δ_n and in $B(0)$.

If $n = 2q$ is even, A_{q+1} is not on Ω_n and so is on Δ_n; indeed it belongs to $B(\infty)$.

Consider now the osculating spaces of Δ_n at A_1. Since, at A_1, $\theta = 0$ and $\xi_j = \varepsilon^j$ the osculating $[s]$ is, as explained in [5], determined by the $n - s$ linear equations

$$\sum (\varepsilon^j)^{r-2} \varepsilon^j x_j = 0 \quad \text{or} \quad \sum \varepsilon^{(r-1)j} x_j = 0 \qquad r = n,\ n-1, \ldots, s+1\,.$$

But this is the equation of that bounding prime of $\mathfrak{S}$ that is opposite to A_{n-r+2} and so one can say, in succession, that

the osculating $[n-1]$ at A_1 is $A_3A_4A_5 \ldots A_nA_0A_1$,

the osculating $[n-2]$ at A_1 is $A_4A_5 \ldots A_nA_0A_1$,

. . . .

the osculating plane at A_1 is $A_nA_0A_1$,

the tangent at A_1 is A_0A_1.

Or, building the figure from the spaces of lower dimension, the tangent of Δ_n at A_1 is A_1A_0, the osculating plane $A_1A_0A_n$, the osculating solid $A_1A_0A_nA_{n-1}$, and so on.

Suppose now that $n = 2p + 1$; the alternative $n = 2q$ will be discussed below. The inversion

$$h_0 h_2 \ldots h_{2p} \equiv h_1 h_2 \ldots h_{2p+1} \tag{18.1}$$

replaces the coordinate ε^{jk} of A_k by $(-1)^j \varepsilon^{jk}$ or $\varepsilon^{j(k+p+1)}$, and so transposes A_k and A_{k+p+1} (suffixes *modulo* $2p + 2$); the vertices of $\mathfrak{S}$ undergo the $(p+1)$-fold transposition

$$(A_0A_{p+1})(A_1A_{p+2}) \ldots (A_pA_{2p+1})\,.$$

Hence the tangent of Δ_{2p+1} at A_{p+2} is $A_{p+2}A_{p+1}$, the osculating plane $A_{p+2}A_{p+1}A_p$, and so on, the osculating $[2p]$ being the bounding prime

$$A_{p+2}A_{p+1} \ldots A_{2p}A_0 \ldots A_{p+4}$$

of $\mathfrak{S}$ opposite A_{p+3}.

Clearly A_1A_{p+2} is a principal chord; but there is more to say. For A_1A_{p+2} is common to

$$A_1A_0A_{2p+1}A_{2p} \ldots A_{p+2} \quad \text{and} \quad A_{p+2}A_{p+1} \ldots A_2A_1$$

which are the osculating $[p + 1]$'s at A_1 and A_{p+2} respectively. Take, for example, $p = 2$. Then the chord A_1A_4 of Δ_5 lies not only in the osculating [4]'s at both A_1 and A_4 but also in the osculating solids

$$A_1A_0A_5A_4 \quad \text{and} \quad A_4A_3A_2A_1\,.$$

A_1A_4 is one of 16 such chords joining pairs in the batch $B(0)$; the members of a pair are images of one another in the inversion $h_0h_2h_4$.

It was seen, in the account of Γ_5', that any point P in $B(0)$ is joined to three other points P_{03}, P_{14}, P_{25} of the batch by principal chords, and that every edge of the tetrahedron $PP_{03}P_{14}P_{25}$ is such a chord. The four points are transforms of any one by h_0h_3, h_1h_4, h_2h_5; e.g.

$$h_2h_5P_{03} = h_2h_5h_0h_3P = h_1h_4P = P_{14}\,.$$

This is therefore true also of Δ_5, since invariance under the involution $\wp^3$ endows Δ_5 with the properties of Γ_5'. But invariance under $\wp^2$ endows Δ_5 with the properties of Γ_5'' also; among them is the fact that there is a fourth principal chord joining P to another point P_{024} of $B(0)$, the transform of P by $h_0h_2h_4$. It is this last chord which, as has just been noted, now lies not only in the osculating [4]'s but also in the osculating solids of Δ_5 at both P and P_{024}. The joins of pairs of points in $B(0)$ include $48 + 16 = 64$ principal chords of Δ_5.

19. – Suppose now, still considering Δ_n, that n is even, say $n = 2q$. The projectivity

$$J: x_j = \varepsilon^{(q+2)j} x_{2q+1-j} \qquad j = 1, 2, \ldots, 2q$$

with the first coordinate x_0 unchanged, is seen to transpose Ω_k and $\Omega_{k'}$ when $k + k' = 2q - 2$. In detail

$$\varepsilon^{jk} x_j^2 \qquad \text{becomes } \varepsilon^{j(k+2q+4)} x_{2q+1-j}^2$$

so that the typical term of the sum into which Ω_k is transformed is, changing j to $2q + 1 - j$,

$$\varepsilon^{(2q+1-j)(k+2q+4)} x_j^2 = \varepsilon^{-j(k+3)} x_j^2 = \varepsilon^{j(2q-k-2)} x_j^2$$

since $\varepsilon^{2q+1} = 1$. Thus J transposes the $n-1$ quadrics through Δ_{2q} in pairs, save that Ω_{q-1} is invariant. J also, as can be similarly verified, transposes Ω_{n-1} and Ω_n.

A matrix form for J is

$$\mathfrak{J} = \begin{bmatrix} \pm 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \varepsilon^{q+2} \\ \cdot & \cdot & \varepsilon^{2q+4} & \cdot \\ \cdot & \varepsilon^{2q(q+2)} & \cdot & \cdot \end{bmatrix}$$

(the sign of the leading element being chosen so that $|\mathfrak{J}| = +1$) having the unit matrix for its square; J is an involution. If P is the $(2q+1)$-rowed permutation matrix

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

imposing $\wp$ it appears, since $\mathfrak{J} P \mathfrak{J} = \varepsilon^{q+2} P^{-1}$ that J and $\wp$ generate a dihedral group of $4q+2$ projectivities. The $4q+2$ matrix forms for these can, on multiplication by the diagonal matrices imposing projectivities of E, have the signs of any of their entries changed and yet still impose projectivities that leave Δ_{2q} invariant; the group of self-projectivities has order $(4q+2)2^{2q}$ and can be imposed by as many matrices of determinant $+1$ (should q be odd write the leading entry of $\mathfrak{J}$ as -1).

This order is 160 when $q=2$. So far as is known Δ_n has never been encountered save in this simplest instance, but Δ_4 was discovered by Wiman in his quest [11] for curves of low genus admitting self-projectivities. He did not, however, exploit his discovery other than by giving this order 160. Some further contributions to the study of Δ_4 have since appeared in [3]; a little more appears below.

The characteristic polynomial of $\mathfrak{J}$ is, if the leading element is $+1$,

$$(1-\lambda)(\lambda^2-1)^q$$

and there is no obstacle to perceiving that, corresponding to the latent root $+1$, $q+1$ linearly independent latent vectors span the $[q]$

$$Q: x_{r+1} = \varepsilon^{(q+2)(r+1)} x_{2q-r}$$

while, corresponding to the latent root -1, q linearly independent latent vectors span the $[q-1]$

$$Q': x_0 = x_{r+1} + \varepsilon^{(q+2)(r+1)} x_{2q-r} = 0 \, .$$

Here r ranges over $0, 1, \ldots, q-1$. Both Q and Q' meet all of

$$X_1 X_{2q}, \; X_2 X_{2q-1}, \; \ldots, \; X_q X_{q+1} \, ;$$

Q also contains X_0, while Q' lies in $x_0 = 0$.

J is the harmonic inversion in Q and Q' which, Ω_{q-1} being invariant, are polar spaces for $\Omega_{q-1}=0$. Furthermore: since Ω_k and $\Omega_{k'}$ are transposed when $k+k'= 2q-2$ any point in which either quadric meets either Q or Q' is on the other quadric.

Hence Δ_{2q} meets Q in the 2^q points

$$\Omega_0 = \Omega_1 = \ldots = \Omega_{q-1} = 0 ;$$

these quadrics have no other common point in Q. Moreover the 2^q tangents of Δ_{2q} at these points, being invariant under J, all meet Q'; the points themselves are paired, the 2^{q-1} joins of the pairs all passing through X_0.

Q and Q' are but one of $(4q+2)2^{q-1}$ pairs of polar spaces for $\Omega_{q-1}=0$ that are axes of involutions leaving Δ_{2q} invariant. For Q, Q' are themselves invariant under

$$h_0,\, h_1h_{2q},\, h_2h_{2q-1},\, \ldots,\, h_qh_{q+1},\; J .$$

The first $q+1$ of these $q+2$ involutions generate an elementary abelian group of order 2^q which is amplified by J to a non-abelian group of order 2^{q+1} whose projectivities are imposed by unimodular matrices either diagonal or having the shape of $\mathfrak{J}$. So there are as many polar pairs Q, Q' as there are cosets of a group of order 2^{q+1} in one of order $(4q+2)2^{2q}$. Each vertex X_j of S is in 2^q of the spaces Q; the polars Q' of these lie in $x_j = 0$.

20. – The points of Δ_{2q} are paired by J; the chords, all transversal to Q and Q', joining these pairs generate a scroll R_0 which meets Q in a directrix curve δ_0, of order m say, with a point of multiplicity 2^{q-1} at X_0. R_0 has a double curve ϱ_0, of order μ say, in Q' since each generator of R_0 meets Q'—which lies in $x_0=0$—in the same point as does its transform by h_0 (under which R_0 is invariant).

A prime through Q' contains those m generators of R_0 that pass through its intersections with δ_0, so that it meets Δ_{2q} in $2m$ points; whence $m = 2^{2q-2}$. But the 2^{2q-1} intersections of Δ_{2q} with a prime through Q consist of the 2^q points in Q itself and two points on each of the 2μ generators of R_0 that pass through the μ intersections of the prime with ϱ_0; whence

$$2^{2q-1} = 2^q + 4\mu , \qquad \mu = 2^{2q-3} - 2^{q-2} .$$

Since the section of R_0 by this prime consists of δ_0 and the 2μ generators the order of R_0 is

$$m + 2\mu = 2^{q-1}(2^q - 1) .$$

A $[q-1]$ through X_0 and lying in Q, unless its position is specialised, meets δ_0 in $2^{2q-2} - 2^{q-1}$ further points collinear with X_0 in pairs. Any contacts of the $[q-1]$ with branches of δ_0 at X_0 must involve further intersections to an even number; one expects therefore all the branches to be inflectional.

The genus π_q' of R_0 is found immediately from Zeuthen's formula. For δ_0 is in $(1, 2)$ correspondence with Δ_{2q}, the branch points being the 2^q points where generators of R_0 are tangents of Δ_{2q}. Hence, using the known value of π_{2q} on the right,

$$4(\pi'_q - 1) + 2^q = 2 \cdot 2^{2q-2}(2q-3) ,$$
$$\pi'_q = 2^{2q-3}(2q-3) - (2^{q-2}-1) .$$

One can also find the genus π''_q of ϱ_0. Any chord PP_0 of Δ_{2q} through X_0 is paired by J with a second such chord $P'P'_0$; PP' and $P_0P'_0$ concur on ϱ_0 and meet $x_0 = 0$ on the doubly covered Δ_{2q-1} that is the projection of Δ_{2q} from X_0. Hence there is a $(1, 2)$ correspondence between ϱ_0 and Δ_{2q-1} and so, the genus of Δ_{2q-1} being known, it only remains to find where the correspondence branches. There are two types of point on ϱ_0 where it does so.

(*a*) If P is one of the 2^q intersections of Δ_{2q} with Q it coincides with P'; only one point of Δ_{2q-1} arises, namely that on $X_0PP_0 \equiv X_0P'P'_0$. The tangents of Δ_{2q} at P and P_0 meet at a branch point on ϱ_0; there are 2^{q-1} of these.

(*b*) The 2^{q-1} generators PP' of R_0 through X_0 each give rise to coincidences $P' \equiv P_0$, $P \equiv P'_0$; such a generator meets ϱ_0 in a branch point, the only point of Δ_{2q-1} corresponding to it being on $X_0PP' \equiv X_0P'_0P_0$. There are 2^{q-1} branch points of this type also.

The Zeuthen formula therefore gives

$$4(\pi''_q - 1) + 2^{q-1} + 2^{q-1} = 2 \cdot 2^{2q-3}(2q-4) ,$$
$$\pi''_q = 2^{2q-3}(q-2) - (2^{q-2}-1) .$$

21. – Δ_{2q} is invariant under J which transforms A_k, having its $(j+1)$-th coordinate ε^{jk} into the point for which

$$x_j = \varepsilon^{(q+2)j+(2q+1-j)k} = \varepsilon^{(q+2-k)j}$$

since $\varepsilon^{2q+1} = 1$; so J transposes A_k and $A_{k'}$ where $k + k' \equiv q + 2 \pmod{2q+1}$. Since A_1 is on Δ_{2q} so, as already seen otherwise in § 18, is A_{q+1}; J transposes not only A_1 and A_{q+1} themselves but also

the tangents A_1A_0 and $A_{q+1}A_{q+2}$

the osculating planes $A_1A_0A_{2q}$ and $A_{q+1}A_{q+2}A_{q+3}$,

....

and finally the osculating primes, these being the bounding primes of $\mathfrak{S}$ opposite A_2 and A_q.

It now appears that

$$A_1 A_0 A_{2q} \dots A_{q+2} A_{q+1}$$

is the osculating $[q+1]$ of Δ_{2q} at both A_1 and A_{q+1}. This $[q+1]$ is fixed under both $\wp$ and J, but it is one of 2^{2q} such biosculating spaces obtainable from any one of them by using E; the two contacts are one in each of $B(0)$ and $B(\infty)$, and the wholes of those two batches are thereby accounted for.

22. – For Wiman's curve Δ_4 $q=2$ and $\varepsilon^5=1$, Δ_4 being defined ([11], p. 39 as modified in [3], p. 1265) by

$$\sum x_j^2 = \sum \varepsilon^j x_j^2 = \sum \varepsilon^{2j} x_j^2 = 0 \tag{22.1}$$

with summations running over $j = 0, 1, 2, 3, 4$. This curve lies on 20 sextic scrolls of genus 2, each of them having a directrix plane quartic and a nodal line which is the polar line of the plane of the quartic in $\sum \varepsilon^j x_j^2 = 0$. Each solid $x_j = 0$ contains four of these lines; they form a skew quadrilateral each pair of whose opposite sides is a pair of polar lines for $\sum \varepsilon^j x_j^2 = 0$; for example, the lines in $x_0 = 0$ are

$$x_1 + \varepsilon^4 x_4 = x_2 + \varepsilon^3 x_3 = 0\,, \qquad x_1 - \varepsilon^4 x_4 = x_2 - \varepsilon^3 x_3 = 0\,,$$

$$x_1 - \varepsilon^4 x_4 = x_2 + \varepsilon^3 x_3 = 0\,, \qquad x_1 + \varepsilon^4 x_4 = x_2 - \varepsilon^3 x_3 = 0\,.$$

Consider now the scroll R_0 with nodal line

$$Q' : x_0 = x_1 + \varepsilon^4 x_4 = x_2 + \varepsilon^3 x_3 = 0\,.$$

The polar plane of Q' in $\Omega_1 = 0$ is

$$Q : x_1 - \varepsilon^4 x_4 = x_2 - \varepsilon^3 x_3 = 0$$

and meets $\Omega_0 = 0$ and $\Omega_2 = 0$ in the same conic; indeed the plane meets Δ_4 in those four points for which

$$x_0^2 + (1+\varepsilon^3) x_4^2 + (1+\varepsilon) x_3^2 = x_0^2 + 2\varepsilon^4 x_4^2 + 2\varepsilon^3 x_3^2 = 0\,.$$

The tangents to Δ_4 at these points all meet Q'. R_0 has a node at X_0 and meets Q in a quartic curve δ_0 with a biflecnode at X_0, each branch having an inflection there.

Since the difference of the two vectors

$$(x_0,\ \ x_1,\ \ x_2,\ \ x_3,\ \ x_4)$$

$$(x_0,\ \varepsilon^4 x_4,\ \varepsilon^3 x_3,\ \varepsilon^2 x_2,\ \varepsilon x_1)$$

registers a point on Q' their sum registers a point in Q; δ_0 is the locus of the point

$$(22.2) \qquad (X, Y, Z, \varepsilon^2 Z, \varepsilon Y) \quad \text{with} \quad X:Y:Z = 2x_0 : x_1 + \varepsilon^4 x_4 : x_2 + \varepsilon^3 x_3$$

as $(x_0, x_1, x_2, x_3, x_4)$ varies on Δ_4. An equation for δ_0 is obtained by eliminating the five x_j from (22.1) and (22.2). As expeditious a way as any is probably to note that

$$(\varepsilon + 1 + \varepsilon^4) x_1^2 x_4^2 + (\varepsilon^3 + 1 + \varepsilon^2) x_2^2 x_3^2 = x_0^4$$

everywhere on Δ_4, for this is seen to hold identically in θ on using (with $n = 4$) the parametrization (17.1). The outcome is

$$\varepsilon^2(\varepsilon + 1 + \varepsilon^4) Y^4 + \varepsilon^4(\varepsilon^3 + 1 + \varepsilon^2) Z^4 = \tfrac{1}{2} X^2\{\varepsilon(\varepsilon^3 - 1 + \varepsilon^2) Y^2 + \varepsilon^2(\varepsilon - 1 + \varepsilon^4) Z^2\},$$

a plane quartic with a biflecnode at $Y = Z = 0$.

23. – The fact of the 16 principal chords joining points in $B(0)$ to points in $B(\infty)$ lying in biosculating solids is an as yet unremarked property of Δ_4; although the fact that a special form of Humbert's plane sextic was a projection of Δ_4 from a chord was noted and exploited in [3] the chord was not, apart from the possibility of its being a tangent, specially selected. Were it, however, to be one of these 16 principal chords the projection H would have two inflections I and I' with the same tangent, this tangent being the intersection of the plane $\tilde{\varpi}$ of H with the biosculating solid.

Project Δ_4 from its chord A_1A_3 onto the opposite plane $\tilde{\varpi} \equiv A_0A_2A_4$ of $\mathfrak{S}$; the biosculating solid, opposite A_2, is $\sigma_3 = 0$ where

$$\sigma_k \equiv \sum_{j=0}^{4} \varepsilon^{jk} x_j \qquad k = 0, 1, 2, 3, 4$$

so that

$$5x_j \equiv \sum_{k=0}^{4} \varepsilon^{-jk} \sigma_k \qquad j = 0, 1, 2, 3, 4$$

and

$$(23.1) \qquad \left\{ \begin{aligned} 5\Omega_0 &\equiv \sigma_0^2 + 2\sigma_2\sigma_3 + 2\sigma_1\sigma_4\,, \\ 5\Omega_1 &\equiv \sigma_3^2 + 2\sigma_2\sigma_4 + 2\sigma_0\sigma_1\,, \\ 5\Omega_2 &\equiv \sigma_1^2 + 2\sigma_0\sigma_2 + 2\sigma_3\sigma_4\,. \end{aligned} \right.$$

In order to obtain the equation of H, the projection of Δ_4 from $\sigma_0 = \sigma_3 = \sigma_1 = 0$, one has only to eliminate σ_2 and σ_4 from the equations got by equating the three quadratic forms simultaneously to zero. The first and third equations can be solved for σ_2 and σ_4 and the solution substituted in the second. The outcome is, writing X, Y, Z for σ_0, σ_1, σ_3,

$$2Z^6 - 5X^2Y^2Z^2 - (X^5 + Y^5)Z + 5X^3Y^3 = 0\,, \tag{23.2}$$

showing clearly that $Z = 0$ is a bitangent with both contacts inflections.

The action of $\wp$ multiplies σ_k by ε^{-k}, inducing the replacement of X, Y, Z by X, $\varepsilon^{-1}Y$, $\varepsilon^{-3}Z$; this accords with H being unchanged under the projectivity of period 5 which replace X, Y, Z by $\varepsilon^3 X$, $\varepsilon^2 Y$, Z. Moreover, since J permutes the σ_k as $(\sigma_0\sigma_1)(\sigma_2\sigma_4)(\sigma_3)$, it induces in $\tilde{\varpi}$ the harmonic inversion which transposes X and Y while leaving Z unchanged. Thus H is invariant under a dihedral group $\mathfrak{D}$ of 10 self-projectivities.

24. – The general properties of Humbert's sextic are of course possessed by this specialised form H, indeed are enhanced by the presence of $\mathfrak{D}$. This is not the place to elaborate them, but one may remark in passing that A_1A_3 is a line on F, the cyclide $\Omega_0 = \Omega_2 = 0$, and that, the tangent plane of F at $x_j = \lambda\varepsilon^j + \mu\varepsilon^{3j}$ on A_1A_3 being

$$\lambda\sigma_1 + \mu\sigma_3 = \lambda\sigma_3 + \mu\sigma_0 = 0\,,$$

the tangent planes to F at the points of A_1A_3 generate the line-cone $\sigma_0\sigma_1 = \sigma_3^2$. Hence (see § 9 of [2]) the conic through the five nodes of H is $XY = Z^2$, and the two tangents at any node of H pass one through I and one through I'.

The operations of E all leave F, as well as its intersection Δ_4 with $\Omega_1 = 0$, invariant, and F is projected onto $\tilde{\varpi}$ from A_1A_3 by a transformation $(1, 1)$ save for five « exceptional » lines on F that meet A_1A_3. So the operations of E induce Cremona transformations in $\tilde{\varpi}$ that leave H invariant: H admits a group of 160 Cremona self-transformations.

The sections of F by solids are projected from A_1A_3 into the adjoint cubics of H. Now while a general canonical curve of genus 5 has 496 quadritangent solids —this being the number $2^{p-1}(2^p - 1)$ when $p = 5$—it was remarked in [2], § 17, that Δ_4, indeed that Γ_4, has ten singly-infinite families: if t is a tangent the solid joining t to the edge X_jX_k of S also contains the tangents t_j, t_k, t_{jk} obtained by applying h_j and h_k to t. Δ_4, however, is now seen to have 16 bisosculating solids quite separate from these ten families of quadritangent solids; $\sigma_3 = 0$ does not contain any edge, nor indeed any vertex, of S. Those 15 of the 16 that do not contain A_1A_3 meet F in elliptic quartics whose projections onto $\tilde{\varpi}$ are adjoint cubics of H each having two 4-point intersections with it (and containing its five nodes).

25. – The involutions in $\mathfrak{D}$ form the coset of its cyclic subgroup and all transpose I and I'; the projectivities of the cyclic subgroup leave both I and I' unmoved so that if I and I' are designated as Poncelet's « circular » points in an extended Euclidean plane this cyclic subgroup will consist of rotations.

If, then, one puts

$$Z=1\,,\qquad X=\mathfrak{X}+i\mathfrak{Y}\,,\qquad Y=\mathfrak{X}-i\mathfrak{Y}$$

in (23.2) one obtains

$$5(\mathfrak{X}^2+\mathfrak{Y}^2)^2(\mathfrak{X}^2+\mathfrak{Y}^2-1)+2=2\mathfrak{X}(\mathfrak{X}^4-10\mathfrak{X}^2\mathfrak{Y}^2+5\mathfrak{Y}^4)$$

and the right-hand side here is

$$32\prod_{j=0}^{4}\left(\mathfrak{X}\cos\frac{2j\pi}{5}-\mathfrak{Y}\sin\frac{2j\pi}{5}\right).$$

Now put $\mathfrak{X}=\mathfrak{R}\cos\Theta$, $\mathfrak{Y}=\mathfrak{R}\sin\Theta$ to obtain

$$5\mathfrak{R}^4(\mathfrak{R}^2-1)+2=32\mathfrak{R}^5\prod_{j=0}^{4}\cos\left(\Theta+\frac{2j\pi}{5}\right)=2\mathfrak{R}^5\cos 5\Theta\,.$$

The invariance under rotation is obvious, as it is under reflection in $\Theta=0$. One is, however, denied the pleasure of a visual representation: the odd function $5(R-R^{-1})+2R^{-5}$ has a single minimum value 2 (when $R=1$) and otherwise, for $R>0$, exceeds 2, so that H has no real points save its five nodes—acnodes according to Cayley's nomenclature.

Each involution of $\mathfrak{D}$ is a central harmonic inversion with centre on II'; its axis joins the harmonic conjugate, with respect to I and I', of this centre to one of the five nodes of H, the other four nodes being transposed in pairs.

Although H has no real points other than its nodes this need not inhibit one from saying that every circle $XY=kZ^2$ cuts H in a decad of points invariant under $\mathfrak{D}$. This decad may, for certain values of k, be a repeated pentad at each point of which the circle touches H; it is just that the points will have complex coordinates. Each such pentad, invariant with H and the circle, must have one vertex on an axis of each of the five involutions. Hence there will be four such « contact circles » because an axis meets H in four points apart from the node through which it passes.

This accords with the Jacobian set of the g^1_{10} cut, apart from I and I', on H by the circles having ([8], § 33) $20+10-2=28$ points. Of these, 20 are on the four contact circles, and the defect is made up if each of I and I' accounts for four members of the Jacobian set. And this they do because the line II' repeated is a member of the pencil of circles, so that one of the decads consists of I and I' both taken five times. Although the assessment of multiplicity can be a subtle problem this present instance is of the « elementary » type whose solution is given in § 36 of [8], and other places.

26. – It is also worth recording that the osculating planes of Δ_4 at A_1 and A_3, lying as they do in a biosculating solid, have a line in common: indeed A_0A_4. The projection of Δ_4 from A_0A_4 onto the opposite plane $A_1A_3A_2$ of $\mathfrak{S}$ is a plane octavic $\mathcal{C}$ with triple points at A_1 and A_3. But there is only a single tangent at each triple point, and this is the same—A_1A_3 itself—at both. The equation of $\mathcal{C}$ is the result of eliminating, now, σ_0 and σ_1 from (23.1). The elimination of σ_0 from the second and third equations gives

$$\sigma_1^3 + 2\sigma_1\sigma_3\sigma_4 - \sigma_2(\sigma_3^2 + 2\sigma_2\sigma_4) = 0 ,$$

while its elimination from the first and third gives

$$(\sigma_1^2 + 2\sigma_3\sigma_4)^2 + 8\sigma_2^2(\sigma_2\sigma_3 + \sigma_1\sigma_4) = 0 .$$

Combinations of these two results furnish the two quadratic equations for σ_1:

$$\sigma_1^2(\sigma_3^2 + 10\sigma_2\sigma_4) + 8\sigma_1\sigma_2^2\sigma_3 + 2\sigma_3\sigma_4(\sigma_3^2 + 2\sigma_2\sigma_4) = 0 ,$$

$$2\sigma_1^2\sigma_3\sigma_4 + \sigma_1\sigma_2(\sigma_3^2 + 10\sigma_2\sigma_4) + 4\sigma_3(\sigma_3\sigma_4^2 + 2\sigma_2^3) = 0 .$$

Elimination of σ_1 between them yields the sought equation for $\mathcal{C}$ namely, replacing σ_2, σ_3, σ_4 by X, Y, Z,

$$Z^8 + 360Z^4X^2Y^2 + 512Z^3(X^5 + Y^5) + 2240Z^2X^3Y^3 + 2000X^4Y^4 = 0 .$$

The triple points A_1, A_3, with their single tangent along $Z = 0$, are manifest, as is invariance under $\mathfrak{D}$.

The pencil of conics $XY = kZ^2$ cut, apart from A_1 and A_3, decads of points on $\mathcal{C}$; the Jacobian set of this g_{10}^1 consists, as did a Jacobian set on H, of 28 points. The singular conic $Z^2 = 0$,a member of the pencil, meets $\mathcal{C}$ at eight points both at A_1 and A_3; removing each of A_1 and A_3 three times leaves a decad consisting of two quintuple points each, as with H, contributing four to the Jacobian set. So 20 points remain, and these will be contact pentads for four of the conics.

Such a pentad invariant, with the conic and $\mathcal{C}$, under $\mathfrak{D}$, must have one member on the axis of each of the five involutions in $\mathfrak{D}$. Each axis, however, meets $\mathcal{C}$ in eight points; the four not belonging to contact pentads are found to coincide in pairs at nodes of $\mathcal{C}$. For example: one axis is $X = Y$ whose eight intersections with $\mathcal{C}$ are pin-pointed by the identity.

$$2000x^8 + 2240x^6 + 1024x^5 + 360x^4 + 1 \equiv (10x^2 + 4x + 1)^2(20x^4 - 16x^3 + 28x^2 - 8x + 1) .$$

The zeros of the biquadratic factor on the right account for one member of each contact pentad; those of the repeated quadratic factor account for the two nodes on $X = Y$, whose (complex) coordinates can thus be given explicitly. A plane octavic of genus 5 with two triple points has to have $21 - 5 - 2.3 = 10$ further double points or their equivalent.

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