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Some implications of the geometry of the 21-point plane

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I

1. The projective plane $\not{p}$ of points whose homogeneous coordinates belong to the Galois Field $\mathrm{GF}(2^2)$, or F_4 as we will call it, consists of 21 points lying 5 on each of 21 lines; 5 lines pass through each point. F_4 is a quadratic extension of that field F_2 which consists only of the zero mark 0 and the unit mark 1; the quadratic x^2+x+1 has its coefficients in F_2, but not its roots. If ω is either root the other is ω^2, and F_4 consists of

$$0, \quad 1, \quad \omega, \quad \omega^2$$

where $1+1=0$ as in F_2, and $\omega+\omega^2=1=\omega^3$.

When the three coordinates of a point are arranged as a column vector, premultiplication by a non-singular matrix M (whose nine elements all belong to F_4) imposes a projectivity in $\not{p}$; matrices M, ωM, $\omega^2 M$ all impose the same projectivity and have equal determinants. The matrices of unit determinant impose the linear fractional group $\mathrm{LF}(3, 2^2)$ of projectivities; it is a simple group Γ of the same order as, but not isomorphic to, the alternating group $\mathscr{A}_8$ and DICKSON, by using canonical forms for the matrices ([3], p. 259), found all its sets of conjugate cyclic subgroups. Yet, although his § 228 shows Γ to be a doubly transitive permutation group of degree 21, there is not so much as a mention, with DICKSON, of $\not{p}$ or its 21 points and 21 lines, let alone of the geometry to be explored here; and this geometry does point directly to some features of Γ. For instance, it throws into prominence 168 subgroups of order 360, 360 subgroups of order 168 and 280 subgroups of order 72. There are in $\not{p}$ 168 hexads (the ovals of B. SEGRE [7], p. 37). There are 360 projectivities that transform any hexad h into any other, but these do not belong to Γ unless of unit determinant. Some projectivities are imposed by matrices whose determinant is ω, others by matrices whose determinant is ω^2; Γ is not transitive on the 168 hexads h but permutes them in three transitive sets of 56. The geometrical version of this fact is that any two hexads that are transforms under Γ share *an even number* of vertices; this number is either 0 or 2 because a quadrangle belongs to one, and only one, hexad. That this relation, of sharing an even number of vertices, is reflexive and symmetric, is obvious. That it is transitive, so that all hexads satisfying it form an equivalence class, is shown below (§ 16): if h and h' share an even number of vertices, as also do h and h'', then so do h' and h''.

The 168 hexads provide three non-equivalent permutation representations of Γ. These, together with the representations of degrees 21 and 105 provided by the points and the flags in $\not{p}$, facilitate calculation of the group characters. This calculation has been carried through, but will not be included here.

2. The order of the subgroup of Γ that leaves a hexad h invariant is $\frac{1}{2}\cdot 8!/56=\frac{1}{2}\cdot 6!$ This subgroup does indeed subject the six vertices of h to the alternating group $\mathscr{A}_6$ of even permutations. Now it happens that there are precisely six lines in $\not{p}$ skew to h, i.e. not containing any of its vertices; these form a hexagram H whose sides undergo, simultaneously with the vertices of h, the permutations of $\mathscr{A}_6$. So $\not{p}$ affords an appropriate setting for the automorphisms, outer as well as inner, of $\mathscr{A}_6$. The symmetric group $\mathscr{S}_6$ is not accessible here as a group of projectivities; but it, and its automorphisms, present themselves when one uses the involution J that replaces each mark of F_4 by its own square.

3. No hexad h in $\not{p}$ has its points all on the same conic; a conic in $\not{p}$ consists of only five points. So h can serve as the set of base points of a web of plane cubic curves mapping the plane sections of a cubic surface ([*1*], p. 360). One naturally takes the liberty of using "curve" and "surface" to denote the sets of zeros of homogeneous ternary and quaternary polynomials although, over finite fields, these sets are finite and possibly vacuous. The particular cubic surface $\mathscr{H}$ so encountered is the one lately studied by HIRSCHFELD [*6*]; since HIRSCHFELD did not find it necessary to use the plane map it seems worth while to obtain some properties of $\mathscr{H}$ by using the geometry in $\not{p}$.

Any (1, 1) transformation of $\mathscr{H}$ is mapped by a transformation of $\not{p}$ which is (1, 1) save perhaps at the points of h; as each point of h maps a whole line on $\mathscr{H}$ it is liable to be transformed into a "curve". In particular: any projectivity leaving $\mathscr{H}$ invariant is mapped by a Cremona transformation of $\not{p}$. All such projectivities are known to form a hyperorthogonal or unitary group ([*3*], p. 309; [*5*], p. 661) of order 25920, so that one can represent this classical group as a group of 25920 Cremona transformations of $\not{p}$. The full "cubic surface group", of order 51840, will not be discussed. One would obtain it by using any of the 25920 correlations that permute the 27 lines on $\mathscr{H}$ among themselves; any one of 36 null polarities ([*6*], p. 87) would serve.

II

4. Each point of the projective plane $\not{p}$ over F_4 has three homogeneous coordinates, all belonging to F_4 but not all simultaneously zero. There are, excluding (0, 0, 0), $4^3-1=63$ coordinate vectors, but scalar multiples of the same vector represent the same point so that, with three non-zero scalars $1, \omega, \omega^2$ in F_4, $\not{p}$ consists of 21 points. It is, like its subplanes over F_2, a Fano plane; the three diagonal points of any quadrangle are collinear in consequence of $1+1$ being 0. A line in $\not{p}$ consists of $(4^2-1)/3=5$ points, and through each point pass 5 lines in $\not{p}$.

Take any quadrangle q; a set of four points, that is, of which no three are collinear. Each of its six *joins* contains two of its four *vertices* and one *diagonal point;* the join is completed by two more points, so that the number of points not on any join is

$$21-(4+3+6\cdot 2)=2.$$

These complete the line λ of diagonal points because these three diagonal points account for the intersections of λ with all the joins of q. As λ does not contain any vertex of q, and as neither of the points supplementing q lies on any join of q, these two supplementary points form a quadrangle with any two vertices of q. Indeed the six points compose a hexad h any four of whose vertices form a quadrangle whose three diagonal points complete the join of the remaining two vertices of h.

Opposite joins of any quadrangle q belonging to h meet at a diagonal point; this is on the join λ of the two remaining vertices of h. Thus h has the Brianchon property, and has it 15 times because there are 15 *synthemes* – to use SYLVESTER's nomenclature ([9], p.91) – or partitions of six objects as three duads. Only over F_4 and its extensions can a hexad have the Brianchon property so multiplied. It involves partitioning six vertices as three duads, but the partitions as two triads are also relevant: if $\Delta \equiv A_1A_2A_3$ and $\Delta' \equiv A_4A_5A_6$ are triangles whose vertices together exhaust h they are in sextuple perspective. The joins of A_1, A_2, A_3 to A_4, A_5, A_6 are, whatever the order of A_4, A_5, A_6, concurrent. Each Brianchon point is a centre of perspective for four of the ten pairs of triangles.

One may take the vertices of the triangle of reference and the unit point to be the vertices of any quadrangle. Its line of diagonal points is then $x+y+z=0$ and the hexad is completed by the other two points on it. Its vertices are

(4.1)

A_1	A_2	A_3	A_4	A_5	A_6
1	.	.	1	1	1
.	1	.	1	ω	ω^2
.	.	1	1	ω^2	ω

5. It is, in plane projective geometry, known that there is a unique projectivity transforming the vertices of a quadrangle into those of (the same or) any other, regard being paid to order. As h is determined by any quadrangle belonging to it there is a group of $6\cdot 5\cdot 4\cdot 3=360$ projectivities permuting its six vertices. Since only the identity can leave four, let alone six, of the vertices unmoved no two projectivities can impose the same permutation; the 360 provide a representation of the alternating group $\mathscr{A}_6$. Since this group is simple every matrix that imposes a projectivity belonging to it has determinant 1.

There is no projectivity transposing two vertices of h and leaving all the other four unmoved, so that the 360 odd permutations are inaccessible if only projectivities are used. But it is clear from (4.1) that the involution J transposes A_5 and A_6 and leaves every other A_i unmoved, so that when J is combined with the projectivities one obtains a representation in $\wp$ of the symmetric group $\mathscr{S}_6$.

6. There are 21 lines in $\not{p}$; of these, 15 are joins of h. Through each vertex of h pass 5 lines, all accounted for by its joins to other vertices, so that there are $21-15=6$ lines not containing any vertex — skew to h one may say. They form a hexagram H, a term justified because no three concur. For, the 15 points not vertices of h being all Brianchon points, were $\mathcal{O}$ a concurrence of three sides of H it would, as a Brianchon point, also be a concurrence of three joins of h. But only 5 lines pass through $\mathcal{O}$ so that at least one join of h would be a side of H whereas, in fact, no side of H contains any point of h. The figure is self-dual; not only does H consist of the lines avoiding every vertex of h but h consists of the points not on any side of H. Any four sides of H form a quadrilateral Q; the remaining two sides complete the set of five lines through the concurrence of the three diagonals of Q. If the sides of H are partitioned as three pairs the three intersections, one for each pair, are collinear; H has the Pascal property, and has it 15 times. The Pascal lines are the 15 joins of h just as the Brianchon points are the 15 intersections of H.

No side of the hexagram associated with the hexad (4.1) passes through any vertex of the triangle of reference; all three coordinates have non-zero coefficients in its equation. Nor, since $x=y=z$ is not to satisfy the equation, can these three coefficients have sum zero — a condition tantamount, over F_4, to the demand that two, but not all, coefficients be equal. The sides of this hexagram are

$$\text{(6.1)} \qquad \begin{cases} \omega x+y+z=0, & x+\omega y+z=0, & x+y+\omega z=0, \\ \omega^2 x+y+z=0, & x+\omega^2 y+z=0, & x+y+\omega^2 z=0. \end{cases}$$

One of its Pascal lines is $x+y+z=0$, this equation being linearly dependent on the pair of any vertical column in (6.1).

7. A Brianchon point is the concurrence of three joins of h. If the vertices of h are labelled 1, 2, 3, 4, 5, 6 each Brianchon point is identified by a syntheme; the fiften synthemes are thus associated one with each Brianchon point. What is the special property of five synthemes when the Brianchon points complete a side of H? Since no two of these five points lie on the same join of h no two of the five synthemes have a duad in common: the five synthemes together include all fifteen duads and compose one of SYLVESTER'S ([9], p. 92) *synthematic totals*. And the six sides of H provide all six of SYLVESTER'S totals.

Whenever six objects are given a second set of six objects, namely the synthematic totals, is associated with them; the groups $\mathscr{A}_6$ and $\mathscr{S}_6$ permute both sets of objects simultaneously. There is a perfect reciprocity between the sets; to prefer either to the other is to take a one-sided view of the group structure and the groups are perhaps better regarded as permutation groups of degree ten, the number of separations of six objects into complementary triads; each such separation of either set induces one, and only one, of the other. In the representation in $\not{p}$ one partitions the vertices of h as triangles Δ, Δ'; their six centres of perspective are themselves vertices of two triangles whose sides compose H, while the six axes of perspective of these latter triangles are the sides of Δ and Δ'. Each of the ten partitions of the vertices of h as two triangles is thus linked unambiguously with one of the ten partitions of the sides of H as two triangles. If, in (4.1), $\Delta \equiv A_1 A_2 A_3$ and $\Delta' \equiv A_4 A_5 A_6$ the centres of perspective are

$$(\omega^2, 1, 1), \quad (1, \omega^2, 1), \quad (1, 1, \omega^2); \quad (\omega, 1, 1), \quad (1, \omega, 1), \quad (1, 1, \omega) \tag{7.1}$$

and the three on either side of the semi-colon are vertices of a triangle whose sides all occur in (6.1) – the left-hand triad has its joins in the upper, the right-hand triad in the lower, stratum of (6.1).

8. The 360 projectivities in $\mathfrak{p}$ that leave h invariant leave H invariant too; but h and H can be transformed into each other by correlations, and it is these correlations that provide outer automorphisms of $\mathscr{A}_6$. A correlation turns points into lines, collinear points into concurrent lines; it is uniquely determined when the quadrilateral Q corresponding to a given quadrangle q is known, the order in which the sides of Q correspond to the vertices of q being relevant. If q is contained in h and Q in H the two vertices of h supplementing q become the two sides of H supplementing Q and since, when the vertices of q are given, there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ choices for the sequence of sides of Q in H there are 360 correlations turning h into H. These, with the 360 projectivities leaving both h and H invariant, make a group of order 720 representing all the automorphisms of $\mathscr{A}_6$. If, to this, one adds the involution J one finds a group of order 1440 representing all the automorphisms of $\mathscr{S}_6$.

Just as *column* vectors identify the *points* so *row* vectors can identify the *lines* of $\mathfrak{p}$. A correlation is given by

$$u' = (u, v, w)' = \Omega \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \Omega\, x,$$

Ω being a non-singular matrix whose elements all belong to F_4; Ω, $\omega\Omega$, $\omega^2\Omega$ all impose the same correlation. It transforms the point x into the line $u = (\Omega x)'$, the point ξ into the line $\rho = (\Omega\xi)'$, and so, if $x = M\xi$,

$$u' = \Omega\, x = \Omega\, M\, \xi = \Omega\, M\, \Omega^{-1} \rho'.$$

To take just one look at an outer automorphism of $\mathscr{A}_6$, and of $\mathscr{S}_6$, in action let

$$M = \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix};$$

it permutes the coordinate vectors of A_1, A_2, A_3 in (4.1) cyclically while merely multiplying those of A_4, A_5, A_6 by scalars; the corresponding projectivity imposes the permutation $(A_1 A_3 A_2)(A_4)(A_5)(A_6)$ on the vertices of h, leaving three of them unmoved.

There is a correlation transforming A_1, A_2, A_3, A_4 respectively into the lines

$$\omega x+y+z=0, \quad x+\omega y+z=0, \quad x+y+\omega z=0, \quad \omega^2 x+y+z=0$$

which appear in (6.1). The first three conditions demand that the rows of Ω are scalar multiples of

$$\omega, 1, 1; \quad 1, \omega, 1; \quad 1, 1, \omega;$$

and the fourth condition thereupon allows

$$\Omega=\begin{bmatrix}\omega^2 & \omega & \omega\\ 1 & \omega & 1\\ 1 & 1 & \omega\end{bmatrix},$$

so that

$$\Omega M \Omega^{-1}=\begin{bmatrix}. & \omega & .\\ . & . & 1\\ \omega^2 & . & .\end{bmatrix}.$$

This matrix imposes the projectivity permuting the A_i in two cycles $(A_1 A_3 A_2)(A_4 A_6 A_5)$, a permutation not conjugate to $(A_1 A_3 A_2)(A_4)(A_5)(A_6)$ whether in $\mathscr{A}_6$ or in $\mathscr{S}_6$.

III

9. So far one has considered only a single h, with its accompanying H, in $\not p$. How many are there? Construct h by selecting successive vertices. Take

A_1, any point of $\not p$ (21 choices);

A_2, any point of $\not p$ other than A_1 (20 choices);

A_3, any point of $\not p$ not on $A_1 A_2$ (16 choices);

A_4, any point of $\not p$ not on any side of $\Delta \equiv A_1 A_2 A_3$ (9 choices).

This fixes h because a quadrangle identifies its two supplementary points, so that the number of h in $\not p$ is

$$21 \cdot 20 \cdot 16 \cdot 9/6 \cdot 5 \cdot 4 \cdot 3=168.$$

Each h has for its stabiliser in Γ a group $\mathscr{A}_6$. No two h have the same stabiliser; such h would have to be disjoint whereas, given h_0, it will be seen that only ten h, none invariant under the stabiliser of h_0, are disjoint from h_0.

Γ has, therefore, 168 alternating subgroups of order 360. It has, too, 360 Klein subgroups of order 168. For there are

$$21 \cdot 20 \cdot 16 \cdot 9/4 \cdot 3 \cdot 2 \cdot 1=2520$$

quadrangles q in $\not p$; each q constitutes, with its three diagonal points, a 7-point subplane of $\not p$ whose group of projectivities is a Klein group. Since each subplane includes $7q$ (namely those consisting of the points not on some one its 7 lines) the number of Klein subgroups in Γ is $2520/7=360$.

10. The argument of §9 shows that there are $20 \cdot 16 \cdot 9/5 \cdot 4 \cdot 3=48h$ with one assigned vertex, $16 \cdot 9/4 \cdot 3=12$ with two and $9/3=3$ with three. When

four vertices are assigned h is, of course, unique. Of the $47h$ that share a vertex V with a given hexad h_0 how many share V only? V and each of the ten pairs of other vertices of h_0 are shared with two more h; V and each of the five other vertices of h_0 are shared with eleven other h, but these include $4 \cdot 2 = 8$ sharing also a third vertex. Thus the number that share V only is

$$47 - [10 \cdot 2 + 5(11-8)] = 12.$$

Let h_0 be the hexagon $A_1 A_2 A_3 A_4 A_5 A_6$ and suppose that a second hexagon shares A_1, and only A_1, with h_0. Each line $A_1 A_i$ through A_1 contains a second vertex B_i of h_1; three joins of h_0 pass through B_i, but neither of the other two lines through B_i contains any vertex of h_0. These two lines are therefore sides of the hexagram H_0 linked with h_0. Now each line through B_i contains a second vertex of h_1, so that each side of H_0 through B_i contains a second B_j (A_1 is not on any side of H_0); moreover no line in $\mathscr{p}$, and so no side of H_0, contains more than two B_i. If, then, one proceeds from B_1, along one of the two sides of H_0 which meet there, to the other vertex of h_1 on this side and then, from this other vertex of h_1, along the other side of H_0 which contains it, and so forth, one returns to B_1 *via* five sides of H_0. One side of H_0 is not used: it is skew to h_1 as well as to h_0. Thus the twelve hexads that share only one given vertex A_i with h_0 fall into six pairs; each pair is associated with one of the six sides of H_0, this side being skew not only to h_0 but to both hexads of the pair.

11. Every set of three vertices of h_0 is shared with two other hexads, each pair of vertices with three others that do not share any third vertex; also, given any single vertex V of h_0 twelve more hexads share only V with h_0. So the number of h disjoint from h_0 is

$$168 - (1 + 20 \cdot 2 + 15 \cdot 3 + 6 \cdot 12) = 10.$$

These are, therefore, the hexads determined one by each of the ten separations into triangles of the sides of H_0; they are permuted among themselves by the $\mathscr{A}_6$ which stabilises h_0. One instance is afforded by the points (7.1), h_0 being given by (4.1).

12. Take any hexad and partition its vertices as triads t, u. Each triad belongs to two further hexads and the triads v, w that so supplement it are found to be the same whether it is t or u that is supplemented. Every two of t, u, v, w compose a hexad, so that there are six hexads

$$\begin{array}{ccc} t+u, & t+v, & t+w, \\ v+w, & w+u, & u+v. \end{array}$$

Since there are ten partitionings of the vertices of any of the 168 hexads as two triads the number of such figures in $\mathscr{p}$ is $1680/6 = 280$. With, for example, the hexad (4.1), and the vertices of the triangle of reference forming t, the triads are

$$\begin{array}{ccc|ccc|ccc|ccc} 1 & . & . & 1 & 1 & 1 & \omega & 1 & 1 & \omega^2 & 1 & 1 \\ . & 1 & . & 1 & \omega & \omega^2 & 1 & \omega & 1 & 1 & \omega^2 & 1 \\ . & . & 1 & 1 & \omega^2 & \omega & 1 & 1 & \omega & 1 & 1 & \omega^2 \end{array}$$

Each three of these four triads consist of those nine points not on any side of the triangle whose vertices are the fourth triad. Through any intersection of sides of any two of these triangles pass sides of the other two. These nine intersections complete the plane and provide an analogue of the classical figure of MACLAURIN: nine points such that the join of any two contains a third. Here the points are

$$(12.1)\qquad \left\{\begin{array}{ccccccccc} . & . & . & 1 & 1 & \omega & 1 & \omega & 1 \\ 1 & 1 & \omega & . & . & . & 1 & 1 & \omega \\ 1 & \omega & 1 & 1 & \omega & 1 & . & . & . \end{array}\right.$$

and, as in the complex field, they are the solutions of

$$(12.2)\qquad xyz = x^3 + y^3 + z^3 = 0.$$

The figure is self-dual: through each of these nine points there passes, in addition to sides of the four triangles, a fifth line; these lines, one through each of the nine points, are such that through the intersection of any two there passes a third, and these intersections are the points in t, u, v, w.

The occurrence of xyz in (12.2) is, over F_4, superfluous: the only cubes in F_4 are 0 and 1, so that every solution of $x^3+y^3+z^3=0$ has one of x, y, z equal to zero. The points (12.1) are just the solutions of

$$x^3+y^3+z^3=0,$$

or of

$$(12.3)\qquad x\bar{x}+y\bar{y}+z\bar{z}=0$$

where the conjugation signified by the bars is pairing in the involution J. The 21 points of $\mathfrak{p}$ consist of 9 for which the unit Hermitian form in (12.3) is zero, 12 for which it is unity; Γ has 280 unitary subgroups. They are of order 72. For each subgroup is transitive on its four triads, and the triangle of reference is stable under projectivities whose matrices are monomial. The number of monomial U such that $\mathsf{U}\bar{\mathsf{U}}'=I$ is $3!\,3^3$. Of these, $3!\,3^2$ have determinant 1; they consist of $3!\,3=18$ sets U, $\omega\mathsf{U}$, $\omega^2\mathsf{U}$.

13. The $15 \cdot 3 = 45$ hexads sharing two vertices (only) with h_0 compose, with h_0 itself and the h disjoint from it, a class of 56. That this is an equivalence class will follow once it is known to be transitive. If two h are both disjoint from h_0 they are quickly seen to have two common vertices. Each of these two h consists of the vertices of a pair of triangles whose sides compose the whole hexagram H_0; if a, b, c and d, e, f are sides of one such pair of triangles one triangle of the other pair has one side among a, b, c and two among d, e, f; the triangles of the second pair could be, say, bdf and ace

and then the vertices common to the hexads would be df and ac. This is the first step towards proving that hexads sharing an even number of vertices form an equivalence class. It remains to find how many vertices are common to

(α) two hexads one of which is disjoint from, and the other shares two vertices with, h_0 and

(β) two hexads both of which have a pair of vertices in common with h_0, and this whether the two pairs are (i) coincident, (ii) disjoint, (iii) overlapping.

14. If A, B are vertices of h_0 three other hexads h_1, h_2, h_3 share A and B (only) with h_0. The remaining vertices of h_0, h_1, h_2, h_3 partition the 16 points off AB as quadrangles q_0, q_1, q_2, q_3; each q_i has D, D', D'' – the points that complete AB – for its diagonal points. If, by way of illustration, h_0 is (4.1) and A, B, are $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ then D, D', D'' are $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ while the quadrangles are

$$\begin{array}{lllll} q_0: & (1,0,0), & (0,1,0), & (0,0,1), & (1,1,1); \\ q_1: & (\omega^2,1,1), & (0,1,\omega), & (0,1,\omega^2), & (\omega,1,1); \\ q_2: & (\omega^2,0,1), & (1,\omega^2,1), & (\omega,0,1), & (1,\omega,1); \\ q_3: & (1,\omega,0), & (1,\omega^2,0), & (1,1,\omega^2), & (1,1,\omega). \end{array}$$

Through each diagonal point pass four lines other than AB. A vertex of any q_i on such a line λ is accompanied on λ by a second vertex of the same q_i; λ is completed by pairs of vertices of two q_i. The remaining vertices of these same two quadrangles complete a second line through this same diagonal point, which is a Brianchon point for all four hexads. Thus each of D, D', D'', divides the four quadrangles into two pairs; of the four lines through it other than AB two are diagonals of one pair, the other two of the other pair. In the above example the lines, other than $x+y+z=0$, through D are

$$\begin{array}{ll} x=0 \text{ and } y=z, & \text{diagonals of } q_0 \text{ and } q_1; \\ \omega x=y+z \text{ and } \omega^2 x=y+z, & \text{diagonals of } q_2 \text{ and } q_3. \end{array}$$

Thus D divides the quadrangles as q_0, q_1 and q_2, q_3. D' gives likewise the division into pairs q_0, q_2 and q_3, q_1, D'' that into q_0, q_3 and q_1, q_2.

The sides of H_0 pass two through each of D, D', D''. The relevant lines are those not diagonals of q_0; at D they are both diagonals of q_2 and q_3, at D' of q_3 and q_1, at D'' of q_1 and q_2. When these six lines, in the four possible ways, are arranged as two triangles each having one side through each of D, D', D'' one obtains four of the ten hexads disjoint from h_0. Each of these four hexads shares two vertices with each of h_1, h_2, h_3; for example, that vertex of either triangle common to sides through D and D'' belongs to q_2, and so to h_2.

Any other hexad disjoint from h_0 consists of vertices of triangles Δ, Δ' such that one of D, D', D'' is a vertex of Δ, another a vertex of Δ', while the third is, as with the four hexads above, common to a side of Δ and a side of Δ'.

If, say, D is a vertex of Δ and D'' of Δ' the hexad is disjoint from h_2 while its vertices, other than D and D'', belong two to h_3 and two to h_1.

An incidental result is that when two hexads share (only) two vertices A, B there are two hexads disjoint from both. These other two hexads also share (only) two vertices C, D; A, B, C, D are collinear and the two pairs of hexads are symmetrically related.

15. Take a hexad $h_0 \equiv A_1A_2A_3A_4A_5A_6$. Let h_1, h_2, h_3 be those hexads sharing only A_1 and A_2 with h_0; h'_1, h'_2, h'_3 those sharing only A_5 and A_6. Quadrangles q_1, q_2, q_3 supplement A_1 and A_2 to complete h_1, h_2, h_3; quadrangles q'_1, q'_2, q'_3 supplement A_5 and A_6 to complete h'_1, h'_2, h'_3. The concurrence B of A_1A_2, A_3A_4, A_5A_6 is a diagonal point of all six quadrangles, a Brianchon point of all seven hexads; it is, too, a diagonal point of $q_0 \equiv A_3A_4A_5A_6$ and of $q'_0 \equiv A_1A_2A_3A_4$.

Choose the suffix 1 so that B divides q_0, q_1 from q_2, q_3 as well as q'_0, q'_1 from q'_2, q'_3; then q_1, q'_1 have common vertices u, v on BA_3A_4 while the vertices of q_2 and q_3, lying as they do on the two lines through B that are not joins of vertices of h_0, are the same eight points as the vertices of q'_2 and q'_3.

Two points D', D'' complete the line BA_1A_2; they are vertices of q'_1. Two lines, say $D'A_3A_6$ and $D'A_4A_5$, through D' are diagonals of both q_0 and q_2; the other two lines through D' are $D'u$ and $D'v$; call their respective intersections with BA_5A_6 C' and C''. These are vertices of q_1. The line $D'uC'$ consists of D', two vertices of q_1 and two of q_3; since u, C' are vertices of q_1 the two remaining points are vertices of q_3. But the same line consists of C', two vertices of q'_1 and either two of q'_2 or two of q'_3; since u, D' are vertices of q'_1 the two remaining points are vertices either both of q'_2 or both of q'_3. There are similar statements for the lines $D'vC''$, $D''uC''$, $D''vC'$. Both q_2 and q_3 share two vertices with both q'_2 and q'_3.

This discussion has shown that h_1 is disjoint from h'_2 and h'_3, h'_1 from h_2 and h_3; on the other hand there are two common vertices for each of the other pairs of hexads:

$$h_1, h'_1; \quad h_2, h'_2; \quad h_2, h'_3; \quad h_3, h'_2; \quad h_3, h'_3.$$

16. Suppose now that, h_0 being given, a hexad h shares (only) A_1 and A_2 with it while k shares (only) A_1 and A_3. It will appear that h and k share one, but only one, vertex in addition to A_1; this will complete the proof that the 168 hexads fall into three equivalence classes of 56.

Let A_1A_i, A_2A_i, A_3A_i meet A_2A_3, A_3A_1, A_1A_2 respectively at P_i, Q_i, R_i; here the suffix i can be any one of 4, 5, 6. The diagonal points of the quadrangle $A_1A_2P_iQ_i$ are A_3, A_i, R_i and this quadrangle is amplified to a hexad when supplemented by the two remaining points of the line $A_3A_iR_i$. The three hexads h_i that can serve for h occur on putting 4, 5, 6 for i. Likewise the three hexads k_i occur on supplementing $A_1A_3P_iR_i$ by the two points which complete the line $A_2A_iQ_i$.

Hexads h_i, k_i with the same suffix share A_1 and P_i; they do not share any other vertex because the points completing $A_3A_iR_i$ are distinct from those completing $A_2 A_i Q_i$. If the suffixes differ take, for definiteness, h_6 and k_4. The former consists of A_1, A_2, P_6, Q_6 and two points on $A_3 A_6 R_6$; the latter of A_1, A_3, P_4, R_4 and two points on $A_2 A_4 Q_4$; hence they share two, and only two, vertices, namely A_1 and the intersection of $A_3 A_6 R_6$ with $A_2 A_4 Q_4$.

IV

17. CLEBSCH's mapping of a cubic surface G on a plane π is (1,1) save that each point A_i of a hexad h in π maps all the points of one of six lines a_i lying wholly on G. These a_i are mutually skew and form one half of a double-six $\mathscr{D}$; any five of them have a single common transversal and these six transversals b_i form the other half of $\mathscr{D}$. Every point of π other than the A_i maps a single point of G.

Every curve in π maps a curve on G; an intersection of two curves in π that is not at any A_i maps an intersection of the corresponding curves on G. Two curves in π through A_i map two curves on G meeting a_i, but they will not meet a_i in the same point unless the curves in π touch at A_i. The plane sections of G are mapped by the web W of cubic curves through all six A_i; every plane section is mapped by such a cubic and every such cubic maps a plane section. If the cubic of W is composite so is the plane section of G.

The order of a curve on G is the number of its intersections with a plane; this is the number of intersections – apart from the A_i – of the mapping curve in π with a cubic of W. Should this number sink to 1 the curve on G is a line. Any line $A_i A_j$ is an obvious instance; it meets a cubic in π three times in all so that, when the cubic belongs to W, there is a single further intersection in addition to A_i and A_j. $A_i A_j$ maps a line $c_{ij} \equiv c_{ji}$ transversal to a_i and a_j, and there are 15 such lines on G. Two of them intersect when they do not share a suffix, but are skew if they have a suffix in common. Again: a conic and cubic in π have six intersections; hence the conic β_i that contains every point of h save A_i has one free intersection with a cubic of W and maps a line b_i on G; b_i is tranversal to all six a_i except that with the same suffix. The map shows that c_{ij} is transversal to b_i and b_j, skew to the remaining b's. Indeed c_{ij} is the line of intersection of the planes $[a_i, b_j]$ and $[a_j, b_i]$.

The lines $A_1 A_2$, $A_3 A_4$, $A_5 A_6$ compose a curve of W; it maps a plane section of G composed of c_{12}, c_{34}, c_{56}. This plane, meeting G in three lines, is a tritangent plane; the three lines in question are not, in general, concurrent, but they would be if $A_1 A_2$, $A_3 A_4$, $A_5 A_6$ were, and so if h had the appropriate Brianchon property. When the three lines on G in a tritangent plane do concur their point of concurrence is called an Eckardt point, say E-pt. for brevity ([*4*], p.229). CLEBSCH himself described a surface – his diagonal surface – with ten E-pts. and remarked on a hexad having the Brianchon property ten times over ([*2*], p.336).

18. All these matters concerning G and π can be translated to finite fields and applied to $\mathscr{H}$ and $\wp$; although one cannot, perhaps, be quite so glib in saying that curves touch one another. Yet here, too, the points on a_i are

mapped by the directions at A_i – there are five of them – and two curves on $\mathscr{H}$ meet a_i at the same point when, and only when, the mapping curves in $\wp$ have the appropriate relation at A_i. Each Brianchon point of h again maps an E-pt. The ten synthemes associated with CLEBSCH's concurrencies at Brianchon points are those exclusive of the five in one synthematic total. In $\wp$, however, all 15 synthemes provide concurrencies so that $\mathscr{H}$ can be said to have the properties of the diagonal surface in sextuplicate – each of the six totals being available to play the part of that one which was exceptional with CLEBSCH's hexad.

An E-pt. mapped at the concurrence of three joins of h is not on any line a_i. An E-pt. on a_i is also on some b_j and then c_{ij}, as well as lying in the plane $[a_i, b_j]$, contains the intersection; in π, $A_i A_j$ touches β_j at A_i ([*10*], p. 206 to 207). This can also happen in a finite plane, and indeed happens in $\wp$ for every pair ij. A conic in $\wp$ consists of five points and has ([*7*], p. 4 and 37) a tangent at each one: namely that one line through the point which is not a chord; thus the five tangents of β_j all contain A_j or, alternatively, $A_i A_j$ touches β_j at A_i (and β_i at A_j). Every point on a_i is, whatever i, an E-pt. So $\mathscr{H}$ has 45 E-pts., and indeed consists of these points only; each point of $\mathscr{H}$ not on any a_i is mapped in π by a point not at any A_i, and so by a Brianchon point of h. The lines c_{ij} cover $\mathscr{H}$.

When the vertices of h are the points (4.1) the conics β_i are

$$(18.1)\qquad \begin{cases} \beta_1\colon\ yz+x^2=0, \quad \beta_2\colon\ zx+y^2=0, \quad \beta_3\colon\ xy+z^2=0, \\ \beta_4\colon\ yz+zx+xy=0, \quad \beta_5\colon\ yz+\omega zx+\omega^2 xy=0, \\ \beta_6\colon\ yz+\omega^2 zx+\omega xy=0. \end{cases}$$

The web W consists of those cubic curves

$$(18.2)\qquad ayz^2+bzx^2+cxy^2+\alpha y^2 z+\beta z^2 x+\gamma x^2 y=0$$

for which

$$(18.3)\qquad a+b+c=\alpha+\beta+\gamma=0.$$

19. Properties of $\mathscr{H}$ accord with the map in $\wp$. As in the classical geometry, the 45 tritangent planes are faces of STEINER trihedra; any such trihedron accounts for nine of the 27 lines, three in each of its faces, and is paired with a second trihedron whose faces account for the same nine lines; each of these nine lines is determined as the intersection of two planes, faces of the two trihedra. Furthermore, such a pair of trihedra is accompanied by two other pairs ([*8*], p. 137) so that the three sets of nine lines, one set determined by each pair, account for all 27. But, with $\mathscr{H}$, the faces of any STEINER trihedron meet not in a point merely but in a line and so, correspondingly, three composite curves of W belong to a pencil. The plane sections

$$(19.1)\qquad \begin{cases} a_1 b_2 c_{12}, & \text{mapped by } (zx+y^2)z=0, \\ a_2 b_3 c_{23}, & \text{mapped by } (xy+z^2)x=0, \\ a_3 b_1 c_{13}, & \text{mapped by } (yz+x^2)y=0, \end{cases}$$

afford an instance; the sum of the three cubic polynomials is identically zero over F_4. The other trihedron of the pair has for its faces the planes of the sections $a_1 b_3 c_{13}$, $a_2 b_1 c_{12}$, $a_3 b_2 c_{23}$ of $\mathscr{H}$. The sections $c_{14} c_{25} c_{36}$, $c_{15} c_{26} c_{34}$, $c_{16} c_{24} c_{35}$ form another STEINER trihedron; the mapping curves are

$$(19.2) \qquad \begin{cases} (y+z)(\omega z+x)(\omega^2 x+y)=0, \\ (\omega y+z)(\omega^2 z+x)(x+y)=0, \\ (\omega^2 y+z)(z+x)(\omega x+y)=0; \end{cases}$$

here too the three polynomials have zero sum. And so on.

20. The axes of three pairs of STEINER trihedra whose faces together account for all 27 lines on $\mathscr{H}$ are the pairs of opposite edges of a tetrahedron $\sum$. $\mathscr{H}$ is linearly dependent on the cubes of the faces of $\sum$; it is an equianharmonic surface ([*4*], p. 265), indeed in 40 different ways since the 240 STEINER trihedra fall into 40 such sets of three pairs. The sections of $\mathscr{H}$ by the faces of $\sum$ are mapped by four curves of W; the cubes of the left-hand sides of their equations are linearly dependent over F_4. For instance:

$$(20.1) \qquad \begin{cases} (y^2 z+\omega z^2 x+\omega^2 x^2 y)^3+(y z^2+\omega z x^2+\omega^2 x y^2)^3 \\ \qquad \equiv (y^2 z+\omega^2 z^2 x+\omega x^2 y)^3+(y z^2+\omega^2 z x^2+\omega x y^2)^3. \end{cases}$$

Here $y^2 z+\omega^2 z^2 x+\omega x^2 y$ is linearly dependent both on the cubic polynomials in (19.1) and those in (19.2); the axes of the two corresponding trihedra meet and the section of $\mathscr{H}$ by their plane is mapped in $\mathscr{p}$ by $y^2 z+\omega^2 z^2 x+\omega x^2 y=0$. There are 40 identities like (20.1).

The fact that the equation of $\mathscr{H}$ can, by choice of $\sum$, be written

$$X^3+Y^3+Z^3+T^3=0$$

or, over F_4

$$X\bar{X}+Y\bar{Y}+Z\bar{Z}+T\bar{T}=0,$$

is a reminder, with the appearance of the unit Hermitian form, that $\mathscr{H}$ is invariant under a unitary group of projectivities ([5], p. 659). Such a projectivity induces a (1,1) transformation on the points of $\mathscr{H}$; this is mapped in $\mathscr{p}$ by a transformation (1,1) save that the A_i are exceptions, i.e. by a Cremona transformation whose fundamental points, if any, are among the A_i and which, since any projectivity turns plane sections of $\mathscr{H}$ into plane sections, leaves the web W unchanged. So one can represent the well-known simple group of order 25920 as a group of Cremona transformations in $\mathscr{p}$.

21. There are involutory quadratic transformations whose fundamental points are the vertices of any triangle Δ belonging to h and which leave one vertex of the residual triangle unmoved while transposing the other two. These transformations map involutory self-projectivities of $\mathscr{H}$. If Δ is the triangle of reference $A_1 A_2 A_3$ a relevant transformation is

$$(21.1) \qquad x:y:z=YZ:ZX:XY.$$

On substituting in (18.2) and cancelling XYZ (this factor, like others to appear in subsequent cancellations, is the Jacobian of the homaloidal net of transforms of lines of $\wp$) the outcome is

(21.2) $$\gamma YZ^2+\alpha ZX^2+\beta XY^2+bY^2Z+cZ^2X+aX^2Y=0,$$

and W is transformed into itself. In order to ascertain the transform of a line on $\mathscr{H}$ under the projectivity mapped in $\wp$ by (21.1) it is enough to take two planes (five are available) through the line, map these sections in $\wp$, transform the maps by (21.1) and note the line on $\mathscr{H}$ that is common to the sections mapped by these transforms. Take, say, c_{12}. The sections

$$a_1 b_2 c_{12}, \quad a_2 b_1 c_{12}, \quad c_{12} c_{34} c_{56}$$

are mapped by

$$(zx+y^2)z=0, \quad (yz+x^2)z=0, \quad z(x+y)(x+y+z)=0$$

whose transforms under (21.1) are, by (21.2),

$$(ZX+Y^2)X=0, \quad (YZ+X^2)Y=0, \quad (X+Y)(YZ+ZX+XY)=0.$$

These map the sections

$$b_2 c_{23} a_3, \quad b_1 c_{13} a_3, \quad c_{34} b_4 a_3,$$

so that the projectivity turns c_{12} into a_3. And so for other lines on $\mathscr{H}$. If, on the other hand, Δ is the triangle $A_4A_5A_6$ a relevant transformation is

$$x:y:z=YZ+X^2:ZX+Y^2:XY+Z^2.$$

On substituting in (18.2) and cancelling $X^3+Y^3+Z^3+XYZ$, i.e. the product of the lines A_5A_6, A_6A_4, A_4A_5 that join the pairs of fundamental points, the outcome is – and here, in order to reach the conclusion, one has to use (18.3) –

$$\beta YZ^2+\gamma ZX^2+\alpha XY^2+cY^2Z+aZ^2X+bX^2Y=0.$$

The involutory quadratic transformations just used, say U and V, do not commute; but UVU and VUV are the same involution, namely

$$x:y:z=X/(YZ+X^2):Y/(ZX+Y^2):Z/(XY+Z^2).$$

When one substitutes $X(ZX+Y^2)(XY+Z^2)$ for x, and the allied polynomials for y and z, in (18.2) the resulting polynomial, of degree 15, includes the six quadratic polynomials in (18.1) as factors. The residual factor is

$$\alpha YZ^2+\beta ZX^2+\gamma XY^2+aY^2Z+bZ^2X+cX^2Y.$$

This last involution maps a projectivity that transposes the two halves of $\mathscr{D}$. It generates, with the 360 projectivities in $\wp$ that leave h invariant, the subgroup of Cremona transformations mapping those 720 projectivities that leave invariant not only $\mathscr{H}$ but also the double-six $\mathscr{D}$.

A multitude of other details could be talked about. But the algebra is routine, the geometry elementary, and matters may be left here.

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