THE KLEIN GROUP IN THREE DIMENSIONS.

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Introduction.

The existence of a simple group of order 168 and the representation of it as a transitive permutation group of degree 7 were known to Galois, but since the group was exhibited as a group of ternary linear substitutions by Klein in his famous paper »Über die Transformation siebenter Ordnung der elliptischen Funktionen» published in Volume 14 of *Mathematische Annalen* (and later in the third volume of his collected mathematical papers) it has been generally known as the Klein group. Klein's paper is indeed so masterful in its handling of material, so penetrating in insight and so rich in its yield of new results that it is only fitting that the group should be associated with his name. The paper was followed, in Volume 15 of *Mathematische Annalen* a few months later, by a second paper »Über die Auflösung gewisser Gleichungen vom siebenten und achten Grade»; this is in the second volume of the collected mathematical papers. It is these two papers, together with the cognate material in the first volume of the Klein-Fricke treatise on elliptic modular functions, that constitute the indispensable sources of the work which follows.

In the second of Klein's papers there is set out a geometrical basis for handling the Klein group operating as a group of quaternary collineations in three dimensions. Even a cursory glance over this paper leaves no doubt of the value, in Klein's estimation, of this approach to the subject. In the introduction he tells us that he has gladly again availed himself of geometrical deliberations for (he says) geometry does not merely make visual and illuminate but serves in these researches the prime purpose of discovery. And later, when he has disclosed the net of quadric loci and the net of quadric envelopes on which the geometrical structure must arise, he says explicitly that they form the proper starting-point for the main treatment of the subject. These are surely authoritative commendations, and Klein must have persisted in his opinion. When the first volume of the Klein-Fricke treatise was published more than ten years later the net of quadric loci, together with the cubic surfaces containing its Jacobian curve, again appeared. And almost so long as half a century later, when the third and last volume of his collected mathematical papers was published in 1923, we find Klein adding a note (pp. 177—8) in which he directs attention to the Jacobian curve and its scroll of trisecants.

While however there can be no doubt of Klein's opinions, and while we may perhaps surmise that he supplemented them with verbal exhortations to his numerous and expert pupils, the fact remains that, except for Baker's »Note introductory to the study of Klein's group of order 168» the geometrical exploration of the three-dimensional figure has been entirely neglected. Baker obtains twenty-one quartic scrolls of genus 1 which contain the Jacobian curve. But Baker builds his arguments on the Klein-Fricke treatise rather than on Klein's two original papers, and as the treatise mentions the net of quadric loci but never the net of quadric envelopes Baker does not mention the reciprocations of the figure into itself. Nor, having signalised the quartic scrolls, does he proceed to deal with any other surfaces or loci. But his results are the only geometrical additions to Klein's own, and while I record this I must also declare that it is to Baker's paper that I owe my introduction to the subject. Perhaps it is of interest to remark that its publication in 1935 at once caused me to consult the original authorities to which I might not have been led otherwise and that, in consequence of this, the generating function $\Phi(x)$ of § 38 was obtained in August 1936.

There is certainly one conspicuous reason why Klein's commendations have not been implemented to better purpose; the geometry of the figure is that of a net of quadrics invariant for a group of 168 collineations whereas, up to some few years ago, very little was known about the geometry of a net of quadrics. When Klein discovered the plane quartic curve that admits a group of 168 collineations he was able to appropriate at once the geometry of a general plane quartic; to lay hands upon its inflections, its bitangents, its sextactic points, its systems of contact cubics, and so on, and the brilliantly effective way in which he did so has been a source of delight to countless mathematicians. When, however, he offered a net of quadrics there was but little geometry ready to hand. This defect is now, since the publication of five notes on a net of quadric surfaces, at least partly remedied and there is a sufficient knowledge of the geometry to enable one to undertake the exploration of Klein's three-dimensional figure. And here I must again acknowledge my debt to Baker's note, for the form of $\Phi(x)$ disclosed by the work which this note instigated was the main impetus to the examination of the geometry of a net of quadric surfaces and the consequent writing of the five notes upon it.

The following abbreviations will be used henceforward to signify certain references.

K.-F. F. Klein: *Vorlesungen über die Theorie der elliptischen Modulfunctionen* (ausgearbeitet und vervollständigt von R. Fricke), Band I. Leipzig, 1890.

K. II. F. Klein: *Gesammelte Mathematische Abhandlungen.* Band II. Berlin, 1922.

K. III. F. Klein: *Gesammelte Mathematische Abhandlungen.* Band III. Berlin, 1923.

B. H. F. Baker: Note introductory to the study of Klein's group of order 168: *Proceedings of the Cambridge Philosophical Society* 31 (1935), 468—481.

O.S. W. L. Edge: Octadic surfaces and plane quartic curves: *Proceedings of the London Mathematical Society* (2), 34 (1932), 492—525.

Note I. W. L. Edge: Notes on a net of quadric surfaces. I. The Cremona transformation: *Proceedings of the London Mathematical Society* (2), 43 (1937), 302—315.

Note II. W. L. Edge: Notes on a net of quadric surfaces. II. Anharmonic Covariants: *Journal of the London Mathematical Society* 12 (1937), 276—280.

Note III. W. L Edge: Notes on a net of quadric surfaces. III. The scroll of trisecants of the Jacobian curve: *Proceedings of the London Mathematical Society* (2), 44 (1938), 466—480.

Note IV. W. L. Edge: Notes on a net of quadric surfaces. IV. Combinantal covariants of low order: *Proceedings of the London Mathematical Society* (2), 47 (1941), 123—141.

The paper falls into four sections, the first of which is concerned with the setting up and description of the figure and an account of some of its primary properties.

The three-dimensional figure has a net of quadric loci as a cardinal feature, and this net may be identified immediately the left-hand side of the equation of Klein's plane quartic k is obtained as a symmetrical four-rowed determinant $\varDelta$. For, the elements of $\varDelta$ being linear forms in the coordinates (ξ, η, ζ), $\varDelta$ may be regarded as the discriminant of a quaternary quadratic form whose coefficients are homogeneous and linear in ξ, η, ζ; this form, when equated to zero, is a quadric which, as ξ, η, ζ vary, varies in a net of quadrics and always passes through eight fixed points P, the base points of the figure. The locus of vertices ot those quadrics of the net which are cones is its Jacobian curve K, which is hereby put into (1, 1) correspondence with k. There is a known procedure for obtaining $\varDelta$ and, moreover, the system of contact cubics associated with $\varDelta$ (the cubics of this system are obtained by bordering $\varDelta$ with a row and column of constants and so are associated each with a plane of the space figure) can be chosen to be that to which the eight inflectional triangles of k belong. It is important to make this choice, for it facilitates the transition from the group of collineations in the plane to the simply isomorphic group $\mathfrak{G}$ of collineations in space. Now an inflectional triangle of k not only has for its vertices points of k but also has for its sides tangents of k: it is an in-and-circumscribed triangle of k. And it was shown in *O.S.* that when a (1, 1) correspondence is set up between a general non-singular plane quartic and the Jacobian curve ϑ of a net of quadrics those eight in-and-circumscribed triangles of the quartic which belong to the system of contact cubics that is, in the setting up of the correspondence, associated with the planes of space are associated with the eight tritangent planes of the scroll of trisecants of ϑ. By applying this result to the (1, 1) correspondence between k and K it follows, in § 4, that each of the eight triads of points on K that correspond to the vertices of an inflectional triangle of k has the properties that the tangent of K at each point of the triad intersects K again in a second point of the same triad and that the plane Π of the triad is tritangent to K. It is shown too, in § 9, that the osculating plane of K at each point of the triad has five-point contact and, in § 10, that the three osculating planes of K at the three points of a triad have one of the eight base points P as their intersection. The eight planes Π are themselves the base planes of a net of quadric envelopes. Each plane Π forms, together with the osculating planes of K at its three contacts with it, a tetrahedron Ω, and with each pair of these eight tetrahedra Ω there is associated a quadric Q with respect to which both tetrahedra of the pair are self-polar (§ 12). The whole figure is its own polar reciprocal with respect to each of these 28 quadrics Q. It is one of the tetrahedra Ω that is tetrahedron of reference for the system of homogeneous coordinates employed to handle the algebraical work.

The second section of the paper is concerned with the deduction by geometrical methods of further properties of the figure. These are obtained by using the subgroups of low order contained in $\mathbf{G}$; the structure of the Klein group is well known, but new results appear in the light of its new representation. The involutions, or collineations of period 2, of $\mathbf{G}$ furnish Baker's quartic scrolls. The nodal lines of these scrolls are the axes of the involutions and are of some importance in the geometry of the figure; they are distributed in 56 coplanar and concurrent triads, each of them belonging to 4 of the triads. The point of concurrence of any triad lies on a join p of two of the base points while the plane of any triad contains an intersection π of two of the base planes (§ 18). Consideration of pairs of permutable involutions discloses two sets each of seven quadrics, with a (3, 3) correspondence between the quadrics of either set and those of the other. Each of the fourteen quadrics possesses, among its self-polar tetrahedra, two whose vertices are complementary tetrads of base points and two whose faces are complementary tetrads of base planes. These fourteen quadrics are connected with the fourteen octahedral groups of collineations that belong to $\mathbf{G}$; each such octahedral group permutes, in all possible ways, a set of four lines p which join the base points in pairs as well as a set of four lines π of intersection of pairs of the base planes. The fourteen quadrics are also very intimately related to fourteen linear complexes that were obtained by Klein, and the geometrical aspect of this relation is considered in §§ 24—27.

The collineations of period 3 which belong to $\mathbf{G}$ also yield interesting information. For example: through each line π pass two planes each of which contains three points P and their three joins p while, dually, on each line p lie two points through each of which pass three planes Π and their three lines of intersection π. The osculating planes of K at its two intersections with a line p meet in the corresponding line π, while the tritangent planes (other than the eight planes Π) of K pass two through each line π; this accounts for every tritangent plane of K. Moreover each line p contains four points at each of which concur three tangents of K.

Section III of the paper is concerned with the invariants of the group. There can be no question of how to obtain these; one must, as Klein surmised

(K. II., 412), consider the combinantal covariants of the net of quadrics. It is now, since the publication of Note IV, practicable to do this. But a few of the invariants have been known for a long time. Invariants, one of each of the degrees 4, 6, 8, 10, 12, occur on p. 242 of Brioschi's paper »Über die Jacobische Modulargleichung vom achten Grade» in Volume 15 of *Mathematische Annalen*; there may not be, at this early date, a full appreciation of the real significance of these invariants but their algebraic forms certainly appear, as indeed also do, on p. 244, equations which are in fact those of cubic surfaces containing K. Two invariants of respective degrees 4 and 14 occur on p. 739 of the Klein-Fricke treatise and here, after having been obtained as linear combinations of modular forms, they are expressly regarded as surfaces which intersect K in invariant sets of 24 and 84 points.

The quartic invariant is the only invariant of this lowest possible degree and the corresponding non-singular quartic surface F^4 has apparently never as yet been subjected to geometrical examination. A few of its many interesting properties are established below, and the surface is obtained by arguments independent of any considerations save those concerned with the geometry of a net of quadrics. It is outpolar to the net of quadric envelopes and has every line π as a bitangent. Its Hessian is a scroll whose nodal curve has its pinch-points at the intersections of F^4 and K; its parabolic curve has at each of these points a triple point with only two distinct tangents.

The group has one, and only one, sextic invariant. But there are several invariants of degree 8 and algebraic expressions for them are given in the table on p. 205. The generating function $\Phi(x)$ of § 38 discloses instantly how many invariants of any given order exist which are not linearly dependent; all invariants of degree 8 are linear combinations of three of them and belong to a net of surfaces of which one member is F^4 repeated. Of other surfaces of this net some six are derived from geometrical definitions of combinantal covariants of the net of quadrics.

Section IV of the paper is concerned with covariant line complexes. This material is perhaps more relevant to a group of 168 substitutions on six variables than to $\mathbf{G}$, and a figure in five dimensions is the chief source of information. The group on six variables has a quadratic invariant which corresponds to the identical relation satisfied by the six Plücker coordinates of a line and any other invariant gives, in combination with this quadratic invariant, a complex covariant for $\mathbf{G}$. Since a little is already known about cubic complexes covariantly related to a net of quadrics it is perhaps not out of place to add this fourth section to the paper and to indicate, in the course of it, equations for and geometrical definitions of some cubic complexes covariant for $\mathbf{G}$. A few remarks are also made about covariant quartic complexes which, so a coefficient in a generating function tells us, are all linearly dependent on two among them.

I.

Primary Properties of the Figure.

1. A plane quartic δ can, as was shown by HESSE[1], be put into birational correspondence with the sextic Jacobian curve ϑ of a net of quadric surfaces; the correspondence is always such that any six coplanar points of ϑ correspond to six points of δ which are the points of contact of δ with a contact cubic, and which do not lie on a conic. There is thus a system of ∞^3 contact cubics of δ associated with the planes of space. There are 36 systems of contact cubics of δ whose sets of six contacts do not lie on conics; in setting up the birational correspondence any one of these 36 systems may be chosen as the one to be associated with the planes of space.

Each of these 36 systems of contact cubics includes eight in-and-circumscribed triangles of δ; the vertices e, f, g of such a triangle lie on δ and the sides fg, ge, ef touch δ at points l, m, n respectively. This was shown in *O.S.*, where it was also shown that if that system Σ of contact cubics is selected to which an in-and-circumscribed triangle efg belongs, and a birational correspondence, associating Σ with the planes of space, is set up between δ and the Jacobian curve ϑ of a net of quadrics *the six coplanar points E, F, G, L, M, N of ϑ which correspond to the six points e, f, g, l, m, n of δ are such that MN, NL, LM pass through E, F, G respectively.* The plane LMN thus contains three trisecants of ϑ. Each of the eight in-and-circumscribed triangles which belong to Σ gives a plane of this kind and there are no planes, other than these eight, containing three trisecants of ϑ.

The polar planes of a point of ϑ with respect to the quadrics of the net all pass through a definite trisecant of ϑ, and each trisecant may be so obtained from one and only one point of ϑ. A point and trisecant of ϑ which correspond in this way are said to be *conjugate* to one another. It was shown (*O.S.* 513—514) that the trisecants conjugate to L, M, N are EMN, FNL, GLM respectively.

[1] *Journal für Math.*, 49 (1855), 279—332; *Gesammelte Werke*, 345.

Other properties of the trisecants, to which it may be convenient to allude subsequently, have been known for a long time. Through any point P of ϑ there pass three trisecants: the plane of any two of these meets ϑ in one further point not on either of them, and this ($O.S.$ 510) is the point which is conjugate to the third trisecant through P. Each pencil of quadrics belonging to the net includes four cones, and the vertices of these cones form a canonical set of points on ϑ; the planes which join those sets of three points of ϑ which make up canonical sets with P all pass through the trisecant conjugate to P and, conversely, any plane through the trisecant conjugate to P meets ϑ further in three points which make up a canonical set with P.

The mention of the canonical sets on ϑ affords an opportunity of stating another fact which will be used later. The Cremona transformation whereby two points correspond to one another when they are conjugate with respect to the net of quadrics was studied in Note I; it was there (p. 310) shown that the locus of those points which are conjugate to the points of a chord PQ of ϑ is a second chord RS of ϑ, and that $PQRS$ is a canonical set.

2. We now consider the plane quartic k which is invariant for a group of 168 collineations. The equation of k may (K. III. 103; K.-F., 701) be written

$$\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3 = 0.$$

The vertices of the triangle of reference are on the curve and, moreover, they are points of inflection; the inflectional tangents at $\eta = \zeta = 0$, $\zeta = \xi = 0$, $\xi = \eta = 0$ are $\zeta = 0$, $\xi = 0$, $\eta = 0$ respectively. The triangle of reference is thus an in-and-circumscribed triangle, but *each of the three vertices coincides with the point of contact of one of the sides.* This happens, indeed, not only for the triangle of reference but also for each of the other seven in-and-circumscribed triangles that belong to the same system of contact cubics (K. III, 116; K.-F., 718).

In order to establish a birational correspondence between k and the Jacobian curve K of a net of quadrics we have to express $\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3$ as a symmetrical determinant Δ, of four rows and columns, whose elements are homogeneous linear forms in ξ, η, ζ; a system of contact cubics is then obtained by bordering Δ with a row and column of constants, and Δ can be found so that the bordering gives any chosen one of the 36 systems, A method of finding Δ was explained by A. C. Dixon[1]; it requires, for the calculation of the linear

[1] *Proc. Cambridge Phil. Soc.* 11 (1902), 350.

forms, prior knowledge of the equations of the contact cubics. Now this knowledge is fully furnished by Klein himself in K. III, 117; the notation used by Klein is changed to the one being used now simply by writing ξ, η, ζ instead of λ, ν, μ respectively. Dixon's argument enables us to assert that if, using the notation[1] of Klein's equations (38) and (39), the elements of

$$\begin{vmatrix} A_1^2 & A_1A_2 & A_1A_3 & A_1A_0 \\ A_1A_2 & A_2^2 & A_2A_3 & A_2A_0 \\ A_1A_3 & A_2A_3 & A_3^2 & A_3A_0 \\ A_1A_0 & A_2A_0 & A_3A_0 & A_0^2 \end{vmatrix}$$

are replaced by their expressions in terms of ξ, η, ζ, and the cofactor of any element divided by the square of $\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3$, then the quotient is that linear form which is to occupy the corresponding position in Δ. The determinant, on using Klein's equations, becomes

$$\begin{vmatrix} -\zeta^3 - \xi\eta^2 & \zeta^2\xi & \xi^2\eta & \zeta\xi^2 \\ \zeta^2\xi & -\eta^3 - \zeta\xi^2 & \eta^2\zeta & \eta\zeta^2 \\ \xi^2\eta & \eta^2\zeta & -\xi^3 - \eta\zeta^2 & \xi\eta^2 \\ \zeta\xi^2 & \eta\zeta^2 & \xi\eta^2 & \xi\eta\zeta \end{vmatrix},$$

and easy calculations then give

$$\Delta \equiv \begin{vmatrix} \eta & . & . & -\xi \\ . & \xi & . & -\zeta \\ . & . & \zeta & -\eta \\ -\xi & -\zeta & -\eta & . \end{vmatrix} \equiv -(\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3).$$

The determinantal form which will be used is, however,

$$\tfrac{1}{2}\Delta \equiv \begin{vmatrix} \eta & . & . & -\tau^{-1}\xi \\ . & \xi & . & -\tau^{-1}\zeta \\ . & . & \zeta & -\tau^{-1}\eta \\ -\tau^{-1}\xi & -\tau^{-1}\zeta & -\tau^{-1}\eta & .. \end{vmatrix}, \qquad 2.1$$

[1] In K. III, 117 there is a misprint in the last of the equations (39); what is printed as A_2A_1 ought to be A_3A_1. In the original paper in *Math. Annalen* 14 there is happily no misprint. Observe the adumbration, in the footnote to these equations, of the cubic envelopes inscribed in a developable of class 6 and genus 3 (cf. K. II, 412).

where $\tau^2 = 2$. The factor τ^{-1} can of course be omitted from the last row and column without altering the equation of k; but it is eminently desirable that it should be retained because its retention enables the self-duality of the space-figure that will be constructed to be more immediately perceived. This square root of 2 was indeed introduced by Klein (K. II, 408), and for the same reason; it does not, however, reappear either in the Klein-Fricke treatise or in Baker's paper.

The curve

$$\begin{vmatrix} \eta & . & . & -\tau^{-1}\xi & u \\ . & \xi & . & -\tau^{-1}\zeta & v \\ . & . & \zeta & -\tau^{-1}\eta & w \\ -\tau^{-1}\xi & -\tau^{-1}\zeta & -\tau^{-1}\eta & . & p \\ u & v & w & p & . \end{vmatrix} = 0$$

is a contact cubic of k, and the ∞^3 different values of the ratios $u:v:w:p$ give all the contact cubics of one of the 36 systems. When $u = v = w = 0$ the contact cubic is $\xi\eta\zeta = 0$; the system of contact cubics is therefore the one to which the triangle of reference belongs. The remaining seven in-and-circumscribed triangles that belong to the system are found by putting $s = 1, 2, 3, 4, 5, 6, 7$ in the identity

$$\begin{vmatrix} \eta & . & . & -\tau^{-1}\xi & \varepsilon^{s} \\ . & \xi & . & -\tau^{-1}\zeta & \varepsilon^{4s} \\ . & . & \zeta & -\tau^{-1}\eta & \varepsilon^{2s} \\ -\tau^{-1}\xi & -\tau^{-1}\zeta & -\tau^{-1}\eta & . & \tau^{-1} \\ \varepsilon^{s} & \varepsilon^{4s} & \varepsilon^{2s} & \tau^{-1} & . \end{vmatrix}$$

$$\equiv \tfrac{1}{2}(\varepsilon^{6s}\xi + \beta\gamma\varepsilon^{5s}\eta + \beta\varepsilon^{3s}\zeta)(\varepsilon^{6s}\xi + \gamma\alpha\varepsilon^{5s}\eta + \gamma\varepsilon^{3s}\zeta)(\varepsilon^{6s}\xi + \alpha\beta\varepsilon^{5s}\eta + \alpha\varepsilon^{3s}\zeta), \qquad 2.2$$

where ε is any primitive seventh root of unity and

$$\alpha = \varepsilon + \varepsilon^6, \qquad \beta = \varepsilon^2 + \varepsilon^5, \qquad \gamma = \varepsilon^4 + \varepsilon^3.$$

These three expressions, as is pointed out by Baker (B., 469), are such that

$$\left.\begin{matrix} \alpha^2 = \beta + 2 & \beta^2 = \gamma + 2 & \gamma^2 = \alpha + 2 \\ \beta\gamma = \alpha + \beta & \gamma\alpha = \beta + \gamma & \alpha\beta = \gamma + \alpha\,; \end{matrix}\right\} \qquad 2.3$$

any polynomial in α, β, γ can thus be expressed as a linear combination of them; and, furthermore, $\alpha + \beta + \gamma = -1$. If we write

$$\theta_1 = \varepsilon + \varepsilon^2 + \varepsilon^4, \qquad \theta_2 = \varepsilon^6 + \varepsilon^5 + \varepsilon^3,$$

where the indices appearing in θ_1 are the quadratic residues and those appearing in θ_2 the quadratic non-residues to modulus 7, then θ_1 and θ_2 are the roots of $(2\theta + 1)^2 + 7 = 0$.

3. The determinant which appears in 2.1 is the discriminant of the quadric

$$\xi(y^2 - \tau tx) + \eta(x^2 - \tau tz) + \zeta(z^2 - \tau ty) = 0. \tag{3.1}$$

This quadric is a cone with vertex (x, y, z, t) if

$$\eta x - \tau^{-1}\xi t = \xi y - \tau^{-1}\zeta t = \zeta z - \tau^{-1}\eta t = \xi x + \eta z + \zeta y = 0. \tag{3.2}$$

These equations give (B., 473), as well as $\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3 = 0$,

$$x : y : z : t = \zeta\xi^2 : \eta\zeta^2 : \xi\eta^2 : \tau\xi\eta\zeta; \tag{3.3}$$

these are therefore the coordinates of that point of K which, in the birational correspondence between k and K, corresponds to the point (ξ, η, ζ) of k.

The quadrics 3.1 for different values of ξ, η, ζ belong to a net $\mathbf{N}$ and the locus of the vertices of the cones of $\mathbf{N}$, obtained by eliminating ξ, η, ζ from 3.2, is

$$\left\| \begin{matrix} -\tau^{-1}t & y & . & x \\ x & . & -\tau^{-1}t & z \\ . & -\tau^{-1}t & z & y \end{matrix} \right\| = 0;$$

these are the equations of K, which is thus common to the cubic surfaces (cf. K. III, 173; K.-F., 728; B., 473)

$$\begin{gathered} X \equiv t^2x + \tau ty^2 + 2yz^2 = 0, \qquad Y \equiv t^2y + \tau tz^2 + 2zx^2 = 0, \\ Z \equiv t^2z + \tau tx^2 + 2xy^2 = 0, \qquad T \equiv 2\tau xyz - t^3 = 0, \end{gathered} \tag{3.4}$$

and to all the cubic surfaces that belong to the linear system, of freedom 3, determined by these. The Cremona transformation in which corresponding points are conjugate with respect to $\mathbf{N}$ is (Note I, 303, footnote) given by

$$x' : y' : z' : t' = X : Y : Z : T. \tag{3.5}$$

All the quadrics of N have in common the eight points $P_0:(0, 0, 0, 1)$ and $P_s(\varepsilon^{6s}, \varepsilon^{3s}, \varepsilon^{5s}, \tau^{-1})$; these will be called the *base points*, and they are Klein's *Hauptpunkten* (K. II, 410). The line $P_i P_j$ will be denoted by p_{ij}; these twenty-eight joins of pairs of base points are, by Hesse's classical discovery, all chords of the Jacobian curve K, and it may be verified, by means of 3.4, that the point $(\varepsilon^{6i} + \lambda\varepsilon^{6j}, \ \varepsilon^{3i} + \lambda\varepsilon^{3j}, \ \varepsilon^{5i} + \lambda\varepsilon^{5j}, \ \tau^{-1} + \lambda\tau^{-1})$ lies on K when λ is a root of the quadratic $7\lambda^2 \pm 4\lambda\sqrt{-7} - 7 = 0$, one or the other sign being taken according as $i - j$ is, or is not, a quadratic residue to modulus 7. Hesse's result is thus verified for the twenty-one joins not passing through P_0; as for the join p_{0s}, it is at once seen that $(\varepsilon^{6s}, \varepsilon^{3s}, \varepsilon^{5s}, \tau^{-1} + \mu)$ lies on K when $2\mu^2 + 4\tau\mu + 7 = 0$.

4. Suppose then, k being a Klein quartic, that the vertices e, f, g of an in-and-circumscribed triangle coincide, respectively, with the points of contact m, n, l of the sides ge, ef, fg; the corresponding phenomenon occurring also for each of the other seven in-and-circumscribed triangles which belong to the same system of contact cubics. Suppose further that, as in the algebra above, this system of contact cubics is associated with the planes of space when k is put into birational correspondence with the Jacobian curve K. Then the remaining intersections E, F, G of any three coplanar trisecants MN, NL, LM with K must coincide with M, N, L respectively. Thus *the sides of the triangle LMN are the tangents of K at its vertices*; MN touches K at M, NL touches K at N and LM touches K at L. Each side of the triangle is a trisecant two of whose three intersections with K coincide, and there are eight triangles of this kind. The equations of their planes are, as is seen on referring to the constants which border the determinant in 2.2, $t = 0$ and $\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t = 0$. We shall call these planes Π_0 and Π_s and speak of them as the *base planes*; they are Klein's *Hauptebenen* (K. II, 410), and the left-hand sides of their equations satisfy the identity

$$\sum_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t)^2 - \tfrac{7}{2} t^2 \equiv 0. \qquad 4.1$$

It follows that the base planes are common to three linearly independent quadric envelopes, and it is at once verified that the quadrics

$$v^2 - \tau p u = 0, \qquad u^2 - \tau p w = 0, \qquad w^2 - \tau p v = 0, \qquad 4.2$$

where (u, v, w, p) are plane coordinates contragredient to the point coordinates

(x, y, z, t), touch all the base planes. The self-duality of our three dimensional figure is now manifest, for these quadric envelopes arise from quadric loci belonging to **N** simply by replacing point coordinates by plane coordinates; this simple replacement, without any further adjustment of coefficients, has been made possible by introducing the irrational τ. It implies that the figure is its own polar reciprocal with respect to the quadric $x^2 + y^2 + z^2 + t^2 = 0$; but it will presently appear that this is only one of twenty-eight quadrics by means of which the figure can be reciprocated into itself.

There is one further matter of nomenclature; the line common to the base planes Π_i and Π_j will be denoted by π_{ij}.

5. When k is subjected to a collineation for which it is invariant contact cubics are transformed into contact cubics. Moreover, since two contact cubics belong to the same system when, and only when, their two sets of six contacts together form the complete intersection of k with a cubic curve, contact cubics which belong to the same system are transformed into contact cubics also belonging to the same system. Thus any system of contact cubics is either unaffected as a whole or else is transformed into a second system; but it is clear that the system to which the eight inflectional triangles belong must be invariant for the collineation. The collineation must, in virtue of the birational correspondence between k and K, induce a birational transformation of K into itself; since a set of coplanar points of K corresponds to a set of points of contact of k with a contact cubic belonging to the same system as do the inflectional triangles it follows that any set of coplanar points of K is transformed into another set of coplanar points and so that the transformation is a collineation.[1] There is thus a group **G** of 168 collineations in space for which K is invariant; it is the Klein group in three dimensions.

6. It is natural to presume that equations for the collineations in space must be obtainable by combining the formulae for the birational correspondence between k and K with equations for the collineations in the plane. The Klein group can, as Dyck pointed out[2], be generated by two operations of periods 7 and 2. These operations were given, as ternary collineations, by Klein (K. III, 107), and two corresponding quaternary collineations will generate **G**. The one

[1] This is a standard proposition; see, for example, SEVERI: *Trattatto di Geometria Algebrica* I (Bologna, 1926), 68. And, for the present application, the first footnote on p. 724 of K.-F.

[2] *Math. Annalen* 20 (1882), 41.

of period 7 is, by using the equation for the collineation in the plane together with equations 3.3, easy to obtain; not so, however, the one of period 2. Klein gives it (K. II, 409) in precisely the form that will be used here, but details as to how he obtained it do not seem to be available. In the Klein-Fricke treatise the quaternary involution is obtained from the ternary one not without transcendental methods, and the result is verified, but *a posteriori*, by Baker. Yet the quaternary collineations are simply those transformations to which, in Klein's notation, A_1, A_2, A_3, A_0 are subjected in consequence of λ, μ, ν undergoing the ternary collineations (K. III, 117), and it ought not to be difficult to obtain the desired equations by purely algebraical methods.

An alternative way of attempting this, depending on the relations between different operations of period 2 belonging to the group, might run on the following lines. Each operation of period 2 is permutable with four others (K. III, 94; K.-F. 382); these fall into two pairs such that each pair, when taken with the original operation and the identity, forms a 4-group — as we shall call the *Vierergruppe* of Klein. An involution j for which k is invariant has (K. III, 102; K.-F., 709) as axis a line containing four sextactic points r_1, r_1', r_2, r_2' of k; the other two involutions of a 4-group containing j permute these four points in pairs, both involutions yielding the same permutation. There thus arise two of the three such permutations, one for each of the two 4-groups, of the four points; there remains a third. Let this outstanding permutation be $(r_1 r_1')(r_2 r_2')$. Then it transpires that the quaternary involution I induced by j is biaxial and has for its axes the two chords $R_1 R_1'$ and $R_2 R_2'$ of K, where, as on other occasions, points of k and K correspond when denoted by the same small and capital letter. Now when j is given its axis can be found, and hence the coordinates of the four sextactic points thereon; equations 3.3 then give the corresponding points of K. When the permutations of the four sextactic points have been determined from a knowledge of the two 4-groups the two chords of K which are axes of I are known and so equations for I can be written down.

7. We now take, as the collineations of periods 7 and 2 that generate the Klein group $\mathbf{G}$ in three dimensions, those whose matrices are (cf. K. II, 409)

$$\mathbf{E} = \begin{bmatrix} \varepsilon^6 & . & . & . \\ . & \varepsilon^3 & . & . \\ . & . & \varepsilon^5 & . \\ . & . & . & 1 \end{bmatrix} \qquad \mathbf{F} = \frac{i}{\sigma} \begin{bmatrix} \beta & \gamma & \alpha & \tau \\ \gamma & \alpha & \beta & \tau \\ \alpha & \beta & \gamma & \tau \\ \tau & \tau & \tau & 1 \end{bmatrix}$$

where $\sigma^2 = 7$ and the other symbols have the same significance as in § 2. It is seen immediately that, if $\mathbf{I}$ is the unit matrix, $\mathbf{E}^7 = \mathbf{I}$ and $\mathbf{F}^2 = -\mathbf{I}$. The point P' which arises from a given point P by applying a collineation is found as follows: the four homogeneous coordinates of P are arranged as a column-vector and this is premultiplied by the matrix of the collineation; the resulting column-vector gives the coordinates of P'. Every point P is invariant for a collineation whenever the matrix is a scalar multiple of the unit matrix. If, however, it is desired to obtain the plane π' which arises by applying the collineation to a plane π (π' is of course the locus of those points which result from applying the collineation to the points of π) then the four homogeneous coordinates of π are arranged as a row-vector; this is postmultiplied, in accordance with the principle of contragredience, by the matrix *inverse* to the matrix of the collineation and the resulting row-vector gives the coordinates of π'. Both $\mathbf{E}$ and $\mathbf{F}$ must be non-singular since $|\mathbf{E}|^7 = |\mathbf{F}|^2 = 1$. Clearly $|\mathbf{E}| = 1$, while

$$|\mathbf{F}| = \frac{1}{\sigma^4}\begin{vmatrix} \beta & \gamma & \alpha & \tau \\ \gamma & \alpha & \beta & \tau \\ \alpha & \beta & \gamma & \tau \\ \tau & \tau & \tau & 1 \end{vmatrix} = \frac{1}{49}\begin{vmatrix} \beta & \gamma & \alpha & . \\ \gamma & \alpha & \beta & . \\ \alpha & \beta & \gamma & . \\ \tau & \tau & \tau & 7 \end{vmatrix}$$

as follows at once on adding the sum of the first three columns, each multiplied by τ, to the last column. Thus

$$\begin{aligned} |\mathbf{F}| &= \tfrac{1}{7}(3\alpha\beta\gamma - \alpha^3 - \beta^3 - \gamma^3) \\ &= \tfrac{1}{7}(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta) = +1. \end{aligned}$$

8. The 168 collineations must permute the base points among themselves, as also the base planes; and so there appears the representation of Klein's group as a permutation group of degree 8. The effect of $\mathbf{E}$ and $\mathbf{F}$ on the base points is to produce the respective permutations, sufficiently described by writing down only the suffixes,

$$\mathbf{e} = (1234567), \qquad \mathbf{f} = (07)(16)(23)(45),$$

and the effect on the base planes is seen, by using the inverse matrices $\mathbf{E}^{-1} = \mathbf{E}^6$ and $\mathbf{F}^{-1} = -\mathbf{F}$, to be precisely the same.

These permutations satisfy the relations

$$\mathbf{e}^7 = \mathbf{f}^2 = (\mathbf{e}\,\mathbf{f})^3 = (\mathbf{e}^4\,\mathbf{f})^4 = 1, \tag{8.1}$$

where, in products of operations, those on the right operate first, and the corresponding relations satisfied by $\mathbf{E}$ and $\mathbf{F}$ are easily obtained. We have

$$\mathbf{E}\,\mathbf{F} = \frac{i}{\sigma}\begin{bmatrix} \beta\varepsilon^6 & \gamma\varepsilon^6 & \alpha\varepsilon^6 & \tau\varepsilon^6 \\ \gamma\varepsilon^3 & \alpha\varepsilon^3 & \beta\varepsilon^3 & \tau\varepsilon^3 \\ \alpha\varepsilon^5 & \beta\varepsilon^5 & \gamma\varepsilon^5 & \tau\varepsilon^5 \\ \tau & \tau & \tau & 1 \end{bmatrix}, \quad (\mathbf{E}\,\mathbf{F})^{-1} = -\mathbf{F}\,\mathbf{E}^6 = -\frac{i}{\sigma}\begin{bmatrix} \beta\varepsilon & \gamma\varepsilon^4 & \alpha\varepsilon^2 & \tau \\ \gamma\varepsilon & \alpha\varepsilon^4 & \beta\varepsilon^2 & \tau \\ \alpha\varepsilon & \beta\varepsilon^4 & \gamma\varepsilon^2 & \tau \\ \tau\varepsilon & \tau\varepsilon^4 & \tau\varepsilon^2 & 1 \end{bmatrix}.$$

If the matrix on the left is squared and the elements of the product simplified by using 2.3, it is found that

$$(\mathbf{E}\,\mathbf{F})^2 = -\frac{1}{\sigma^2}(1 + 2\,\varepsilon^6 + 2\,\varepsilon^5 + 2\,\varepsilon^3)(\mathbf{E}\,\mathbf{F})^{-1}\,i\,\sigma.$$

Now $\varepsilon^6 + \varepsilon^5 + \varepsilon^3$ is a root of the quadratic $(2\theta + 1)^2 + 7 = 0$, and σ can therefore[1] be chosen to be that square root of 7 such that $1 + 2\,\varepsilon^6 + 2\,\varepsilon^5 + 2\,\varepsilon^3 = -\,i\,\sigma$; it then follows that $(\mathbf{E}\,\mathbf{F})^2 = -(\mathbf{E}\,\mathbf{F})^{-1}$, or $(\mathbf{E}\,\mathbf{F})^3 = -\mathbf{I}$. We next remark that

$$\mathbf{E}^4\,\mathbf{F} = \frac{i}{\sigma}\begin{bmatrix} \beta\varepsilon^3 & \gamma\varepsilon^3 & \alpha\varepsilon^3 & \tau\varepsilon^3 \\ \gamma\varepsilon^5 & \alpha\varepsilon^5 & \beta\varepsilon^5 & \tau\varepsilon^5 \\ \alpha\varepsilon^6 & \beta\varepsilon^6 & \gamma\varepsilon^6 & \tau\varepsilon^6 \\ \tau & \tau & \tau & 1 \end{bmatrix}, \quad (\mathbf{E}^4\,\mathbf{F})^{-1} = -\mathbf{F}\,\mathbf{E}^3 = -\frac{i}{\sigma}\begin{bmatrix} \beta\varepsilon^4 & \gamma\varepsilon^2 & \alpha\varepsilon & \tau \\ \gamma\varepsilon^4 & \alpha\varepsilon^2 & \beta\varepsilon & \tau \\ \alpha\varepsilon^4 & \beta\varepsilon^2 & \gamma\varepsilon & \tau \\ \tau\varepsilon^4 & \tau\varepsilon^2 & \tau\varepsilon & 1 \end{bmatrix}.$$

If these two matrices are squared then it is found, on using equations 2.3, that the sum of the two squares is, element by element, identically zero. Hence

$$(\mathbf{E}^4\,\mathbf{F})^2 + (\mathbf{E}^4\,\mathbf{F})^{-2} = 0,$$
$$(\mathbf{E}^4\,\mathbf{F})^4 = -\mathbf{I}.$$

The unimodular matrices $\mathbf{E}$ and $\mathbf{F}$ are therefore such that

$$\mathbf{E}^7 = \mathbf{I},\ \mathbf{F}^2 = (\mathbf{E}\,\mathbf{F})^3 = (\mathbf{E}^4\,\mathbf{F})^4 = -\mathbf{I}.$$

9. Let us return now, for the moment, to the Jacobian curve ϑ which is in birational correspondence with a general plane quartic δ. It is known[2] that there are 24 trisecants of ϑ which are also tangents. We may therefore specify four sets of 24 points on ϑ, namely

[1] It is of course of little importance which square root of 7 is chosen for σ, since $\mathbf{F}$ can at any time be replaced by $-\mathbf{F}$; the choice made here gives a slightly better appearance to the sets of coordinates of the points which appear in § 28.

[2] Zeuthen: *Annali di mat.* (2), 3 (1869), 186; Baker: *Principles of Geometry* 6 (Cambridge, 1933), 32.

(i) the points of contact of those trisecants which touch ϑ;
(ii) the remaining intersections of these trisecants with ϑ;
(iii) the points of ϑ to which these trisecants are conjugate;
(iv) the points of ϑ that correspond to the 24 inflections of δ.

For a general curve δ there thus arise four distinct sets of points on ϑ.

Now suppose that δ is a Klein quartic k; then the corresponding Jacobian curve K is invariant for a group of 168 collineations and each of the above four sets of points on K must also be invariant for this group. But there is only *one* set of 24 points on K that is invariant for the group (K. III, 101; K.-F., 696), namely the set of points which corresponds to the inflections of k. Hence, for the special Jacobian curve K, *the four sets of points must all be the same set.* This statement can be verified; for it has been shown (*O.S.*, 497 and 509) that a point I of K corresponds to an inflection of k when and only when the plane which joins I to its conjugate trisecant touches K at I. The set (iv) of points on K consists therefore of the eight triads of points such as L, M, N; for we have seen that the plane which joins L to its conjugate trisecant MN contains the tangent LM of K at L. But L belongs to the set (i) because it is the point of contact of K with the trisecant LM; it belongs to the set (ii) because it is the intersection of K with the trisecant NL, and, lastly, it belongs to the set (iii) because it is conjugate to the trisecant MN. We will call these 24 points the points c on K, as does Klein (K.-F., 727).

The plane of two trisecants which pass through a point P of the general Jacobian curve ϑ meets ϑ in a sixth point which is not on either of these two trisecants, and this point, as remarked in § 1, is conjugate to the third trisecant of ϑ that passes through P. Suppose then that TU is one of those trisecants which touch ϑ, T being its point of contact and U its remaining intersection with ϑ; let P be that point of ϑ to which TU is conjugate. Then, of the three trisecants through P, two coincide with the trisecant PQR conjugate to T; the tangents of ϑ at Q and R are coplanar, and their plane meets ϑ (apart from its contacts at Q and R and its intersection at P) in U.

Apply this now to the curve K, supposing T and U to be the points M and N respectively; then P is the point L, and PQR is the trisecant NL which is conjugate to M. Thus Q and R both coincide with N, and the plane containing the tangents of ϑ at Q and R becomes a plane having four-point contact with K at N. But this is not all, for U also coincides with N; hence the osculating plane of K at N has *five-point contact*, and similarly for the osculating planes of K at L and M. Thus *the osculating planes of K at the 24 points c all have five-point contact.*[1] Each of these osculating planes has one intersection with K other than its point of contact, and this intersection also is one of the points c; the osculating plane at M, for example, meets K again in N. These 24 planes may be described as stationary planes, and will be called the planes Γ.

There is a formula[2] for the number of stationary osculating planes (i. e. of osculating planes which have four-point contact instead of the usual three-point contact) of a twisted curve. It is found, by applying this formula, that K has 48 stationary osculating planes; these consist of the 24 planes Γ each counted twice. There can be no other stationary osculating planes of K.

10. Each base plane is, by § 4, tritangent to K and there is a point common to the osculating planes at its three contacts. There thus arises a set of eight points which, like the set of base planes, must be invariant for $\mathbf{G}$. But such a set of points, to wit the base points, has already been encountered, and if there is only one such set it follows that *the three stationary planes of K at its points of contact with a base plane have one of the base points as their intersection.*

This is easy to verify. It is sufficient to give the verification for Π_0 since it will then follow for the other planes Π_s by applying the collineations of $\mathbf{G}$.

The equation of Π_0 is $t = 0$; 3.4 shows that it touches K at the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and that the respective tangents are $t = z = 0$, $t = x = 0$, $t = y = 0$. The line $t = z = 0$, for example, is common to the surfaces $X = 0$, $T = 0$ and to the tangent planes $z = 0$ of $Y = 0$ and $t = 0$ of $Z = 0$ at $(1, 0, 0, 0)$. The three stationary osculating planes of K therefore pass one through each of these lines; if it can be shown that they are $z = 0, x = 0, y = 0$ the verification will be completed because these intersect at P_0. But $x = 0$, for example, has, from the form of T, all its intersections with K on the plane $t = 0$ and since it contains both $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ but not the tangent of K at the second of these it must have five-point contact at the first of them.

The verification can also be carried out by using the equations 3.3 for the

[1] This is proved, by considering the zeros of modular forms, in K.-F., 727; the planes are there spoken of as *Nebenebenen.*

[2] This formula, which goes back to Cayley and Zeuthen (see Cayley's *Collected Mathematical Papers*, 5, 516) is given in the standard treatises: Bertini: *Geometria proiettiva degli iperspazi* (Messina 1923), 492; Baker: *Principles of Geometry* 5 (Cambridge 1933), 191.

birational correspondence between k and K, and this method too gives a further substantiation of the five-point contact.

The tangent and osculating plane at a point of K are determined, together with the multiplicities of their contacts, from the expansions for the coordinates of points of K in the neighbourhood of the given point as power series. These expansions are at once calculated by 3.3 provided that corresponding expansions are known for k. Consider, for example, the points of k near $\eta = \zeta = 0$. Putting $\xi = 1$ the equation of k becomes $\eta^3 + \eta\zeta^3 + \zeta = 0$, so that we have the expansions

$$\xi = 1, \quad \eta = \eta, \quad \zeta = -\eta^3 + \eta^{10} - 3\eta^{17} + \cdots,$$

thus verifying incidentally that the tangent of k is $\zeta = 0$ and that it has three-point contact. The equations 3.3 now give

$$x : y : z : t = -\eta^3 + \eta^{10} : \eta^7 - 2\eta^{14} : \eta^2 : -\eta^4 + \eta^{11},$$

where only the first two terms of the power series for the coordinates have been written down; we may therefore take

$$x = -\eta + \eta^8, \quad y = \eta^5 - 2\eta^{12}, \quad z = 1, \quad t = -\eta^2 + \eta^9,$$

so proving that the corresponding point on K is $(0, 0, 1, 0)$, that the tangent there is $y = t = 0$ with ordinary contact and that the osculating plane is $y = 0$ with five-point contact.

It is thus clear that the three stationary planes which osculate K at its points of contact with the base plane Π_i have as their intersection the base point P_i; there thus arise, for $i = 0, 1, 2, 3, 4, 5, 6, 7$, eight tetrahedra Ω_i. These may be called *osculating tetrahedra*; each has a base plane (tritangent to K) and a base point for opposite face and vertex while the three remaining faces osculate K with five-point contact.

11. A collineation of $\mathbf{G}$ which transforms P_i into P_j and so Π_i into Π_j also transforms Ω_i into Ω_j; for the three planes Γ which intersect at P_i must be transformed into those which intersect at P_j. Thus the tetrahedra Ω_i also afford a representation of the Klein group as a permutation group of degree 8; but, as will soon be understood, they do more.

The bottom row of the matrix $\mathbf{F}$, or its inverse $-\mathbf{F}$, shows that the corresponding collineation of period 2 interchanges Π_0 and Π_7. More generally: the collineation corresponding to $\mathbf{E}^s\mathbf{F}\mathbf{E}^{-s}$ is also of period 2 and, since

$$\mathbf{E}^s\mathbf{F}\mathbf{E}^{-s} = \frac{i}{\sigma}\begin{bmatrix} \varepsilon^{6s} & . & . & . \\ . & \varepsilon^{3s} & . & . \\ . & . & \varepsilon^{5s} & . \\ . & . & . & 1 \end{bmatrix}\begin{bmatrix} \beta & \gamma & \alpha & \tau \\ \gamma & \alpha & \beta & \tau \\ \alpha & \beta & \gamma & \tau \\ \tau & \tau & \tau & 1 \end{bmatrix}\begin{bmatrix} \varepsilon^{s} & . & . & . \\ . & \varepsilon^{4s} & . & . \\ . & . & \varepsilon^{2s} & . \\ . & . & . & 1 \end{bmatrix}$$

$$= \frac{i}{\sigma}\begin{bmatrix} \beta & \gamma\varepsilon^{3s} & \alpha\varepsilon^{s} & \tau\varepsilon^{6s} \\ \gamma\varepsilon^{4s} & \alpha & \beta\varepsilon^{5s} & \tau\varepsilon^{3s} \\ \alpha\varepsilon^{6s} & \beta\varepsilon^{2s} & \gamma & \tau\varepsilon^{5s} \\ \tau\varepsilon^{s} & \tau\varepsilon^{4s} & \tau\varepsilon^{2s} & 1 \end{bmatrix},$$

it transforms Π_0 into Π_s and therefore Ω_0 into Ω_s. The equations of the three planes Γ which belong to Ω_s are therefore

$$\begin{aligned} \beta x + \gamma\varepsilon^{3s}y + \alpha\varepsilon^{s}z + \tau\varepsilon^{6s}t &= 0, \\ \gamma\varepsilon^{4s}x + \alpha y + \beta\varepsilon^{5s}z + \tau\varepsilon^{3s}t &= 0, \\ \alpha\varepsilon^{6s}x + \beta\varepsilon^{2s}y + \gamma z + \tau\varepsilon^{5s}t &= 0, \end{aligned} \qquad 11.1$$

where s may have any one of the values 1, 2, 3, 4, 5, 6, 7. The equations of all the planes Γ have thus been found. The coordinates of all the points c can be deduced, since each point c is the intersection of a base plane Π and two planes Γ.

Now it will be observed that

$$\begin{aligned} \varepsilon^{2s}(\beta x + \gamma\varepsilon^{3s}y + \alpha\varepsilon^{s}z + \tau\varepsilon^{6s}t)^2 + \varepsilon^{s}(\gamma\varepsilon^{4s}x + \alpha y + \beta\varepsilon^{5s}z + \tau\varepsilon^{3s}t)^2 \\ + \varepsilon^{4s}(\alpha\varepsilon^{6s}x + \beta\varepsilon^{2s}y + \gamma z + \tau\varepsilon^{5s}t)^2 + 2(\varepsilon^{s}x + \varepsilon^{4s}y + \varepsilon^{2s}z + \tau^{-1}t)^2 \\ \equiv 7(\varepsilon^{2s}x^2 + \varepsilon^{s}y^2 + \varepsilon^{4s}z^2 + t^2), \end{aligned}$$

so that there is a quadric Q_{0s} with respect to which both Ω_0 and Ω_s are self-polar. This fact affords another means of determining the coordinates of the points c, for those three points c which are vertices of Ω_s are the poles with respect to Q_{0s} of the planes 11.1; their coordinates are therefore given by the rows of

$$\begin{matrix} \beta\varepsilon^{6s} & \gamma\varepsilon^{3s} & \alpha\varepsilon^{5s} & \tau \\ \gamma\varepsilon^{6s} & \alpha\varepsilon^{3s} & \beta\varepsilon^{5s} & \tau \\ \alpha\varepsilon^{6s} & \beta\varepsilon^{3s} & \gamma\varepsilon^{5s} & \tau. \end{matrix} \qquad 11.2$$

Not only do P_0 and P_s have, for their polar planes with respect to Q_{0s}, Π_0 and Π_s but the polar plane of any base point is a base plane; the polar plane of $(\varepsilon^{6u}, \varepsilon^{3u}, \varepsilon^{5u}, \tau^{-1})$, which is P_u, is

$$\varepsilon^{2s+6u}x + \varepsilon^{s+3u}y + \varepsilon^{4s+5u}z + \tau^{-1}t = 0,$$

which is Π_{2s+6u}; the suffix may of course be reduced by any multiple of 7. Moreover, since $2s + 6(2s + 6u) = 14s + 36u \equiv u$, the polar plane of P_{2s+6u} is Π_u. Thus the vertices P_u and P_{2s+6u} of Ω_u and Ω_{2s+6u} have for their polar planes with respect to Q_{0s} the faces Π_{2s+6u} and Π_u of the same two osculating tetrahedra. Indeed all four vertices of either Ω_u or Ω_{2s+6u} have for their polar planes with respect to Q_{0s} faces of the same two osculating tetrahedra; this is quickly verified by using 11.1 and 11.2; the vertices of Ω_u are found by writing u for s in 11.2, and the polars of these points with respect to Q_{0s} are precisely the planes got by writing $2s + 6u$ for s in 11.1. The osculating tetrahedra other than Ω_0 and Ω_s are therefore interchanged in pairs by reciprocation in Q_{0s}. Reciprocation in Q_{0s} thus subjects the osculating tetrahedra to a permutation; but it will be noticed that this is an odd permutation, whereas all 168 permutations in the representation of the Klein group as a permutation group of degree 8 are even.

12. The Klein group, when represented as a permutation group of degree 8, is doubly transitive[1]; hence there are collineations of **G** which transform P_0 and P_s into any specified pair P_i and P_j of the base points; these collineations transform Π_0 and Π_s into Π_i and Π_j, and Ω_0 and Ω_s into Ω_i and Ω_j. They therefore transform Q_{0s} into a quadric Q_{ij} with respect to which Ω_i and Ω_j are both self-polar; the existence of Q_{ij} could of course also have been established directly, from the identity which must connect the squares of the left-hand sides of the equations of the faces of Ω_i and Ω_j, without appealing to the transitivity of **G**. And we have the result that *any two of the eight osculating tetrahedra are self-polar for a quadric*. There thus arise twenty-eight quadrics Q_{ij}, and *the whole figure is its own reciprocal with respect to each one of them.*

The reciprocations with respect to the quadrics Q_{ij} give twenty-eight correlations of period 2 for which the figure is invariant. Now let Q be any correlation, not necessarily one of these twenty-eight or of period 2, for which the figure is invariant. Then, if C is a collineation of **G**, QC is a correlation for which the figure is invariant; there are 168 correlations of this kind. Conversely: if Q' is any correlation for which the figure is invariant $Q^{-1}Q'$ must, since it

[1] Burnside: *Theory of Groups* (Cambridge, 1911), 218.

is a collineation for which the figure is invariant, belong to $\mathbf{G}$ and Q', when written in the form $Q(Q^{-1}Q')$, is seen to be one of the above 168 correlations. Hence *the collineations of* $\mathbf{G}$ *and the correlations for which the figure is invariant together form a group* $\mathbf{H}$ *of 336 transformations.* If h is any operation, whether collineation or correlation, of $\mathbf{H}$ and C a collineation of $\mathbf{G}$, then hCh^{-1} is a collineation and so belongs to $\mathbf{G}$; thus $\mathbf{H}$ contains $\mathbf{G}$ as a self-conjugate subgroup.

Every operation of $\mathbf{H}$ permutes the tetrahedra Ω_i among themselves, and so we have a geometrical representation of $\mathbf{H}$ as a permutation group of degree 8; all permutations effected by the collineations of $\mathbf{G}$ are even, whereas the correlations which make up the other half of the operations of $\mathbf{H}$ all produce odd permutations. It is known that this permutation group contains precisely twenty-eight odd permutations of period 2, so that there cannot be any quadrics in addition to the Q_{ij} for which the polars of the eight base points are the eight base planes.[1]

13. When the figure is subjected to one of the correlations of $\mathbf{H}$ the base points are turned into the base planes and the quadrics 3.1 into the quadric envelopes linearly dependent on those appearing in 4.2. The curve K, the locus of vertices of cones which pass through all the base points, is turned into a developable $\varkappa$, the envelope of planes of conics which touch all the base planes, while the planes Γ, having five-point contact with K, become the points c through which pass five »consecutive» planes of $\varkappa$. The cuspidal edge of $\varkappa$ is of order 30, since this is the number of osculating planes of K passing through an arbitrary point; the order of the surface Σ^{16} generated by the tangents of this cuspidal edge is the same as that of the surface S^{16} generated by the tangents of K. There must be in all 96 intersections of K and Σ^{16}, and, since both K and Σ^{16} are invariant for $\mathbf{G}$, these intersections form an invariant set of points on K. But the only invariant sets of points on K are made up, apart from sets of 168 points arising from one another by the different collineations of $\mathbf{G}$, of multiples of three groups of 24, 56, 84 points corresponding respectively to the inflections, contacts of bitangents and sextactic points of k; an invariant set of 96 points can therefore only consist of the 24 points c taken four times, and Σ^{16} must have fourpoint contact with K at each point c.

[1] The permutation group is given by Burnside: *loc. cit.*, 219. This is the same as the extended congruence group discussed by Fricke: K.-F., 445, with $h = 2$. Just as $\mathbf{H}$ is obtainable by adjoining a single reciprocation to $\mathbf{G}$, so the representation as an extended congruence group is obtainable by adjoining a single inversion to a group of bilinear transformations.

II.

Geometrical Developments.

14. Since **G**, operating as a permutation group on the base points, is doubly transitive those collineations for which two given base points are both unchanged form a subgroup of **G** of order $168/(8 \times 7) = 3$; that is they form a cyclic group consisting of the identical operation and of two collineations of period 3, each the square of the other. Moreover there are also three collineations of **G** which *interchange* these two given base points; these, taken with the above three collineations, constitute the set of collineations for which the two given base points are either both invariant or are interchanged with one another, and so form a subgroup of **G** of order 6. It has the cyclic group of order 3 as a subgroup and is a dihedral group. This association of a cyclic group of order 3 and a dihedral group of order 6 with a pair of base points corresponds to the association (K. III, 104; K.-F., 707, 710) of isomorphic groups of ternary substitutions with a bitangent of k.

15. An involution (i. e. a collineation of period 2) that permutes the base points among themselves must either interchange them in four pairs or leave an even number of them unchanged. But the only collineations, other than identity, for which any two base points are unchanged are, as has just been explained, of period 3; thus every involution belonging to **G** must interchange the base points in four pairs, and of course the base planes and osculating tetrahedra correspondingly.

Involutions in space are of two kinds, central and biaxial; those lines which are invariant for a central involution consist of all the lines through its centre and all the lines in a certain plane, while those which are invariant for a biaxial involution consist of the transversals of its two axes. Since an involution belonging to **G** leaves four lines p_{ij} unchanged, and since no two of these lines can intersect because no four of the base points are coplanar, it follows that every involution belonging to **G** must be biaxial.

The number of involutions belonging to **G** is twenty-one, and they are all conjugate to one another in the group (K. III, 93; K.-F., 381). Their matrices must, since the involutions are all conjugate, all have the same trace, and an involution is biaxial or central according as its trace does or does not vanish. But the trace of **F**, which is the matrix of one of the involutions, vanishes with

$\alpha + \beta + \gamma + 1$, so that we have a second proof that all involutions of **G** are biaxial.

16. It is convenient, in order to obtain the first properties of the figure which arises by considering an involution of **G**, to be able to allude to the corresponding figure in the plane. Each of the twenty-one involutory collineations in the plane of k is a harmonic perspectivity (K. III, 102; K.-F., 709) with a point O for centre and a line w for axis. Any two corresponding points of k in the involution are collinear with O and harmonic with respect to O and the point where their join meets w. Each line through O meets k in four points consisting of two pairs of corresponding points, and there are four lines through O for which these two pairs coincide with one another — these lines being four of the bitangents of k. There pass also through O four ordinary tangents of k, namely those which touch k at its four intersections with w; these are four of the sextactic points of k and are the only points of k which are invariant for the involution.

The locus of chords which join pairs of points of K that correspond to one another in an involution I of **G** is a scroll S containing K and the two axes of I; the two intersections of a generator of S with K are harmonic with respect to the intersections of this generator with the axes. Now when the quadrics of the net, of which K is the Jacobian curve, are represented by points of the plane those which are represented by the points of a line through O are the members of a pencil which includes the quadric q represented by O. The intersections of the line with k represent the four cones of the pencil, whose vertices form a tetrahedron self-polar for q. But these vertices are four points of K which consist of two corresponding pairs of points in I, so that *the generators of S are polars of one another with respect to q*. In particular; those four bitangents of k which pass through O give rise to four chords of K which are generators of S and are their own polars with respect to q, so that they are generators also of q. No point of S, other than the points of these four common generators of S and q, can lie on q unless it lies on one of the two axes which, as transversals of the four common generators, both belong to the opposite regulus on q. The four common generators are, since they join pairs of points of K that correspond to pairs of points of contact of bitangents of k, four of the lines p_{ij}.

The order n of S is the number of its generators which meet an arbitrary line λ. Now λ is transformed by I into a line μ, and λ, μ and the two axes are on a quadric (not necessarily belonging to $\mathbf{N}$). Any generator of S which meets λ must also meet μ and so, since all generators of S meet both axes, lie on this quadric. The curve common to S and the quadric therefore consists of n generators and the two axes, so that $2n = n + 2s$ when s is the multiplicity of the axes for S. But, since the common curve of S and q consists of the axes and four of the lines p_{ij}, $2n = 4 + 2s$. It follows that *S is a quartic scroll having the two axes of I as nodal lines.* This result is due to Baker (B, 476) who obtained it by a different method.

It has been observed that the generators of S can be paired so that those of a pair are opposite edges of a tetrahedron whose vertices form a canonical set on K; such a pair of generators is a pair of conjugate chords of K in the sense of Note I, p. 310. It follows that S is, in the sense of Note I, a self-conjugate surface. The knowledge of this fact facilitates the task of finding the equation of S; since the scroll also contains all the base points its equation has for its left-hand side (Note I, 308, footnote) a skew bilinear form in the two sets of variables x, y, z, t and X, Y, Z, T.

17. Four of the lines p_{ij} are invariant for I, and the point which is paired by I with a point of any one of these four lines is a point of the same line; I induces an ordinary involution of pairs of points on each of the four lines. Let p_{ab} be one of the four lines; two of the pairs of the involution on p_{ab} are at once identified, namely the pair of intersections of p_{ab} with K and the pair of base points P_a and P_b. The pair which is harmonic to both these pairs constitutes the double points of the involution, and is thus the pair of points where the axes of I meet p_{ab}. Now it will be observed that these intersections have been identified quite independently of which other three of the twenty-eight lines p_{ij} are invariant for I, and it will be remembered that there are three different involutions of $\mathbf{G}$, belonging to a dihedral group of order 6, which interchange P_a and P_b. It must follow that through any point where an axis of an involution of $\mathbf{G}$ meets a line p_{ij} axes of three involutions pass.

Perhaps an example of a dihedral group, with its three involutions, may appropriately be given here. The permutation $\mathbf{f}$ of the base points, as has been mentioned in § 8, is (07) (16) (23) (45), and it is at once verified that $\mathbf{e}^4\mathbf{f}\mathbf{e}^4$ is the permutation (043) (257). It therefore follows that those collineations of $\mathbf{G}$ which either interchange P_1 and P_6 or leave them both unchanged subject the base points to the following permutations in addition to the identical permutation:

$$(043)\,(257) \qquad\qquad (034)\,(275)$$
$$(07)\,(16)\,(23)\,(45) \qquad (16)\,(02)\,(35)\,(47) \qquad (16)\,(05)\,(24)\,(37).$$

The last three permutations arise from those involutions which interchange P_1 and P_6, and the axes of each of these three involutions must meet p_{16} in the same two points.

18. The discussion has so far been concerned with the effect of an involution I on the base points; it is time to recollect that I has the corresponding effect on the base planes and to enquire what further information can be obtained thereby.

The axes of I, then, are nodal lines not only of a quartic scroll S generated by chords of the curve K but also of a quartic scroll Σ generated by axes of the developable $\varkappa$; the axes of I are themselves also axes of $\varkappa$. If p_{ab} is a generator of S then π_{ab} is a generator of Σ, and the generators of Σ are polars of one another with respect to a quadric which touches all the base planes and meets Σ in its two nodal lines and in the four lines π_{ij} which lie upon it. And the plane which joins an axis of I to a line π_{ab} which meets it is the same for each of the three involutions that interchange Π_a and Π_b and so contains an axis of each of the three involutions. The two planes so arising through π_{ab} are harmonic both to Π_a and Π_b and to the two planes of $\varkappa$ that pass through π_{ab}.

The involutions which interchange Π_a and Π_b are of course the same as those which interchange P_a and P_b, so that their three pairs of axes give two triads of lines such that the lines of either triad not only concur in a point of p_{ab} but also lie in a plane through π_{ab}. We therefore can say that

The forty-two axes of the involutions belonging to G *consist of fifty-six triads of lines, the lines of each triad being both concurrent and coplanar. Each axis belongs to four of the triads.*

19. Two quartic scrolls S and Σ which arise from the same involution have the same nodal lines, and therefore meet residually in eight common generators. Since there are twenty-one involutions there thus appear 168 lines which are both chords of K and axes of $\varkappa$; the axes of the involutions also have this double property, so that $168 + 42 = 210$ such lines are accounted for. And there are no others. For the chords of K form a congruence of order 7 and class 15

while the axes of $\varkappa$ form a congruence of order 15 and class 7, so that Halphen's formula tells us that the number of lines common to the two congruences is $7.15 + 15.7$, which is 210.

20. Suppose now that one axis of an involution I meets K in R_1 and R_1' while the companion axis meets K in R_2 and R_2'. These four points correspond to collinear points of k and so constitute a canonical set; it follows that the three further intersections of any face of the tetrahedron $R_1 R_1' R_2 R_2'$ with K lie on the trisecant which is conjugate to the opposite vertex, the trisecant t_1 conjugate to R_1 lying in the plane $R_1' R_2 R_2'$, and so on. Now R_1 is invariant for I, so that t_1 must also be invariant and hence meet both $R_1 R_1'$ and $R_2 R_2'$; thus t_1 passes through R_1', which therefore counts twice among the six intersections of the plane $R_1' R_2 R_2'$ with K. This in agreement with the fact that the tangent of K at R_1' is invariant for I and so meets $R_2 R_2'$. We thus see that the plane which joins the tangent at R, an intersection of K with either axis of any involution belonging to $\mathbf{G}$, to the companion axis meets K in two further points collinear with R and harmonic with respect to R and the intersection of their join with the companion axis.

The osculating plane of K at R_1 is invariant for I, so that it passes either through $R_1 R_1'$ or through $R_2 R_2'$. But it has just been explained that the plane $R_1 R_2 R_2'$, which contains the tangent of K at R_1, also contains one of the trisecants through R_1; it cannot therefore be the osculating plane at R_1 unless the trisecant and tangent coincide which, since R_1 is not one of the points c, they do not. Hence the osculating plane at R_1 contains $R_1 R_1'$. Similarly the osculating plane at R_1' also contains $R_1 R_1'$. The chord $R_1 R_1'$ is therefore the intersection of the osculating planes of K at R_1 and R_1', and so a *principal chord* of K. The same argument applies to either axis of any involution belonging to $\mathbf{G}$: *the axes of the involutions are forty-two of the principal chords of K.* They are also, by dual reasoning, principal axes of $\varkappa$.

21. If two biaxial involutions are permutable the axes of either are interchanged by or else both invariant for the other. In the second event the involutions are such that both axes of either meet both axes of the other, so forming two pairs of opposite edges of a tetrahedron; their product is then the third involution whose axes are the remaining pair of opposite edges. If this were to occur for three of the involutions of $\mathbf{G}$ the vertices of the tetrahedron would be common to three quadrics of $\mathbf{N}$; for it has been explained that the

two axes of an involution of **G** lie (together with four of the lines p_{ij}) on such a quadric. If these three quadrics were to belong to a pencil there would correspond to them three points O, centres of harmonic perspectivities for which k is invariant, that are collinear; but it is known (K.-F., 712) that the centres of any three such transformations that are permutable are not collinear. The three quadrics cannot therefore belong to a pencil and any points common to them must be among the base points. But no base point lies on an axis of an involution of **G**, so that the assumption that the axes of two permutable involutions are edges of the same tetrahedron must be false.

It must be then that when two involutions of **G** are permutable each interchanges the axes of the other; the four axes must belong to a regulus and constitute two harmonic pairs thereof. Now the product of the two permutable involutions is a third involution permutable with both of them, and the three involutions constitute, together with identity, a 4-group. The axes of the third involution must belong to the same regulus and be harmonic to each of the pairs of axes of the other two; the three pairs of axes of the involutions of the 4-group are thus all on the same quadric Ψ and constitute a *regular sextuple*, each pair being harmonic, in a regulus, to both the others. The twelve intersections of Ψ with K lie two on each of the six axes which lie on Ψ, and so are all of them points a corresponding to sextactic points of k.

It is known (K. III, 94; K.-F., 383, 712) that there are fourteen 4-groups belonging to **G** and falling into two conjugate sets of seven; any involution belongs to one 4-group of each set. There are thus fourteen quadrics Ψ falling into two sets of seven, the axes of an involution of **G** lying on one quadric of each set. The two quadrics Ψ, Ψ' containing the axes of an involution I have in common two lines of the opposite regulus *which meet ten axes in all*, for they must meet both axes of any involution belonging to either of the 4-groups containing I as well as the axes of I itself. The seven quadrics of either set constitute a composite surface of order 14 which cuts out on K the whole set of a-points in which the axes of the twenty-one involutions meet it. Let the quadrics of one set be denoted by the symbol Ψ and those of the other set by the symbol Ψ'. Since each quadric Ψ or Ψ' contains three pairs of axes, and since each pair of axes lies on one quadric of each system, with each quadric Ψ there is associated a triplet of quadrics Ψ' while with each quadric Ψ' is associated a triplet of quadrics Ψ; there is a (3,3) correspondence between the quadrics Ψ and the quadrics Ψ'. Thus not only are the seven quadrics of each set permuted among themselves by the collineations of $\mathbf{G}$ but so are the seven triplets of quadrics of each set. This feature of the Klein group, when regarded as a permutation group of degree 7, of permuting not only seven objects but also, at the same time and by the same operations, seven triplets of these objects, was signalised by Nöther.[1]

22. Each involution of $\mathbf{G}$ interchanges the base points and the base planes in pairs. Since the product of two different involutions which interchange the same pair is a collineation, other than the identical collineation, which leaves each member of the pair unchanged this product must be of period 3; it follows that no two pairs that are interchanged by involutions belonging to the same 4-group can be identical. Now if an involution I_2 transposes the suffixes a, b while an involution I_3 transposes a, c the collineation $I_2 I_3$ transforms c into b; if, then, I_2 and I_3 are permutable the collineation $I_3 I_2$ must also transform c into b so that if d is the suffix which I_2 transposes with c the same suffix d must be the one which I_3 transposes with b. Thus I_2 induces, in addition to two further transpositions, the transpositions $(a\,b)\,(c\,d)$, I_3 induces $(a\,c)\,(b\,d)$, while $I_2 I_3 = I_1 = I_3 I_2$ induces $(b\,c)\,(a\,d)$. The remaining four suffixes are subjected to similar sets of transpositions by the three involutions. The six axes of I_1, I_2, I_3 lie on a quadric, Ψ say.

Consider now the tetrahedron formed by the base planes Π_a, Π_b, Π_c, Π_d. Take any one of the six edges of this tetrahedron, say π_{bc}. Since Π_b and Π_c are transposed by I_1, π_{bc} is invariant for I_1 and meets both its axes; moreover Π_a and Π_d are also transposed by I_1 so that the two vertices joined by π_{bc} are harmonic to the pair of points in which their join meets the axes of I_1 and so are conjugate points for Ψ. The same argument is applicable to every edge of the tetrahedron, any two vertices of which are therefore conjugate points for Ψ; the tetrahedron is therefore self-polar for Ψ. The same result holds for the tetrahedron formed by the other four base planes, which undergo corresponding transpositions when subjected to the 4-group of collineations. So that we have

> To each of the fourteen quadrics Ψ there corresponds a division of the base planes into two sets of four, and the tetrahedra formed thereby are both self-polar for Ψ. The same result holds for the two tetrahedra whose vertices are the two corresponding sets of base points.

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[1] *Math. Annalen* 15 (1879), 94.

There thus arise four tetrahedra self-polar for Ψ, two of them with base planes for faces and two of them with base points for vertices. And each of the fourteen quadrics Ψ or Ψ' is both inpolar to all the quadrics 3.1 as well outpolar to all the quadrics 4.2.

There are thirty-five different ways of dividing the base planes into complementary sets of four; the two tetrahedra so arising are self-polar for the same quadric whose point equation is obtainable forthwith from 4.1 by performing the corresponding division, into complementary sets of four, of the eight terms appearing there. And so, when the effect of the involutions of a 4-group of $\mathbf{G}$ is known, the point equation of the quadric Ψ associated therewith can be found quickly. The plane equation is similarly obtainable from the identity which connects the squares of the equations of the eight base points, which is

$$\sum_{s=1}^{7}(\varepsilon^{6s}u + \varepsilon^{3s}v + \varepsilon^{5s}w + \tau^{-1}p)^2 - \frac{7}{2}p^2 \equiv 0.$$

The point equation and plane equation of any quadric Ψ are obtainable from one another by interchanging point and plane coordinates and, at the same time, replacing ε by its reciprocal, ε^6.

23. As examples, the two 4-groups which contain the involution $I(\mathbf{F})$ associated with the permutation $\mathbf{f}$ may be given. It is found, on referring to § 8, that

$$\mathbf{f} = (07)(23)(16)(45), \quad \mathbf{e}^6\mathbf{f}\mathbf{e}^2\mathbf{f}\mathbf{e}^2 = (02)(37)(14)(56), \quad \mathbf{e}\mathbf{f}\mathbf{e}^3\mathbf{f}\mathbf{e}^2 = (03)(27)(15)(46).$$

These constitute, with the identical permutation, a 4-group, and the quadric which contains the six axes of the associated involutions has, for self-polar tetrahedra, the four arising from the division 0237/1456. The point equation of this quadric is therefore, from 4.1,

$$(\varepsilon^2 x + \varepsilon y + \varepsilon^4 z + \tau^{-1} t)^2 + (\varepsilon^3 x + \varepsilon^5 y + \varepsilon^6 z + \tau^{-1} t)^2$$
$$= \frac{7}{2}t^2 - (x + y + z + \tau^{-1} t)^2. \qquad 23.1$$

The other 4-group containing $\mathbf{f}$ consists of identity and the three permutations

$$\mathbf{f} = (07)(45)(16)(23), \quad \mathbf{e}\mathbf{f}\mathbf{e}^5\mathbf{f}\mathbf{e}^5 = (05)(47)(12)(36), \quad \mathbf{e}^2\mathbf{f}\mathbf{e}^3\mathbf{f}\mathbf{e} = (04)(57)(13)(26).$$

The quadric which contains the six axes of the associated involutions is now associated with the division 0457/1236, and so has the equation

$$(\varepsilon^4 x + \varepsilon^2 y + \varepsilon z + \tau^{-1} t)^2 + (\varepsilon^5 x + \varepsilon^6 y + \varepsilon^3 z + \tau^{-1} t)^2$$
$$= \frac{7}{2} t^2 - (x + y + z + \tau^{-1} t)^2. \quad 23.2$$

The two axes of $I(\mathbf{F})$ must lie on both the above quadrics.

These two axes, incidentally, also lie on the quadric which contains the four lines p_{07}, p_{16}, p_{23}, p_{45}. This quadric has an equation of the form 3.1, and the coefficients are easily determined, for example from the conjugacy of P_0 and P_7 and of three other pairs of base points; the equation is

$$(\beta - \gamma)(y^2 - \tau t x) + (\gamma - \alpha)(x^2 - \tau t z) + (\alpha - \beta)(z^2 - \tau t y) = 0.$$

The equations 23.1 and 23.2 are the *point equations* of those quadrics Ψ and Ψ' which contain the two axes of $I(\mathbf{F})$; the plane equations are deducible immediately. From these equations it is quickly verified that the two quadrics are both inpolar and outpolar to one another, and that each of them is its own polar reciprocal with respect to the other. And these geometrical relations hold between any one of the fourteen quadrics Ψ or Ψ' and each member of its associated triplet.

24. Much more could be written about these fourteen quadrics, and while an exhaustive account of their properties would be inordinately long it is important to mention some of their more fundamental relations to the lines of the figure. It would indeed be wrong to omit all mention of those features of the three-dimensional figure that are related to the formation of equations of the seventh degree whose Galois group is of order 168. It will be found that the subjoined tables are useful. Herein the operations of period 2 of $\mathbf{G}$, regarded as permutations of eight objects, are exhibited in commutative triads so that each horizontal layer of each table corresponds to a 4-group; the seven 4-groups for each table constitute a conjugate set. In the first column the permutations are given in terms of the two generating permutations $\mathbf{e}$ and $\mathbf{f}$; the second column shows their effect on the base points, base planes or osculating tetrahedra. In the third column appears that division, of the base points or base planes, into two sets of four which gives tetrahedra self-polar for that quadric Ψ or Ψ' on which lie the six axes of the three involutions of the 4-group; this division is instantly read off from the three permutations of the eight digits. Below this

division of eight digits into two tetrads appears another division into four duads; this requires for its explanation, which is given subsequent to the tables, a reference to the associated triplet of quadrics.

When $\mathbf{e}$ and $\mathbf{f}$ satisfy the relations 8.1 the operation $\mathbf{e}^{p-1}\mathbf{f}\mathbf{e}^{1-p}$, where p is any integer, is of period 2. Of the two 4-groups which include this and the identical permutation one arises by taking the two permutations

$$\mathbf{e}^{p+5}\mathbf{f}\mathbf{e}^{2}\mathbf{f}\mathbf{e}^{3-p} \text{ and } \mathbf{e}^{p}\mathbf{f}\mathbf{e}^{3}\mathbf{f}\mathbf{e}^{3-p}$$

while the other arises by taking the two permutations

$$\mathbf{e}^{p+1}\mathbf{f}\mathbf{e}^{2}\mathbf{f}\mathbf{e}^{-p} \text{ and } \mathbf{e}^{p+1}\mathbf{f}\mathbf{e}^{3}\mathbf{f}\mathbf{e}^{2p}.$$

$\mathbf{f}$	(07) (23) (16) (45)	0237/1456 04/12/36/57
$\mathbf{e^6fe^2fe^2}$	(02) (37) (14) (56)	
$\mathbf{efe^3fe^2}$	(03) (27) (15) (46)	
$\mathbf{efe^6}$	(01) (34) (27) (56)	0134/2567 05/23/16/47
$\mathbf{fe^2fe}$	(03) (14) (25) (67)	
$\mathbf{e^2fe^3fe}$	(04) (13) (26) (57)	
$\mathbf{e^2fe^5}$	(02) (45) (13) (67)	0245/1367 06/15/27/34
$\mathbf{efe^2f}$	(04) (25) (17) (36)	
$\mathbf{e^3fe^3f}$	(05) (24) (16) (37)	
$\mathbf{e^3fe^4}$	(03) (56) (17) (24)	0356/1247 07/13/26/45
$\mathbf{e^2fe^2fe^6}$	(05) (36) (12) (47)	
$\mathbf{e^4fe^3fe^6}$	(06) (35) (14) (27)	
$\mathbf{e^4fe^3}$	(04) (67) (12) (35)	0467/1235 01/24/37/56
$\mathbf{e^3fe^2fe^5}$	(06) (47) (15) (23)	
$\mathbf{e^5fe^3fe^5}$	(07) (46) (13) (25)	
$\mathbf{e^5fe^2}$	(05) (17) (23) (46)	0157/2346 02/14/35/67
$\mathbf{e^4fe^2fe^4}$	(07) (15) (26) (34)	
$\mathbf{e^6fe^3fe^4}$	(01) (57) (24) (36)	
$\mathbf{e^6fe}$	(06) (12) (34) (57)	0126/3457 03/17/25/46
$\mathbf{e^5fe^2fe^3}$	(01) (26) (37) (45)	
$\mathbf{fe^3fe^3}$	(02) (16) (35) (47)	

$\mathbf{f}$	(07) (45) (16) (23)	0457/1236 03/14/27/56
$\mathbf{e^2fe^2fe^6}$	(05) (47) (12) (36)	
$\mathbf{e^2fe^3fe}$	(04) (57) (13) (26)	
$\mathbf{efe^6}$	(01) (56) (27) (34)	0156/2347 04/13/25/67
$\mathbf{e^3fe^2fe^5}$	(06) (15) (23) (47)	
$\mathbf{e^3fe^3f}$	(05) (16) (24) (37)	
$\mathbf{e^2fe^5}$	(02) (67) (13) (45)	0267/1345 05/17/24/36
$\mathbf{e^4fe^2fe^4}$	(07) (26) (15) (34)	
$\mathbf{e^4fe^3fe^6}$	(06) (27) (14) (35)	
$\mathbf{e^3fe^4}$	(03) (17) (24) (56)	0137/2456 06/12/35/47
$\mathbf{e^5fe^2fe^3}$	(01) (37) (26) (45)	
$\mathbf{e^5fe^3fe^5}$	(07) (13) (25) (46)	
$\mathbf{e^4fe^3}$	(04) (12) (35) (67)	0124/3567 07/23/15/46
$\mathbf{e^6fe^2fe^2}$	(02) (14) (37) (56)	
$\mathbf{e^6fe^3fe^4}$	(01) (24) (36) (57)	
$\mathbf{e^5fe^2}$	(05) (23) (17) (46)	0235/1467 01/26/34/57
$\mathbf{fe^2fe}$	(03) (25) (14) (67)	
$\mathbf{fe^3fe^3}$	(02) (35) (16) (47)	
$\mathbf{e^6fe}$	(06) (34) (12) (57)	0346/1257 02/16/37/45
$\mathbf{efe^2f}$	(04) (36) (17) (25)	
$\mathbf{efe^3fe^2}$	(03) (46) (15) (27)	

The fourteen 4-groups which belong to $\mathbf{G}$ are all accounted for by giving to p the values 1, 2, 3, 4, 5, 6, 7. The tables can now be compiled forthwith and are as shown.

The quadrics Ψ and Ψ' which contain the axes of the involution $I(\mathbf{E}^{p-1}\mathbf{F}\mathbf{E}^{1-p})$ will be denoted by Ψ_p and Ψ'_p; the quadrics Ψ_p correspond to the 4-groups of the first table and the quadrics Ψ'_p to those of the second table.

The divisions, in the last columns of the tables, of the eight digits into four duads are now easily explained. Consider the three involutions in the top layer of the first table; their axes lie on Ψ_1 and the second table shows that they also lie one pair on each of Ψ'_1, Ψ'_5, Ψ'_7 which therefore constitute the triplet of quadrics Ψ' associated with Ψ_1. The three divisions into two tetrads that arise for the quadrics Ψ' of this triplet show that the four lines

$$p_{04}, \quad p_{12}, \quad p_{36}, \quad p_{57}$$

have the property that when they are divided into two pairs the lines of each pair are polars of one another with respect to a quadric of the triplet, the three quadrics of the triplet corresponding to the three divisions of the four lines into two pairs. And the same is true of the four lines

$$\pi_{04}, \quad \pi_{12}, \quad \pi_{36}, \quad \pi_{57}.$$

This explains the division into four duads that appears in the first layer of the first table. The thirteen similar divisions admit similar explanations.

The division is also associated with another property that will be met later when those operations of $\mathbf{G}$ which are of period 3 are considered, namely that each of the four pairs of lines

$$\pi_{04}, p_{04}; \quad \pi_{12}, p_{12}; \quad \pi_{36}, p_{36}; \quad \pi_{57}, p_{57}$$

is a pair of polar lines for Ψ_1.

25. Before we proceed to deduce further geometrical results another important aspect of these divisions into four duads must be signalised. It is known (K. III, 95; K.-F., 384) that each of the fourteen 4-groups is self-conjugate in one octahedral group of $\mathbf{G}$; *the four lines p that correspond to a division into four duads undergo all possible permutations when subjected to the corresponding octahedral group of collineations*, while the same of course holds for the four lines π. Such a set of four lines may be called an octahedral set (of lines p or of lines π). The lines p of an octahedral set, since they join the eight base points

in pairs, correspond to a set of four bitangents of k whose points of contact lie on a conic; that each octahedral subgroup of the Klein group of ternary collineations has associated with it such a set of four bitangents on which it imposes all possible permutations is well known (K. III, 103; K.-F., 712). The conics through the contacts of these octahedral sets of bitangents were first obtained by Klein (K. III, 106, 108) and then figured in a paper by Gordan[1]. Since the four bitangents of an octahedral set are not concurrent the four lines p of an octahedral set do not belong to a regulus and so have only two transversal lines. The same is true of an octahedral set of lines π.

26. Suppose now that two pairs of lines l, l' and m, m' both consist of polar lines with respect to any non-singular quadric surface q. Then, if the four lines do not belong to a regulus, their two transversals n, n' are also polar lines for q. Of several short proofs of this result we select one which uses the representation of the lines of three-dimensional space by points of a quadric Ω in five-dimensional space since we shall presently find it convenient to use this representation for another purpose. The two reguli on q are represented by the conics γ_1 and γ_2 in which Ω is met by two planes ϖ_1 and ϖ_2 that are polars of each other; the lines l, l', m, m' are represented by points L, L', M, M' such that LL' meets both ϖ_1 and ϖ_2, say in L_1 and L_2 (not points of Ω) while MM' meets them in M_1 and M_2. The transversals n, n' are represented by the two intersections N, N' of Ω with that line which is the polar of the space $LL'MM'$: this space is $L_1L_2M_1M_2$ and, since ϖ_1 and ϖ_2 are polars of one another, its polar line joins the pole N_1 of L_1M_1 with respect to γ_1 to the pole N_2 of L_2M_2 with respect to γ_2. And therefore, since NN' meets both ϖ_1 and ϖ_2, n and n' are polar lines with respect to q.

It follows that the two transversals of an octahedral set of lines are polar lines with respect to each of the three quadrics of a triplet; for example, the transversals of $p_{04}, p_{12}, p_{36}, p_{57}$ are polar lines for each of $\Psi'_1, \Psi'_5, \Psi'_7$, as also are the transversals of $\pi_{04}, \pi_{12}, \pi_{36}, \pi_{57}$.

Denote, for the moment, the two axes of $I(\mathbf{F})$ by x and x'; they lie on Ψ_1 and on Ψ'_1.

Since p_{04} and p_{57} are interchanged by $I(\mathbf{F})$ they belong to a regulus with x and x'; for the same reason p_{12} and p_{36} belong also to a regulus with x and x'. The quadric surfaces on which these reguli lie are, since an octahedral set

[1] *Math. Annalen* 20 (1882), 515—530.

of lines does not belong to a regulus, distinct and so have in common, in addition to x and x', two lines t and t'; these are therefore the two transversals of the octahedral set p_{04}, p_{57}, p_{12}, p_{36} and, since they meet x and x', are both invariant for $I(\mathbf{F})$. Similar reasoning shows that t and t' also meet the axes of those other two involutions whose axes lie on Ψ_1. Hence t, t' themselves must lie on Ψ_1 and belong to ϱ_1, the regulus complementary to that which contains the axes of the three involutions.

Each quadric Ψ' of the triplet associated with Ψ_1 contains two lines of ϱ_1 which are harmonically conjugate in ϱ_1 to t and t', for, as has been shown, t and t' are polar lines for each quadric of the triplet. Hence we have:

> Any one of the fourteen quadrics Ψ or Ψ' has among the lines of one of its reguli axes of three permutable involutions of $\mathbf{G}$. Through each of the three pairs of axes passes a quadric of the associated triplet and this quadric contains two lines of the complementary regulus ϱ. *The three pairs of lines so arising have a common harmonic pair and so belong to an involution in* ϱ.

The double elements of this involution were obtained as the transversals of an octahedral set of lines p. But the corresponding octahedral set of lines π clearly yields, through the same associated triplet of quadrics, the same involution and so *two corresponding octahedral sets of lines* p *and lines* π *constitute eight lines with two common transversals.*

27. Yet another connection can now be established with Klein's work. When Klein (K. II, 413) put to himself the question of finding the simplest constructs that only admit *seven* different positions when subjected to the collineations of $\mathbf{G}$ he answered it by giving two sets of linear complexes. A second answer has now been provided by obtaining the quadrics Ψ and Ψ', and it may be submitted that these too have Klein's desired criterion of simplicity. The quadrics and the linear complexes are intimately related, and this relation will now be set forth.

Let us use again the representation of lines of three-dimensional space by points of Ω; any quadric q has two complementary reguli represented on Ω by those conics γ_1 and γ_2 in which it is met by two planes ϖ_1 and ϖ_2 that are polars of one another. Suppose now that any two lines α and β, represented by points A and B of γ_1, are chosen in one of the reguli: what conics of Ω through A and B represent reguli on quadrics q' that are self-reciprocal with respect to q? (It will be remembered that quadrics Ψ and Ψ' are self-reciprocal with respect to one another when each of them is a member of the triplet associated with the other.) The lines other than α and β of such a regulus are polars of one another with respect to q, and any such pair of lines is harmonic in the regulus to α and β. If then L is any point of an eligible conic γ' through A and B the polar line with respect to q of the line which is represented by L must be represented by the other intersection M of γ' with the line which joins L to T, the intersection of the tangents of γ' at A and B. Since L and M represent lines that are polars of one another with respect to q the tangent primes of Ω at L and M must intersect γ_1 in the same pair of points, and since the line joining these, being the polar of the intersection of LM and AB with respect to γ_1, is not AB, the polar prime of T, which certainly contains AB, must contain the whole of ϖ_1. Wherefore T must lie in ϖ_2. The locus of the eligible conics γ' is the section of Ω by the prime which joins ϖ_2 to AB.

Suppose now that a linear complex Δ contains reguli r and r' which, having two common lines, lie on q and q' respectively. The prime section of Ω which represents Δ must contain, in the above notation, both γ_1 and γ' so that the pole of this prime must lie in ϖ_2 on the polar of T with respect to γ_2. The two points of γ_2 which lie on this polar represent the two lines common to those reguli on q and q' that are complementary to r and r'. If then a single linear complex Δ contains not only r but also three further reguli r'_1, r'_2, r'_3 lying respectively on quadrics q'_1, q'_2, q'_3 that are all self-reciprocal with respect to q (each of r'_1, r'_2, r'_3 containing two lines of r) it must be that those three lines in ϖ_2, which join the pairs of points of γ_2 representing the pairs of lines of the regulus ϱ complementary to r that lie respectively on q'_1, q'_2, q'_3, are concurrent; or that the three pairs of lines belong to an involution in ϱ. And the converse is true also. Now this, as has been shown, is precisely what happens for any quadric Ψ or Ψ' and its associated triplet, and so

> On any one of the quadrics Ψ_p is a regulus r_p containing the axes of three permutable involutions of $\mathfrak{G}$. Each pair of axes belongs also to a regulus on a quadric of the associated triplet, and these three reguli belong to a linear complex Δ_p which also contains r_p. And from each quadric Ψ'_p there arises similarly a linear complex Δ'_p.

This describes how to obtain the linear complexes from the quadrics. As with the quadrics, so with the linear complexes: they form two sets of seven and associated with each complex of either set is a triplet of complexes of the other set. The quadrics are obtained from the linear complexes very simply: given one of the complexes the corresponding quadric is that which contains the regulus which is common to the three complexes of the associated triplet.

The fourteen complexes Λ and Λ' are Klein's *linear complexes.* Since equations for all the quadrics Ψ and Ψ' are, as remarked in § 22, immediately available there is no hindrance to the verification of this statement. But space will be saved here if the verification is suppressed, and perhaps countenance will also be given to the suppression as some retaliation for the tantalising withholding (cf. K. II, 413) by Klein of the geometrical discussions by which he obtained the linear complexes.

Since each linear complex contains a regulus on the corresponding quadric, and since this regulus also belongs to each of the three associated linear complexes, each of the fourteen complexes must be connected with the members of its associated triplet by a linear identity. These identities are not mentioned by Klein, but they are immediately derivable from equations (23) on p. 414 of K. II. The corresponding identities between conics which pass through the contacts of octahedral sets of bitangents of k are given by Gordan (*loc. cit.* 520).

28. Nothing has yet been said about equations for the axes, or about coordinates of points which lie upon them. It is however quite easy to approach these matters with the help of the definition, given in § 17, of the pair of intersections of p_{ij} with axes as the common harmonic pair of two known pairs of points on p_{ij}. For it follows, from the remarks at the end of § 3, that the point $(\varepsilon^{6i} + \lambda\varepsilon^{6j},\ \varepsilon^{3i} + \lambda\varepsilon^{3j},\ \varepsilon^{5i} + \lambda\varepsilon^{5j},\ \tau^{-1} + \tau^{-1}\lambda)$ lies on an axis when λ is a root of $\lambda^2 + 1 = 0$ and that $(\varepsilon^{6s}, \varepsilon^{3s}, \varepsilon^{5s}, \tau^{-1} + \mu)$ lies on an axis when μ is a root of $2\mu^2 - 7 = 0$; here i and j are any two different digits, and s any one digit, among 1, 2, 3, 4, 5, 6, 7. The fifty-six intersections of the axes with the lines p_{ij} are thus identified, and the fifty-six planes which join the axes to the lines π_{ij} are identified just as simply.

We are, for example, led to the following points on the axes of $I(\mathbf{F})$:

on p_{07};

$$(1,\ 1,\ 1,\ \tau^{-1} + \sigma\tau^{-1}) \quad \text{and} \quad (1,\ 1,\ 1,\ \tau^{-1} - \sigma\tau^{-1}).$$

on p_{16};

$$(\varepsilon + i\varepsilon^6,\ \varepsilon^4 + i\varepsilon^3,\ \varepsilon^2 + i\varepsilon^5,\ \tau^{-1} + i\tau^{-1}) \text{ and } (\varepsilon - i\varepsilon^6,\ \varepsilon^4 - i\varepsilon^3,\ \varepsilon^2 - i\varepsilon^5,\ \tau^{-1} - i\tau^{-1}).$$

on p_{23};

$$(\varepsilon^4 + i\varepsilon^5,\ \varepsilon^2 + i\varepsilon^6,\ \varepsilon + i\varepsilon^3,\ \tau^{-1} + i\tau^{-1}) \text{ and } (\varepsilon^4 - i\varepsilon^5,\ \varepsilon^2 - i\varepsilon^6,\ \varepsilon - i\varepsilon^3,\ \tau^{-1} - i\tau^{-1}).$$

on p_{45};

$(\varepsilon^2 + i\varepsilon^3,\ \varepsilon + i\varepsilon^5,\ \varepsilon^4 + i\varepsilon^6,\ \tau^{-1} + i\tau^{-1})$ and $(\varepsilon^2 - i\varepsilon^3,\ \varepsilon - i\varepsilon^5,\ \varepsilon^4 - i\varepsilon^6,\ \tau^{-1} - i\tau^{-1})$.

The four points written on the left are collinear, lying on one axis of $I(\mathbf{F})$, while the four points on the right are also collinear and lie on the companion axis.

29. It is known (K. III, 93; K.-F., 381) that $\mathbf{G}$ contains forty-two operations of period 4. Any one of them, say J, generates a cyclic subgroup of $\mathbf{G}$, the square of J, which is the same as the square of its inverse J^{-1} or J^3, being one of the involutions. If the square of J is I any united point of J is also a united point of I and so must lie on an axis of I. It cannot be that every point of an axis is a united point of J, for each axis is a chord of K and it is known (K. III, 98; K.-F., 381—2) that J has no united point on K; thus each axis contains the two united points of the projectivity induced thereupon by J which, since every point of the axis is invariant for $I = J^2$, is of period 2 and so an ordinary involution.

The united points of J are thus the vertices of a tetrahedron, which will be called a *fundamental tetrahedron* and denoted by T, two of whose opposite edges are axes. Since each point of either axis is transformed by J into its harmonic conjugate with respect to the two united points on this axis the position of the united points is known as soon as two pairs of corresponding points on the axis are known. But one such pair, R_1, R_1', consists of the intersections of the axis with K while a second such pair, S_1, S_1', consists of the intersections of the axis with those trisecants of K that are conjugate to the points R_2, R_2' in which K meets the companion axis; these two pairs are sufficient to determine two vertices V_1, V_1' of T and the vertices V_2, V_2' on the companion axis are determined similarly. Other facts are available for finding these vertices, or for verifying their positions when already found. There are, for instance, four lines p_{ij} which meet both axes of I; these are interchanged in pairs by J and the vertices of T on either axis must be harmonic to its intersections with either of the two pairs. And there is an analogous statement involving the four corresponding lines π_{ij}.

Those edges of T which are not axes will be called *transversals*. Each point on a transversal belongs to a unique set of four points, obtained by subjecting it repeatedly to the collineation J, on the transversal; this set is invariant for J, is linearly dependent on two particular sets consisting of a vertex of T counted four times, and so is harmonic. Moreover, since $J^2 = I$, each set consists of two pairs both harmonic to the pair of vertices of T and so constitutes with these vertices a regular sextuple. These remarks have some significance when surfaces are considered which are invariant for the collineations of $\mathbf{G}$.

30. Suppose now that C is one of the two collineations of period 3 for which both P_i and P_j are invariant. The assumption that a third base point is also invariant for C implies that at least five of the base points are invariant, and so that C is the identical collineation which it is assumed not to be. The remaining six base points are therefore permuted by C in two cycles of three: say $(P_a P_b P_c)(P_d P_e P_f)$. There are then at least four united points of C on p_{ij}, namely P_i, P_j and the intersections of p_{ij} with the two planes $P_a P_b P_c$ and $P_d P_e P_f$; it follows that *every individual point of p_{ij} is a united point of C.* If a line is invariant for C but yet is not such that every point on it is a united point then it joins two united points of C, these being the united points of the projectivity induced on the line by C. This projectivity is of period 3, and each point of the line belongs to a unique triad of points that arises from any one of its members by repeated applications of C; these triads all have the pair of united points as their Hessian duad and are linearly dependent on the two particular triads which consist of a united point taken three times. Such a line is that common to the planes $P_a P_b P_c$ and $P_d P_e P_f$; one triad of points thereon consists of the intersections with p_{bc}, p_{ca}, p_{ab} and a second of the intersections with p_{ef}, p_{fd}, p_{de}; the two united points U_{ij} and U'_{ij} are found as the Hessian duad of either of these triads. The *fundamental spaces* of C, to use the standard nomenclature[1], are p_{ij}, U_{ij}, U'_{ij}; the united points of C are U_{ij}, U'_{ij} and every point of p_{ij} while the united planes of C are $U_{ij} p_{ij}$, $U'_{ij} p_{ij}$ and every plane through $U_{ij} U'_{ij}$. Now among the united planes of C are Π_i and Π_j, neither of which passes through p_{ij}; it follows that the line $U_{ij} U'_{ij}$ must be π_{ij}. Reciprocally it appears that p_{ij} joins the point of intersection of Π_a, Π_b, Π_c to the point of intersection of Π_d, Π_e, Π_f. And we have

> If any two base points, say P_i and P_j, are chosen the remaining six are thereby divided into two triads and the line common to the planes of these triads is π_{ij}. Dually, if Π_i and Π_j are selected the remaining six base planes are thereby divided into two triads, and the point of intersection of either of these triads is on p_{ij}.

[1] BERTINI: *Geometria proiettiva degli iperspazi* (Messina, 1923), 76.

Thus every line π_{ij} is met by six of the lines p_{ij}, every line p_{ij} by six of the lines π_{ij}.

As an example we may use the permutations given in § 17, by which it appears that the two planes which pass through π_{16} are $P_0 P_4 P_3$ and $P_2 P_5 P_7$. This is easily verified. The equation of $P_0 P_4 P_3$ is

$$0 = \begin{vmatrix} x & y & z & t \\ 0 & 0 & 0 & 1 \\ \varepsilon^4 & \varepsilon^2 & \varepsilon & \tau^{-1} \\ \varepsilon^3 & \varepsilon^5 & \varepsilon^6 & \tau^{-1} \end{vmatrix} = (\varepsilon - \varepsilon^6)x + (\varepsilon^4 - \varepsilon^3)y + (\varepsilon^2 - \varepsilon^5)z,$$

while that of $P_2 P_5 P_7$ is

$$0 = \begin{vmatrix} x & y & z & \tau t \\ \varepsilon^2 & \varepsilon & \varepsilon^4 & 1 \\ \varepsilon^5 & \varepsilon^6 & \varepsilon^3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z & \alpha x + \gamma y + \beta z + \tau t \\ \varepsilon^2 & \varepsilon & \varepsilon^4 & \alpha\varepsilon^2 + \gamma\varepsilon + \beta\varepsilon^4 + 1 \\ \varepsilon^5 & \varepsilon^6 & \varepsilon^3 & \alpha\varepsilon^5 + \gamma\varepsilon^6 + \beta\varepsilon^3 + 1 \\ 1 & 1 & 1 & \alpha + \beta + \gamma + 1 \end{vmatrix},$$

where the lower three constituents of the last column are now all zero, so that the equation is $\alpha x + \gamma y + \beta z + \tau t = 0$. But

$$(\varepsilon - \varepsilon^6)x + (\varepsilon^4 - \varepsilon^3)y + (\varepsilon^2 - \varepsilon^5)z \equiv \varepsilon x + \varepsilon^4 y + \varepsilon^2 z + \tau^{-1} t - (\varepsilon^6 x + \varepsilon^3 y + \varepsilon^5 z + \tau^{-1} t),$$
$$\alpha x + \gamma y + \beta z + \tau t \equiv \varepsilon x + \varepsilon^4 y + \varepsilon^2 z + \tau^{-1} t + (\varepsilon^6 x + \varepsilon^3 y + \varepsilon^5 z + \tau^{-1} t),$$

and the passage of the two planes through π_{16} is manifest, as is the fact that they are harmonically conjugate to Π_1 and Π_6.

These two harmonic pairs of planes through π_{16} determine a third pair, harmonic to both of them, with equations

$$\varepsilon x + \varepsilon^4 y + \varepsilon^2 z + \tau^{-1} t \pm i(\varepsilon^6 x + \varepsilon^3 y + \varepsilon^5 z + \tau^{-1} t) = 0. \qquad 30.1$$

This pair of planes, being harmonic both to Π_1 and Π_6 and to $P_0 P_4 P_3$ and $P_2 P_5 P_7$, must be the pair of united planes through π_{16} for each of the three involutions of G which interchange Π_1 and Π_6. Each plane of the pair therefore contains axes of these three involutions and is determined as the join of π_{16} to one of the two points of p_{16} where three axes meet. The coordinates of the two points in question have been given in § 28, and it is at once verified that they lie one in each of the two planes 30.1.

Dually there is, on each line p_{ij}, a regular sextuple of points. The three pairs of the sextuple are: (1) P_i and P_j, (2) the points of concurrence of two triads of base planes, (3) the intersections of p_{ij} with axes of involutions.

31. Certain facts about the intersections of π_{ij} with any surface F that is invariant for the collineations of $\mathbf{G}$ can be predicted in consequence of its invariance for the collineation C of period 3. For the intersections of F with π_{ij} must, as a whole, be unchanged by C and so consist of the united points U, U' and a certain number of triads of points. Moreover, since there are involutions of $\mathbf{G}$ which interchange U and U', F must have equal multiplicities at these two points and also equal orders of contact, or multiplicities of intersection, with UU'. Thus if the order of F is congruent to 1 to modulus 3 its multiplicities of intersection at U and U' must be congruent to 2, while if the order of F is congruent to 2 to modulus 3 these multiplicities must both be congruent to 1.

32. A reference to the tables on p. 184 shows that the polar planes of P_1 and P_6 with respect to Ψ_2 are $P_0P_3P_4$ and $P_2P_5P_7$ while the polar planes of the same two points with respect to Ψ'_7 are, although in the opposite order, the same two planes. Hence p_{16} and π_{16} are polar lines with respect to both Ψ_2 and Ψ'_7 — the two quadrics that correspond to the octahedral sets in which p_{16} and π_{16} occur. The analogous property holds for any pair of lines p_{ij} and π_{ij}: they are polar lines with respect to those two quadrics Ψ and Ψ' that correspond to the octahedral sets in which p_{ij} and π_{ij} occur. Also: given any quadric Ψ or Ψ' each of the four pairs p_{ij} and π_{ij} that belongs to the octahedral set associated with this quadric is a pair of polar lines with respect to it. Each pair is a pair of polar lines also with respect to a quadric of the other set of seven, and the four quadrics so arising are those which do *not* belong to the associated triplet.

33. Suppose now that b, b' are the intersections of K with a line p (suffixes may be dropped for the time being) and U, U' are the united points of C on the corresponding line π. Since b is a united point of C the tangent of K at b is unchanged by C and so, as it is distinct from p, must pass through U or U'; let it pass through U. The plane Up has, apart from its contact at b and its intersection at b', three further intersections with K which, since $2b + 2b'$ is a canonical set on K, are all on the trisecant conjugate to b'; this trisecant, since it is unchanged by C, must pass through U. Similar statements hold for the

trisecant conjugate to b, which passes through U' and meets there the tangent of K at b'. The scroll R^8 generated by the trisecants of K is of order 8 and invariant for $\mathfrak{G}$; that it should pass through both U and U' is in accordance with § 31.

The osculating plane of K at b is a united plane of C which does not pass through p and therefore must pass through π; the same is true of the osculating plane at b'. Hence *π is the line common to the osculating planes of K at its intersections with p.*

34. Consider now the relation of π to the tangents of K, which generate a developable S^{16}. Both U and U' lie, as has just been shown, on S^{16}; not only does this happen but the tangent planes of S^{16} at U and U', being the osculating planes of K at b and b', both contain π which therefore touches S^{16} at both U and U'. This, too, is in accordance with § 31. There remain twelve further intersections of π with S^{16}; the tangent of K which passes through any one of these gives rise, by repetitions of C, to a triad of tangents which are cyclically permuted by C and which, since the plane joining any one of them to π is a united plane of C, lie in a plane through π. Two such planes are indeed familiar — the base planes which pass through π; there remain two others. Thus, apart from the base planes Π, there are two further tritangent planes of K passing through each of the twenty-eight lines π; there are thus obtained, in addition to the eight base planes, fifty-six tritangent planes of K. Formulae[1] that connect the different singularities of a twisted curve show that K has sixty-four tritangent planes altogether; we have therefore accounted for them all. The dual result is that the triple points of the cuspidal edge of the developable $\varkappa$ consist of the eight base points P and of two further points on each of the lines p.

C permutes the tangents of K in triads, except for the two tangents at b and b' which it leaves unchanged; if a tangent of K meets p in a point other than b or b' all tangents that arise from it by repeated applications of C must meet p in the same point. Now p meets twelve tangents of K other than those at b and b'; these must therefore consist of four concurrent triads. The lines p therefore account for 112 of those points through which pass three tangents of K.

[1] Zeuthen: *Nouvelles Annales de mathématique* (2), 7 (1868), 402; Cayley: *Collected Mathematical Papers*, 8, 77.

35. There are collineations of period 7 which belong to $\mathbf{G}$; there are forty-eight in all and they are distributed in eight conjugate cyclic groups (K. III, 93; K.-F., 375). Each such collineation has for united points the vertices of an osculating tetrahedron Ω; one example is the collineation whose matrix is $\mathbf{E}$ or one of its powers, for the matrix, being in diagonal form, is the instrument of a collineation whose united points are the vertices of Ω_0. Any point, other than a vertex, on an edge of Ω belongs to a unique heptad of points on this edge which arises from any one of its members by repetition of the corresponding collineation. It follows that any surface F which is invariant for $\mathbf{G}$ can only meet the edge in the two vertices lying thereon and in a number of these heptads; if the order of F is congruent to n to modulus 7 then the sum of the multiplicities of its intersections with any edge of Ω at the two vertices is also congruent to n.

III.

The Invariants.

36. We pass now to the consideration of invariants of the Klein group in three dimensions; by an invariant of $\mathbf{G}$ is meant any surface which is transformed into itself by every one of the 168 collineations of $\mathbf{G}$. There are two ways of approaching this matter, and they supplement one another most conveniently. The first approach may be described as geometrical, and springs from the fact that all combinantal covariants and contravariants of the net 3.1 give surfaces with the desired property of invariance. Since several combinants of a general net of quadrics have been identified the corresponding combinants for the special net can be obtained; all combinants of the special net are obtainable in this way, although the specialisation may cause some to vanish identically, others to coincide one with another, and relations of linear dependence to hold which do not hold for the general net. The second approach, which emerges from the theory of group characters and depends on properties of groups of linear substitutions, may be described as algebraical. The geometrical approach leads to the actual invariants themselves and yields polynomials that give equations for the surfaces. The algebraical approach gives no information about the form of these polynomials, but it tells precisely how many there are of any given order which are linearly independent.

37. A linear substitution both determines and is determined by a unique square matrix, the number of rows and columns in the matrix being the number of variables subjected to the substitution. In the geometrical representation of the substitution as a collineation the coordinates of a point P are constituents of a column vector which is premultiplied by the matrix; the constituents of the product, which is another column vector, are the coordinates of the point P' into which the collineation transforms P. Now this geometrical representation, abundantly advantageous though it be, carries with it a certain complication for, since the position of a point is not altered when all its homogeneous coordinates are multiplied by one and the same factor, a collineation, viewed as a geometrical operation which changes the position of a point, is not altered when every element of its matrix is multiplied by one and the same factor. Thus, while a substitution determines a unique matrix and so a unique collineation, the converse does not hold: a collineation need not determine a unique substitution. A group $\mathbf{G}$ of collineations may not determine a group, of the same order, of substitutions but some group $\mathbf{G}^*$ of substitutions whose order is a multiple of that of $\mathbf{G}$. It may be possible to find in $\mathbf{G}^*$ a subgroup which is simply isomorphic with $\mathbf{G}$ but, on the other hand, it may well be impossible. The existence of simply isomorphic groups $\mathbf{G}$ and $\mathbf{G}^*$ is indeed of proved impossibility even for the elementary example of the 4-group, which does not admit a representation as a group of four binary substitutions[1]; eight binary substitutions inevitably arise, forming a group in (2, 1) isomorphism with the 4-group. And an analogous situation prevails for the Klein group $\mathbf{G}$; it is not possible to obtain a representation of $\mathbf{G}$ as a group of 168 quaternary substitutions, and the attempt to do so inevitably produces a group $\mathbf{G}^*$ of 336 substitutions in (2, 1) isomorphism with $\mathbf{G}$ (K. II, 409; K.-F., 724). And it cannot have escaped notice that in the group generated by the matrices $\mathbf{E}$ and $\mathbf{F}$ the identical collineation corresponds to two different matrices $\mathbf{E}^7 = \mathbf{I}$ and $\mathbf{F}^2 = -\mathbf{I}$; to every one of the 168 operations of the permutation group generated by $\mathbf{e}$ and $\mathbf{f}$ there correspond two operations, of the substitution group generated by $\mathbf{E}$ and $\mathbf{F}$, whose matrices are negatives of one another. It may be remarked in passing that $\mathbf{G}^*$ is not the group $\mathbf{H}$ encountered in § 12; the substitution $-\mathbf{E}$ belongs to $\mathbf{G}^*$ and has period 14 while no operation of $\mathbf{H}$, which is a permutation group of degree 8, can have this period.

[1] Klein: *Vorlesungen über das Ikosaeder* (Leipzig, 1884), 37 and 46.

38. A substitution group G^* has invariants; by this is meant that there are polynomials, in the variables which are subjected to the substitutions, which are unchanged by all the substitutions of G^*. If an invariant of G^* is equated to zero the resulting surface is an invariant of G. A generating function which yields important information about the invariants of G^* can be constructed, and we have the following theorem[1]:

If there are N operations in G^* then the number of linearly independent invariants which are of degree m is the coefficient of x^m in

$$\frac{1}{N}\sum\frac{1}{(1-\lambda_1 x)(1-\lambda_2 x)\ldots(1-\lambda_n x)},$$

summed over all the substitutions of G^*, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the multipliers (i. e. the latent roots of the matrix) of a substitution.

Two substitutions which are conjugate in G^* have the same set of multipliers, so that the terms of the sum can, if desired, be assembled in sets of equal terms corresponding to the conjugate sets of G^*.

We have then to construct a function

$$\Phi(x) \equiv \frac{1}{336}\sum\frac{1}{(1-\lambda_1 x)(1-\lambda_2 x)(1-\lambda_3 x)(1-\lambda_4 x)}.$$

The multipliers of the various substitutions of G^* are given in the following table. The number at the left-hand end of each line is the number of substitutions which have the same set of multipliers as does the substitution chosen for illustration; the number at the right-hand end of each line is the period of these substitutions. A few lines of explanation follow the table.

1	$\mathbf{I}$	$1,$	$1,$	$1,$	1	1
1	$-\mathbf{I}$	$-1,$	$-1,$	$-1,$	-1	2
56	$-\mathbf{EF}$	$1,$	$1,$	$\omega,$	ω^2	3
42	$\mathbf{F}$	$i,$	$i,$	$-i,$	$-i$	4
56	$\mathbf{EF}$	$-1,$	$-1,$	$-\omega,$	$-\omega^2$	6
24	$\mathbf{E}$	$\varepsilon^6,$	$\varepsilon^3,$	$\varepsilon^5,$	1	7
24	$\mathbf{E}^3$	$\varepsilon^4,$	$\varepsilon^2,$	$\varepsilon,$	1	7
84	$\mathbf{E}^4\mathbf{F}$	$\tau^{-1}(1+i),$	$-\tau^{-1}(1+i),$	$\tau^{-1}(1-i),$	$-\tau^{-1}(1-i)$	8
24	$-\mathbf{E}$	$-\varepsilon^6,$	$-\varepsilon^3,$	$-\varepsilon^5,$	-1	14
24	$-\mathbf{E}^3$	$-\varepsilon^4,$	$-\varepsilon^2,$	$-\varepsilon,$	-1	14

[1] Burnside: *Theory of Groups* (Cambridge, 1911), 301.

The multipliers of $\mathbf{I}$, $-\mathbf{I}$, $-\mathbf{E}$ and all its powers are obvious, since these matrices are all in diagonal form. Those of $\mathbf{F}$ are determined by two facts: first that $\mathbf{F}^2 = -\mathbf{I}$, so that the square of each multiplier of $\mathbf{F}$ must be -1, and secondly that the sum of the four multipliers vanishes with the trace $\frac{i}{\sigma}(\beta + \alpha + \gamma + 1)$ of $\mathbf{F}$. The multipliers of $\mathbf{E}^4\mathbf{F}$ are found similarly; the squares of two of them must be i and of the remaining two $-i$, while the sum of the four must vanish with the trace $\frac{i}{\sigma}(\beta\varepsilon^3 + \alpha\varepsilon^5 + \gamma\varepsilon^6 + 1)$ of $\mathbf{E}^4\mathbf{F}$. Since $(-\mathbf{E}\mathbf{F})^3 = \mathbf{1}$ the multipliers of $-\mathbf{E}\mathbf{F}$ must all be cube roots of 1 and their sum must be equal to the trace $-\frac{i}{\sigma}(\beta\varepsilon^6 + \alpha\varepsilon^3 + \gamma\varepsilon^5 + 1)$ of $-\mathbf{E}\mathbf{F}$; this trace is equal to 1, so that the four multipliers must be as in the table. Those of $\mathbf{E}\mathbf{F}$ are got by a change of sign immediately.

As for the numbers of substitutions with given multipliers, they can be verified by appealing to the $(2, 1)$ isomorphism between $\mathbf{G}$ and $\mathbf{G}^*$ and the known distribution of the operations of $\mathbf{G}$ in different conjugate sets[1].

The table shows that

$$\Phi(x) \equiv \frac{1}{336}\{f(x) + f(-x)\}$$

where

$$f(x) \equiv \frac{1}{(1-x)^4} + \frac{56}{(1-x)(1-x^3)} + \frac{21}{(1+x^2)^2} + \frac{24}{(1-\varepsilon^6 x)(1-\varepsilon^3 x)(1-\varepsilon^5 x)(1-x)}$$

$$+ \frac{24}{(1-\varepsilon^4 x)(1-\varepsilon^2 x)(1-\varepsilon x)(1-x)} + \frac{42}{1+x^4}$$

$$\equiv \frac{1}{(1-x)^4} + \frac{56(1+x+x^2)}{(1-x^3)^2} + \frac{21}{(1+x^2)^2} + \frac{24(2+x-x^2-2x^3)}{1-x^7} + \frac{42}{1+x^4}.$$

On expansion it is found that

$$\Phi(x) \equiv 1 + x^4 + x^6 + 3x^8 + 2x^{10} + 5x^{12} + 5x^{14} + 10x^{16} + 9x^{18} + \cdots$$

39. The first item of information furnished by $\Phi(x)$ is that *there is one, and only one, quartic surface invariant for* $\mathbf{G}$. This surface appears in the Klein-Fricke treatise on p. 739, where it is actually derived as a surface which intersects K at the 24 points c. The corresponding polynomial appears on p. 242

[1] See, for example, the table of group characters of $\mathbf{G}$ given by D. E. LITTLEWOOD: *Proc. London Math. Soc.* (2) 39 (1935), 188 and 192.

of Brioschi's paper[1]; there is no thought there of geometrical interpretation, but Brioschi's A is, save for the difference in notation, identical with the $\mathrm{X}'(A)$ in K.-F. Brioschi obtains this polynomial, as he obtains further invariants, as a coefficient in a modular equation, while in K.-F. it is obtained as a symmetric function of modular forms; it is proposed here to derive this quartic polynomial by the geometrical method. It was explained in Note IV (p. 135) that there exist quartic surfaces which are combinantal concomitants of a net of quadrics; when the net is a Klein net those of these concomitants which do not vanish identically must, since x^4 appears with coefficient $+1$ in $\Phi(x)$, all coincide with a surface F^4. Now there is, according to Note IV, a quartic surface F^4 having all the lines π_{ij} as bitangents; this is in agreement with the remarks in § 31 concerning the intersections of covariant surfaces with π_{ij} and these remarks tell us further that the two points of contact of F^4 with π_{ij} must be the united points of the two collineations of period 3 for which P_i and P_j are invariant. F^4 was found in Note IV as the dual of Gundelfinger's contravariant ϕ^4; we will then obtain ϕ^4, which must also be invariant for $\mathfrak{G}$, directly. Once ϕ^4 has been found reciprocation with respect to Q_{07}, which amounts to replacing plane coordinates u, v, w, p by point coordinates x, y, z, t, gives the equation of F^4.

The quadrics 3.1 of N meet a plane $ux + vy + wz + pt = 0$ in conics which are projected from P_0 by the cones

$$\begin{aligned}
\tau u x^2 + p y^2 \qquad\qquad\qquad\; &+ \tau w z x + \tau v x y = 0,\\
\tau v y^2 + p z^2 + \tau w y z \qquad\qquad &+ \tau u x y = 0,\\
p x^2 \qquad + \tau w z^2 + \tau v y z + \tau u z x \qquad &= 0.
\end{aligned}$$

Gundelfinger's contravariant is the envelope of those planes which are such that Sylvester's invariant for the above three conics vanishes. This invariant is given in full by Salmon in § 389 of his *Conic Sections* for any three conics whatever, so that its value can be found for the above three particular ones. If we denote, for the moment, by $S\psi(u, v, w)$ the sum of three terms derived from one another by cyclic permutations of u, v, w the value of the invariant is

$$\begin{aligned}
&(p^3 + 2\tau uvw)^2 + 2\tau S u(vp - \tau w^2)(2uv^2 + wp^2) + 2 S u^2(\tau v^2 - up)(\tau w^2 + vp)\\
&\qquad - 2\tau p S u(\tau u^2 - wp)(\tau w^2 - vp) + 2\tau uvw(p^3 + 2\tau uvw) - 4u^2v^2w^2.
\end{aligned}$$

The terms independent of p herein are seen to cancel one another, as also the terms of the first degree in p. If the factor p^2 is removed and the resulting

[1] *Math. Annalen* 15 (1879).

expression equated to zero we find

$$p^4 + 6\tau uvwp + 2(v^3 w + w^3 u + u^3 v) = 0$$

for the equation of ϕ^4. The equation of F^4 is therefore

$$t^4 + 6\tau xyzt + 2(y^3 z + z^3 x + x^3 y) = 0.$$

This polynomial is, save for the modification caused by the introduction of the irrationality τ, the same as that obtained by Brioschi; Brioschi's notation is changed into the present notation by replacing a_0, a_1, a_2, a_3 by $\tau^{-1}t$, x, y, z respectively.

It is seen from its equation that F^4 is outpolar to every quadric 4.2; it must therefore[1] be linearly dependent on the fourth powers of the base planes. And in fact

$$14\{t^4 + 6\tau xyzt + 2(y^3 z + z^3 x + x^3 y)\} \equiv \sum_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t)^4 + (i\sigma\tau^{-1} t)^4.$$

It was indeed by observing that the right-hand side of this identity is a modular form invariant for $\mathfrak{G}^*$ that the quartic surface was obtained in K.-F.

The base plane $t = 0$ meets F^4 in a Klein quartic; the lines of intersection of $t = 0$ with the remaining seven base planes are, in accordance with the reasoning of Note IV, an Aronhold set of bitangents of this curve. Their points of contact could easily be found. By applying suitable collineations of $\mathfrak{G}$ it follows that every base plane meets F^4 in a Klein quartic.

40. There is, on a surface F^n of order n, a *flecnodal curve*, or locus of points whereat one of the inflectional tangents has four-point contact; it was shown by Salmon that this curve is the intersection of F^n with a covariant surface of order $11n - 24$. The flecnodal curve of F^4 is therefore obtained as its intersection with a covariant surface of order 20, which must have the property of being invariant for $\mathfrak{G}$. This surface, by § 31, passes through the fifty-six points U which lie two on each of the lines π_{ij}, and these points are therefore, since F^4 also passes through them, on the flecnodal curve. Moreover the Hessian H^8 of F^4, which is of order 8 and also invariant for $\mathfrak{G}$, must also, by § 31, pass through these points. The points U are thus among the points of contact of

[1] Reye: *Journal für Math.* 78 (1874), 112—113.

the parabolic and flecnodal curves of F^4. It was shown by Clebsch[1] that the parabolic and flecnodal curves of a surface touch wherever they meet, and that on a quartic surface there are 320 of these contacts. The points U account for 56 of them, leaving 264 others. These outstanding points include the 24 points c, and moreover include them multiply; it is sufficient to establish this fact for any one of the points c, since it will then also be established for the others, which can all be obtained from the one by collineations of $\mathfrak{G}$. Consider, then, the point $y=z=t=0$. That this is on the flecnodal curve of F^4 is obvious because the line $y=z=0$ has all its four intersections with F^4 at this point. As for it lying on the parabolic curve, it is sufficient to remark that the tangent plane $y=0$ of F^4 meets it in the curve $y=t^4+2z^3x=0$ having a triple point; it follows, from a result of C. Segre's[2], that the point is a multiple point on the parabolic curve. Segre's italicised statement (pp. 174—5) establishes indeed that the parabolic curve has a *triple* point and also gives the necessary information for determining its three tangents there; this information will be used when the nature of the singularity possessed by the Hessian surface has been determined. The necessary and sufficient conditions given by Segre (p. 175) for the parabolic curve to have a point of higher multiplicity than 3 are not here satisfied.

41. The second item of information furnished by $\Phi(x)$ is that there is one, and only one, sextic surface invariant for $\mathfrak{G}$. The geometrical approach leads instantly to this surface for, by § 18 of Note IV, the Jacobian of any net of quadrics and a quartic covariant is, unless it vanishes identically, a sextic covariant. The Jacobian of F^4 and the net 3.1 is seen not to vanish identically, and is the surface

$$\begin{vmatrix} -t & \cdot & \tau x & z^3+3x^2y+3\tau yzt \\ \tau y & -t & \cdot & x^3+3y^2z+3\tau zxt \\ \cdot & \tau z & -t & y^3+3z^2x+3\tau xyt \\ -x & -y & -z & 2t^3+3\tau xyz \end{vmatrix} = 0.$$

This surface contains the whole of K, since the four minors formed from the first three columns of the determinant all vanish along the curve. On expansion, and division by 2, there results

[1] *Journal für Math.* 58 (1861), 105.
[2] *Rom. Accad. Lincei Rendiconti* (5) 6^2 (1897), 173—5.

$$B \equiv -t^6 + 5\tau xyzt^3 + 5t^2 \Sigma y^3 z + 5\tau t \Sigma y^2 z^3 + \Sigma yz^5 + 15x^2y^2z^2 = 0,$$

where Σ denotes the sum of three terms obtained from any one of them by cyclic permutation of x, y, z. The polynomial on the left is, save for sign, precisely Brioschi's B; he was the first to find the polynomial, after which it seems to have faded into oblivion and not to have been encountered by anyone since. It will be observed that F^6 also is outpolar to the quadrics 4.2, and that

$$42\,B \equiv \sum_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t)^6 + (i\sigma\tau^{-1} t)^6.$$

It is clear from the form of B that $y = z = 0$ has all its six intersections with F^6 at the point $y = z = t = 0$; thus the line which joins any base point to any one of the three points c associated with is has all its six intersections with F^6 at this latter point. This is in agreement with § 35. F^6 must circumscribe each of the 21 fundamental tetrahedra (§ 29) and meet each transversal in points forming a regular sextuple of which the vertices of the fundamental tetrahedron joined by the transversal constitute one of the three pairs.

F^6 is the locus of a point O such that the point O' which, in the sense of Note I, is conjugate to it with respect to the net 3.1 lies in the polar plane of O with respect to F^4. The curve common to F^6 and F^4 is the locus of points where F^4 is touched by quadrics of this net.

42. The generating function $\Phi(x)$ shows that $\mathfrak{G}$ has invariants of degree eight and that these are linearly dependent on three of them; thus octavic surfaces exist which are invariant for $\mathfrak{G}$ and they all belong to a net N. Three octavic surfaces not belonging to the same pencil have, in general, 512 common points. But among the octavic surfaces invariant for $\mathfrak{G}$ is the square of F^4 so that the base points of N consist of the 256 points common to F^4 and to two further surfaces of N, this set of points being counted twice over. These 256 points themselves may well include multiple sets, and more will be said about them presently.

The geometrical approach furnishes at once several octavic invariants. There are, besides $(F^4)^2$,

Π^8: the product of the base planes,

S^8: the locus of conics which touch all the base planes,

E^8: the locus of equianharmonic base curves,

R^8: the scroll of trisecants of K,

H^8: the Hessian of F^4,

J^8: the Jacobian of F^6,

and others. These must all belong to N. It is known from Note IV that S^8 is a linear combination of Π^8 and $(F^4)^2$; other relations of linear dependence will appear when equations for the surfaces have been obtained. There is no difficulty in finding these equations, although some patience is necessary to work them out to the last detail. The final forms are given, as compendiously as possible, in the table.

It will be convenient to denote the left-hand sides of the equations of the octavic surfaces by the same symbols as the surfaces themselves. We take

$$\tau \Pi^8 \equiv 16\, t \prod_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t),$$

the numerical multiplier being inserted so that t^8 does not have a fractional coefficient.

S^8 is known to have triple points at the base points; this fact was not mentioned in Note IV, but it is known (cf. *O. S.*, 514) that the contravariant dual to S^8, namely the envelope of the cones which belong to 3.1, has the base planes as tritangent planes. Thus, since $S^8 \equiv \Pi^8 - \lambda (F^4)^2$ for some constant λ, it is only necessary to choose λ so that the term in t^8 disappears, whereupon

$$4\, S^8 \equiv 8\, \tau\, t \prod_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t) - (t^4 + 6\, \tau\, x y z t + 2\, y^3 z + 2\, z^3 x + 2\, x^3 y)^2.$$

The surface generated by equianharmonic base curves of 3.1 is obtained by writing $y^2 - \tau t x$, $x^2 - \tau t z$, $z^2 - \tau t y$ instead of the line coordinates l, m, n in the equation of the equianharmonic envelope of the Klein quartic k (cf. Note II, § 2). Since this envelope is

$$l m^3 + m n^3 + n l^3 = 0$$

$$E^8 \equiv (y^2 - \tau t x)(x^2 - \tau t z)^3 + (x^2 - \tau t z)(z^2 - \tau t y)^3 + (z^2 - \tau t y)(y^2 - \tau t x)^3. \qquad 42.1$$

This surface has all the base points P_i as quadruple points.

The equation of R^8 can quickly be found in at least two ways. In the first place R^8 is the fundamental surface for the Cremona transformation 3.5, and so must be the Jacobian of the homaloidal web of cubic surfaces:

$$-3R^8 \equiv \frac{\partial(X, Y, Z, T)}{\partial(x, y, z, t)} \equiv \begin{vmatrix} t^2 & 2\tau t y + 2z^2 & 4yz & 2tx + \tau y^2 \\ 4zx & t^2 & 2\tau t z + 2x^2 & 2ty + \tau z^2 \\ 2\tau t x + 2y^2 & 4xy & t^2 & 2tz + \tau x^2 \\ 2\tau y z & 2\tau z x & 2txy & -3t^2 \end{vmatrix}.$$

In the second place this Cremona transformation, placing two points in correspondence when they are conjugate for the net 3.1, is involutory; when it is applied twice to any surface this surface does not change. Take, as the simplest example, the plane $t = 0$; its conjugate surface is $T = 0$, or $t^3 = 2\tau xyz$. Performing the transformation a second time produces the surface $T^3 = 2\tau XYZ$, which must therefore consist of R^8 and of the plane $t = 0$. Hence

$$tR^8 \equiv 2\tau(t^2x + \tau ty^2 + 2yz^2)(t^2y + \tau tz^2 + 2zx^2)(t^2z + \tau tx^2 + 2xy^2) - (2\tau xyz - t^3)^3.$$

For the Hessian of F^4 we take

$$H^8 \equiv \begin{vmatrix} 2xy & \tau zt + x^2 & \tau yt + z^2 & yz \\ \tau zt + x^2 & 2yz & \tau xt + y^2 & zx \\ \tau yt + z^2 & \tau xt + y^2 & 2zx & xy \\ yz & zx & xy & t^2 \end{vmatrix}$$

The Jacobian of F^6 need not be explicitly written down; when worked out its coefficients are found to be as in the subjoined table.

This table simply exhibits each octavic invariant, placed on the left, as a linear combination of the terms which appear along the top, the coefficients being entered in the body of the table. Two invariants, in addition to those already mentioned, have been added. The first of these, C^8, is Brioschi's polynomial. Since there was only one invariant of degree four and only one of degree six Brioschi was predestined to encounter them; but the invariant of degree eight which he was to encounter need not by any means have been among the simplest, as indeed it now proves not to have been, nor has it necessarily any convenient geometrical meaning. If in his polynomial C the letters a_0, a_1, a_2, a_3 are replaced by $\tau^{-1}t, x, y, z$ and the result multiplied by the factor 8 the outcome is C^8. The second invariant which has been added is the outpolar invariant of degree eight, and is given by

$$\frac{7}{2}\Omega^8 \equiv \sum_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t)^8 + (i\sigma\tau^{-1} t)^8.$$

	t^8	$4\tau xyzt^5$	$4t^4\Sigma y^3z$	$2\tau t^3\Sigma y^2z^3$	$2x^2y^2z^2t^2$	$2t^2\Sigma yz^5$	$\tau xyzt\Sigma y^3z$	$\tau t\Sigma x^7$	Σy^6z^2	$2xyz\Sigma y^2z^3$
$(F^4)^2$	1	3	1	.	36	.	24	.	4	4
Π^8	1	7	−7	28	−28	−28	56	8	.	.
S^8	.	1	−2	7	−16	−7	8	2	−1	−1
E^8	.	.	1	−4	9	3	−3	−1	1	
R^8	1	−1	1	4	28	4	16	.	.	8
H^8	.	1		−1	2	−1	2	.	1	−1
J^8	6	−3	5	25	165	18	105	1	2	45
C^8	15	−63	7	140	252	84	168	8	−56	168
Ω^8	43	21	35	140	1260	84	840	8	56	280

43. These invariants are, by the theory expounded above, linearly dependent on three of them, and the table vindicates this dependence. Any three which are not themselves linearly dependent may be chosen for a basis in terms of which all others can be linearly expressed. It is convenient to choose E^8, the only one of the surfaces with quadruple points at the base points P_i, as one member of this basis. Either S^8 or H^8 might then be added, both these surfaces having triple points at P_i, and it is obvious that $H^8 - S^8 \equiv 2\,E^8$. We will choose H^8, both because its coefficients are somewhat lower than those of S^8 and because of its intimate relation with F^4. As the last member of the basis we add $(F^4)^2$. The expressions for the other invariants in terms of the three selected ones are as follows.

$$\Pi^8 \equiv (F^4)^2 + 4\,H^8 - 8\,E^8, \qquad S^8 \equiv H^8 - 2\,E^8,$$
$$R^8 \equiv (F^4)^2 - 4\,H^8 \qquad J^8 \equiv 6\,(F^4)^2 - 21\,H^8 - E^8,$$
$$C^8 \equiv 15\,(F^4)^2 - 108\,H^8 - 8\,E^8, \qquad \Omega^8 \equiv 43\,(F^4)^2 - 108\,H^8 - 8\,E^8.$$

It is noticeable that $\Omega^8 - C^8 \equiv 28\,(F^4)^2$ and this, when written in the form

$$\frac{7}{2}\,C^8 \equiv \frac{7}{2}\,\Omega^8 - \frac{1}{2}\,(14\,F^4)^2$$

gives a simple expression for Brioschi's polynomial in terms of the fourth and eighth powers of the base planes, an expression which could indeed be obtained quite easily from Brioschi's standpoint.

44. We proceed now with the discussion of the base points of the net of octavic invariants; there are 256 points, common to F^4 and two octavic surfaces, to be accounted for.

The points U, two of which are on each line π, are, by § 31, on F^4 and all the octavic surfaces. The curve common to F^4 and Π^8 consists of eight plane quartics, one in each base plane and each bitangent to seven lines π at points U. The multiplicity of a point U among the set of 256 is the multiplicity of the intersection of this composite curve with an octavic invariant not containing the whole of it, say with R^8. It is quickly verified that R^8 has a simple point at U and that its tangent plane there does not pass through π, so that its intersection with the composite curve, two components of which touch π at U, counts for two among the points $F^4 = \Pi^8 = R^8 = 0$. Consider, for example, π_{07}, whose equations are $t = x + y + z = 0$; it meets R^8 where

$$t = x + y + z = x\,y\,z\,(y^2 z^3 + z^2 x^3 + x^2 y^3) = 0,$$

a set of eight distinct points.

The points U are therefore to be reckoned twice among the 256 points: there remain 144 points to be accounted for, and it is found that these consist of the 24 points c reckoned six times. This is easily shown to hold for the three points c in the base plane $t = 0$; it follows, as usual, for the remaining points c by applying the collineations of $\mathfrak{G}$. For $t = 0$ belongs to Π^8, meets F^4 in the Klein quartic $\Sigma y^3 z = 0$ and R^8 in the composite curve $x y z \Sigma y^2 z^3 = 0$. The part $x y z = 0$ of this curve is one of the inflectional triangles of the quartic, so that each of its vertices (which are points c on K) counts for four among the twelve intersections of the triangle and the quartic. Each of the three points also counts for two among the intersection of the quartic and $\Sigma y^2 z^3 = 0$; at $(1, 0, 0, 0)$, for instance, the quartic has an inflection with tangent $y = 0$ while the quintic has a cusp with tangent $z = 0$. The section of R^8 by a base plane has therefore for its intersections with the section of F^4, in addition to fourteen points U, three points c counted six times. All points c are then to be reckoned six times among the points $F^4 = \Pi^8 = R^8 = 0$, and all 256 points are now accounted for.

45. An invariant octavic surface which does not contain K touches it at each of the 24 points c, the tangent plane of the surface there being the corresponding base plane. The point counts for twelve among the 512 base points of the net N of octavic surfaces. While the general surface of N has no multiple point there, certain particular surfaces do have singularities; R^8 has K for a triple curve and, at a point c, two of its three tangent planes coincide with the base plane through the point, while H^8 has a uniplanar point, or unode, there. The possession by H^8 of this singularity proves again what has already been established in § 40: that the points c are multiple points on the parabolic curve of F^4. The equations for F^4 and H^8 show that the tangent plane of F^4 at a point c, for instance at $(1, 0, 0, 0)$, is also the plane of lines that have three-point intersection with H^8 there; this plane $(y = 0)$ meets H^8 in a curve having a triple point and so (Segre: *loc. cit.* 174) the parabolic curve of F^4 has also a triple point at which the tangents are the same as those of the section of H^8 by $y = 0$. It is seen, on closer examination, that two of these three tangents coincide; if y is put equal to zero in H^8 the terms of the lowest degree in z

and t jointly are of the third degree, and there is only one of them, namely $-2zt^2x^5$.

The parabolic curve of F^4 has therefore the following property: it has a triple point at each of the points c, one of the three tangents being that edge, of the osculating tetrahedron to which c belongs, which passes through the associated base point. The other two tangents coincide with that tangent of K which passes through c but whose point of contact with K is a different point c, this being that edge of the osculating tetrahedron which does not join c to the base point but which passes through c and lies in the tangent plane of F^4 there.

46. The scroll R^8 has K for a triple curve; moreover the base planes are tritangent planes of R^8, each of them containing three edges of an osculating tetrahedron which are generators of R^8. It follows that the reciprocal of R^8 with respect to any one of the fundamental quadrics Q_{ij} is a scroll ϱ^8 which has $\varkappa$ for a tritangent developable and the base points for triple points, the three generators of ϱ^8 through any base point being the three edges of an osculating tetrahedron which meet there. Now ϱ^8 is an invariant surface and so linearly dependent on $(F^4)^2$, H^8 and E^8; since it has triple points at the base points it must be linearly dependent on H^8 and E^8 only. The fact that it contains edges of an osculating tetrahedron which lie on H^8 and not on E^8 (the equation of E^8 shows instantly that no edge of the tetrahedron of reference can lie on it) means that ϱ^8 can only be H^8 itself. This establishes the fact that *the Hessian of F^4 is a scroll.* Since $R^8 + 4H^8 \equiv (F^4)^2$ the two scrolls R^8 and H^8 touch wherever they meet; every generator of either must be a quadritangent line of the other and H^8 is, for the special net of quadrics that admits a Klein group of collineations, the scroll ϱ^8 whose existence was established in Note III. The curve of contact of the two scrolls lies on F^4 and is its parabolic curve, while the developable of tangent planes to the two scrolls at the points of this curve is circumscribed to Gundelfinger's contravariant.

Since H^8 is a scroll of order 8 and genus 3 it has, by the standard formulae for scrolls, a nodal curve of order 18 with twenty-four pinch points and eight triple points. These points have already been identified; the pinch points are the unodes at the points c while the triple points are the base points. The polar quadric of any base point with respect to F^4 is the associated base plane taken twice; this is obvious, from the equation of F^4, for P_0 and is therefore true also for the remaining base points by invariance under the group $\mathbf{G}$.

47. The term $2x^{10}$ in $\Phi(x)$ shows that all invariants of degree ten are linearly dependent on two of them, and so belong to a pencil p of surfaces. One surface of p is the product of F^4 and F^6, and all invariants of degree ten must be linear combinations of $F^4 F^6$ and some second invariant.

The geometrical approach leads to a second invariant instantly, namely the surface F^{10} which is conjugate to F^6 in the sense of Note I; since F^6 passes simply through K F^{10} has K for a triple curve (Note I, p. 304). The curve common to F^6 and F^{10} consists, apart from K counted three times, of a curve C^{42} which, since it is a self-conjugate curve, must, by p. 305 of Note I, have 84 intersections with K. This set of 84 points on K, which is invariant for $\mathfrak{G}$, must be the set of intersections of K with the axes of the involutions. A second set of 84 points on C^{42} consists of the vertices of the fundamental tetrahedra, for these tetrahedra must, by § 29, all be inscribed in F^{10} as well as in F^6. Every surface of p must pass through C^{42}, while K must count triply as part of its intersection with F^6; every surface of p must also pass through the curve of intersection of F^{10} and F^4 which, like F^{10} and F^4 themselves, must touch every line π_{ij} at both points U which lie on it.

The Jacobian of any invariant of degree eight either vanishes identically or is an invariant of degree ten; thus several invariants of degree ten appear, all of them linearly dependent on the Jacobians of $(F^4)^2$, H^8, E^8. But, by 42.1, the Jacobian of E^8 vanishes identically; also the Jacobian of $(F^4)^2$ is a numerical multiple of $F^4 F^6$. The Jacobian of any invariant of degree eight must therefore be linearly dependent on $F^4 F^6$ and J^{10}, the Jacobian of H^8. Since the base points are triple points of H^8 and lie on all the quadrics 3.1 they are triple points of J^{10}. It must of course be that J^{10} is a linear combination of $F^4 F^6$ and F^{10}; which precise linear combination it is can be found from the fact that K is a triple curve on F^{10}, and this investigation is carried out below.

When it is desired to find the linear relation which connects three different invariants of degree ten it is sufficient to know the coefficients of the two highest powers of t (the tenth and seventh powers as they prove to be) in these invariants; it will not then be necessary to write out any of the invariants in full, but if they are extended beyond the two highest terms in t convincing means of verification are thereby afforded.

48. If we use the form for H^8 given in the table above we find, introducing the multiplier 4 for convenience, that

$$4J^{10} \equiv \begin{vmatrix} -t & \cdot & \tau x & \begin{array}{l} 4\tau yzt^5 - 2\tau t^3(3z^2x^2 + 2xy^3) \\ \qquad - 2t^2(5zx^4 + y^5 - 4xy^2z^2) + \cdots \end{array} \\ \tau y & -t & \cdot & \begin{array}{l} 4\tau zxt^5 - 2\tau t^3(3x^2y^2 + 2yz^3) \\ \qquad - 2t^2(5xy^4 + z^5 - 4x^2yz^2) + \cdots \end{array} \\ \cdot & \tau z & -t & \begin{array}{l} 4\tau xyt^5 - 2\tau t^3(3y^2z^2 + 2zx^3) \\ \qquad - 2t^2(5yz^4 + x^5 - 4x^2y^2z) + \cdots \end{array} \\ -x & -y & -z & \begin{array}{l} 20\tau xyzt^4 - 6\tau t^2\Sigma y^2z^3 - 4t\Sigma yz^5 \\ \qquad + 8x^2y^2z^2t + \cdots \end{array} \end{vmatrix}$$

which gives

$$J^{10} \equiv -2\tau xyzt^7 + 2t^6\Sigma y^3z + \tau t^5\Sigma y^2z^3 + 15x^2y^2z^2t^4 - 4t^4\Sigma yz^5 + \cdots.$$

Also, by direct multiplication,

$$F^4F^6 \equiv -t^{10} - \tau xyzt^7 + 3t^6\Sigma y^3z + 5\tau t^5\Sigma y^2z^3 + 75x^2y^2z^2t^4 + t^4\Sigma yz^5 + \cdots.$$

The invariant F^{10} is a linear combination of these two and has K as a triple curve; the linear combination must therefore be such that the plane $t = 0$ meets it in a curve with (at least) triple points at the vertices of the triangle of reference in that plane. Now if t is put equal to zero the two above invariants become

$$4J_0^{10} \equiv \begin{vmatrix} \cdot & \cdot & \tau x & 6x^5y^2 - 2z^6x - 2y^3z^4 - 8x^3yz^3 - 6x^2y^4z \\ \tau y & \cdot & \cdot & 6y^5z^2 - 2x^6y - 2z^3x^4 - 8x^3y^3z - 6xy^2z^4 \\ \cdot & \tau z & \cdot & 6z^5x^2 - 2y^6z - 2x^3y^4 - 8xy^3z^3 - 6x^4yz^2 \\ -x & -y & -z & 2\tau xyz\Sigma y^3z \end{vmatrix},$$

which gives

$$J_0^{10} \equiv -\Sigma y^4z^6 - xyz\Sigma x^7 - 2x^2y^2z^2\Sigma xz^3,$$

and

$$\begin{aligned} F_0^4F_0^6 &\equiv 2\Sigma y^3z\,\Sigma yz^5 + 30x^2y^2z^2\Sigma y^3z \\ &\equiv 2\Sigma y^4z^6 + 2xyz\Sigma z^7 + 32x^2y^2z^2\Sigma y^3z. \end{aligned}$$

The linear combination of J_0^{10} and $F_0^4F_0^6$ must now be selected so that none of x, y, z occurs to a power higher than the seventh, and the desired combination is $2J_0^{10} + F_0^4F_0^6$. Hence

$$\begin{aligned} F^{10} \equiv 2J^{10} + F^4F^6 \equiv -t^{10} - 5\tau xyzt^7 + 7t^6\Sigma y^3z + 7\tau t^5\Sigma y^2z^3 \\ + 105x^2y^2z^2t^4 - 7t^4\Sigma yz^5 + \cdots + 28x^2y^2z^2\Sigma y^3z. \end{aligned}$$

Perhaps it is worth while to find the expressions in terms of F^{10} and $F^4 F^6$ of Brioschi's polynomial D and the outpolar invariant of degree ten. For D (*loc. cit.*, 242) we find instantly

$$\begin{aligned} 2\,D &\equiv t^{10} - 23\,\tau\,x\,y\,z\,t^7 + 21\,t^6\,\Sigma\,y^3 z + 7\,\tau\,t^5\,\Sigma\,y^2 z^3 + 105\,x^2 y^2 z^2 t^4 - 49\,t^4\,\Sigma\,y\,z^5 + \cdots \\ &\equiv 6\,F^{10} - 7\,F^4 F^6. \end{aligned}$$

For the outpolar invariant we take

$$105\,\Omega^{10} \equiv \sum_{s=1}^{7} (\varepsilon^s x + \varepsilon^{4s} y + \varepsilon^{2s} z + \tau^{-1} t)^{10} + (i\,\sigma\,\tau^{-1}\,t)^{10},$$

$$\begin{aligned} \Omega^{10} &\equiv -5\,t^{10} + 3\,\tau\,x\,y\,z\,t^7 + 7\,t^6\,\Sigma\,y^3 z + 21\,\tau\,t^5\,\Sigma\,y^2 z^3 + 315\,x^2 y^2 z^2 t^4 + \cdots \\ &\equiv 5\,F^4 F^6 - 4\,J^{10} \\ &\equiv 7\,F^4 F^6 - 2\,F^{10}. \end{aligned}$$

IV.

Covariant Line Complexes.

49. It was explained in Note III how a net of quadric surfaces gives rise to a five-dimensional figure consisting essentially of a quadric Ω and two Veronese surfaces v and w that are polar reciprocals of one another with respect to Ω. This figure will now be set up for the net $\mathbf{N}$ (3.1) by following the procedure of Note III. The Plücker coordinates λ, μ, ν, λ', μ', ν' will have the following signification; if a line is the join of two points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) then

$$\begin{aligned} \lambda &= y_1 z_2 - y_2 z_1, & \mu &= z_1 x_2 - z_2 x_1, & \nu &= x_1 y_2 - x_2 y_1, \\ \lambda' &= x_1 t_2 - x_2 t_1, & \mu' &= y_1 t_2 - y_2 t_1, & \nu' &= z_1 t_2 - z_2 t_1, \end{aligned}$$

while if it is the intersection of two planes (u_1, v_1, w_1, p_1) and (u_2, v_2, w_2, p_2),

$$\begin{aligned} \lambda &= u_1 p_2 - u_2 p_1, & \mu &= v_1 p_2 - v_2 p_1, & \nu &= w_1 p_2 - w_2 p_1, \\ \lambda' &= v_1 w_2 - v_2 w_1, & \mu' &= w_1 u_2 - w_2 u_1, & \nu' &= u_1 v_2 - u_2 v_1. \end{aligned}$$

The trisecant conjugate to a point (x_0, y_0, z_0, t_0) of K is the line of intersection of the polar planes of this point with respect to any two quadrics of $\mathbf{N}$, and so has equations

$$\begin{aligned} -\tau\,x_0\,x \qquad + t_0\,z + z_0\,t &= 0, \\ t_0\,y - \tau\,z_0\,z + y_0\,t &= 0; \end{aligned}$$

its Plücker coordinates are therefore

$$-\tau x_0 y_0, \quad -z_0 t_0, \quad \tau z_0^2 + y_0 t_0, \quad -t_0^2, \quad -2 z_0 x_0, \quad -\tau x_0 t_0. \qquad 49.1$$

These are seen, by using equations 3.3, to be proportional to

$$\tau \xi^2 \eta \zeta^3, \quad \tau \xi^2 \eta^3 \zeta, \quad -\tau \xi^2 \eta^4 - \tau \xi \eta^2 \zeta^3, \quad 2 \xi^2 \eta^2 \zeta^2, \quad 2 \xi^3 \eta^2 \zeta, \quad 2 \xi^3 \eta \zeta^2,$$

where (ξ, η, ζ) is the point of k corresponding to the point (x_0, y_0, z_0, t_0) of K. But, since $\xi \eta^3 + \eta \zeta^3 + \zeta \xi^3 = 0$, the binomial is equal to $\tau \xi^4 \eta \zeta$, so that the Plücker coordinates of the trisecant can be taken to be

$$\zeta^2, \quad \eta^2, \quad \xi^2, \quad \tau \eta \zeta, \quad \tau \xi \eta, \quad \tau \zeta \xi. \qquad 49.2$$

This result can also be obtained from 49.1 without appealing to 3.3; it is perhaps more in harmony with the discussion in Note III to obtain it in this way, which makes use of one of Hesse's results. For Hesse showed[1] that

$$x_0^2, \; y_0^2, \; z_0^2, \; t_0^2, \; y_0 z_0, \; z_0 x_0, \; x_0 y_0, \; x_0 t_0, \; y_0 t_0, \; z_0 t_0$$

are proportional to the cofactors of the appropriate elements of the determinant 2.1, and so to

$$-\xi \eta^2 - \zeta^3, \quad -\zeta \xi^2 - \eta^3, \quad -\eta \zeta^2 - \xi^3, \quad 2 \xi \eta \zeta, \quad \eta^2 \zeta, \quad \xi^2 \eta, \quad \zeta^2 \xi, \quad \tau \zeta \xi^2, \quad \tau \eta \zeta^2, \quad \tau \xi \eta^2.$$

Thus the Plücker coordinates of a trisecant are proportional to

$$\zeta^2 \xi, \quad \xi \eta^2, \quad \xi^3, \quad \tau \xi \eta \zeta, \quad \tau \xi^2 \eta \quad \tau \zeta \xi^2,$$

and so again are given by 49.2.

50. The six expressions 49.2 for the line-coordinates of a trisecant of K have been calculated on the supposition that (ξ, η, ζ) is a point of k. But if we suppose that (ξ, η, ζ) is any point of ϖ, not necessarily constrained to lie on k, then the six expressions 49.2 are the coordinates of a point of a Veronese surface v in a five-dimensional space Σ; calling the expressions $y_1, y_2, y_3, y_4, y_5, y_6$ respectively they can, in matrix fashion, be arranged as the constituents of a column vector

$$y = \mathbf{M} [\xi^2, \; \eta^2, \; \zeta^2, \; \tau \eta \zeta, \; \tau \zeta \xi, \; \tau \xi \eta]' = \mathbf{M} \Xi,$$

where

[1] *Journal für Math.* 49 (1855), 288.

$$\mathbf{M} = \begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix}.$$

If (ξ, η, ζ) lies on k then

$$y_1 y_4 + y_2 y_5 + y_3 y_6 = \tau(\eta \zeta^3 + \zeta \xi^3 + \xi \eta^3) = 0,$$

so that the point y is on the curve Γ in which v is met by the quadric Ω whose equation is $y' \mathbf{A} y = 0$, with

$$\mathbf{A} = \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}.$$

This is of course the quadric in Σ whose points represent the lines of the three-dimensional space.

The surface w which is the polar reciprocal of v with respect to Ω, and whose section by Ω represents the generators of the scroll H^8, is, by § 6 of Note III,

$$y = (\mathbf{M}' \mathbf{A})^{-1} [l^2, m^2, n^2, \tau m n, \tau n l, \tau l m]' = (\mathbf{M}' \mathbf{A})^{-1} \Lambda'.$$

Since $\mathbf{M}$ is here orthogonal, $\mathbf{M}' = \mathbf{M}^{-1}$ and therefore

$$(\mathbf{M}' \mathbf{A})^{-1} = \mathbf{A}^{-1} (\mathbf{M}')^{-1} = \mathbf{A} \mathbf{M} = \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

so that the parametric form of w is

$$y = [\tau m n, \tau l m, \tau n l, n^2, m^2, l^2]'. \qquad 50.1$$

51. There are set up, through the intermediary of the space figure, a (1. 1) correspondence between the points (ξ, η, ζ) of the plane ϖ and the points of v and a (1, 1) correspondence between the lines (l, m, n) of ϖ and the points of w. We pass, starting from a point (ξ, η, ζ) of k, to K by the birational transformation 3.3, and the trisecants of K are then, when the lines of space are represented in the standard fashion by the points of Ω, represented by Γ, the intersection of Ω and v. It might at first be said that a (1, 1) correspondence is set up only between k and Γ; but once Γ is given there is only one Veronese surface v which contains it and this must correspond to ϖ. It is true that different (1, 1) correspondences can be set up between v and ϖ in which Γ corresponds, as it has to do, with k; but all these correspondences associate the prime sections of v with the conics of ϖ so that any two of them can only differ by a collineation of ϖ for which k is invariant, that is by a collineation of the Klein group. It is presumed then that, k having been related to Γ, any definite one of the 168 correspondences is chosen to relate v and ϖ. This being so, any line (l, m, n) of ϖ is represented by a conic of v; there is a prime which touches v all along this conic (i. e. contains all the tangent planes of v at points of the conic) and the pole of this prime with respect to Ω is a point of w; this is the definition of w, whose points are thus in (1, 1) correspondence with the lines of ϖ. A curve on v corresponds to a locus in ϖ, a curve on w to an envelope in ϖ.

The curve of intersection of Ω with v, being given by $\Xi' \mathbf{M}' \mathbf{A} \mathbf{M} \Xi = 0$, is, since

$$\mathbf{M}'\mathbf{A}\mathbf{M} = \begin{bmatrix} . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ . & . & . & 1 & . & . \\ . & . & 1 & . & . & . \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \end{bmatrix},$$

$\xi\eta^3 + \eta\zeta^3 + \zeta\xi^3 = 0$ and corresponds to k; this is of course to be expected. The curve Δ of intersection of Ω with w is $\Delta(\mathbf{M}'\mathbf{A}\mathbf{M})^{-1}\Delta' = 0$; this, since $\mathbf{M}'\mathbf{A}\mathbf{M}$ is, like the matrices $\mathbf{A}$ and $\mathbf{M}$, self-inverse, is $lm^3 + mn^3 + nl^3 = 0$ and corresponds to the equianharmonic envelope χ of k. The relationship between χ and k is symmetrical; not only is χ the equianharmonic envelope of k but k is the equianharmonic locus of χ.

These results can now be interpreted in the space figure. There corresponds to a line of χ a point of $\varDelta$, and the tangent prime of Ω there touches v along a conic such that the four intersections of Ω with it form an equianharmonic tetrad; thus a generator of H^8 is quadritangent to R^8 and the four generators of R^8 through the points of contact are, in the regulus to which they belong, equianharmonic. The relation between the two scrolls is mutual: the four generators of either which meet any generator of the other are equianharmonic. The generators of both scrolls belong to the quartic complex of lines that are cut equianharmonically by F^4.

An incidental consequence of these remarks is that the parabolic curve of F^4 has two complementary sets of quadrisecants, generating respectively the scrolls R^8 and H^8; every generator of either scroll is cut by the curve in an equianharmonic tetrad of points.

52. There is a simple and direct method of passing from ϖ and k to v and Γ and Ω without the intervention of the space figure. This method is to consider the group of substitutions which

$$\xi^2,\ \eta^2,\ \zeta^2,\ \tau\eta\zeta,\ \tau\zeta\xi,\ \tau\xi\eta,$$

the constituents of Ξ, undergo when ξ, η, ζ are themselves subjected to the group of 168 ternary substitutions. These six constituents of Ξ will be named

$$y_3,\ y_2,\ y_1,\ y_4,\ y_6,\ y_5$$

in this order so that they agree precisely with the six coordinates of a point of v. Now it is known[1] that the group $\mathbf{J}$ of substitutions induced on the y_i is one of the irreducible representations of the 168 group, so that there is a Klein group $\mathbf{J}$ of substitutions in Σ. As for Ω, it arises at once on unravelling the left-hand side of the equation of k by Sylvester's process[2]. For it is known that the quadric which arises by Sylvester's unravelment cuts v in the same curve Γ, corresponding to k, as does Ω and, further, that it is the unique quadric through Γ which is outpolar to v. If, then, it can be shown that Ω is outpolar to v it must be Ω that arises on unravelment. But the row vector of coordinates of a prime touching v along a conic has the form

$$[n^2,\ m^2,\ l^2,\ \tau mn,\ \tau lm,\ \tau nl],$$

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[1] Burnside: *Theory of Groups* (Cambridge, 1911), 371, Ex. 8.
[2] Edge: *Proc. Roy. Soc. Edinburgh*, A, 61 (1942), 247.

so that the quadrics inscribed in v are all linearly dependent on the six

$$2\,v_1\,v_2 = v_4^2 \qquad 2\,v_2\,v_3 = v_5^2 \qquad 2\,v_3\,v_1 = v_6^2$$
$$v_4\,v_5 = \tau\,v_2\,v_6 \qquad v_5\,v_6 = \tau\,v_3\,v_4 \qquad v_6\,v_4 = \tau\,v_1\,v_5$$

where the v_i are prime coordinates in Σ. *All* these quadrics are inpolar to Ω, since the equation of Ω is

$$y_1\,y_4 + y_2\,y_5 + y_3\,y_6 = 0.$$

J may be defined as that group of substitutions of the y_i for which v and Ω are both invariant. Now each substitution of the group $\mathbf{G}^*$ in three dimensions induces a substitution on the line coordinates; it is thereby represented as a collineation in Σ for which Ω is invariant and, since R^8 is unchanged by all substitutions of $\mathbf{G}^*$, for which v, the only Veronese surface through Γ, is also invariant. Two substitutions of $\mathbf{G}^*$ which are, in the (2, 1) isomorphism between $\mathbf{G}^*$ and $\mathbf{G}$, associated with the same substitution of $\mathbf{G}$ yield the same collineation in Σ; for the two substitutions only differ by the signs they give to the point coordinates, and therefore do not differ at all in their effect upon the line coordinates. Thus the substitutions induced upon the line coordinates by $\mathbf{G}^*$ give a Klein group **J** of 168 substitutions in the five-dimensional space Σ, while **J** is simply isomorphic with $\mathbf{G}$ (cf. K. II, 413).

53. **J** is a group of substitutions, and the results of the theory of group characters are available to give information about its invariants; the relevant generating function $\Psi(x)$, in which the coefficient of x^m is equal to the number of linearly independent invariants of degree m, is easily obtained from a knowledge of the structure of the Klein group. On the other hand the geometry of the figure in Σ allows many invariants to be detected instantly and the two approaches, algebraical and geometrical, supplement one another as they did in the study of the invariants of $\mathbf{G}^*$.

Any invariant of **J**, when equated to zero, gives a primal which, if Ω does not wholly belong to it, meets Ω in a locus whose points represent the lines of some complex in the three-dimensional space; the order of the complex is the degree of the invariant of **J**. This complex must be unaffected by the collineations of $\mathbf{G}$, and so must be a covariant complex. Conversely: the lines of any complex of order m which is covariant for the three-dimensional configuration are represented by the points common to Ω and a primal of order m which is invariant for **J**.

We now obtain the generating function $\Psi(x)$. In the table the various rows refer to the six different conjugate sets of operations in the Klein group. The number on the extreme left of a row is the number of operations in the set; this is followed by the three multipliers of any operation of the set when it belongs to the ternary group of substitutions. From these the six multipliers of the corresponding operation of **J** are immediately deduced, and the number at the extreme right of a row is the period of the corresponding operations.

1	$1,$	$1,$	1	$1,$	$1,$	$1,$	$1,$	$1,$	1	1
21	$1,$	$-1,$	-1	$1,$	$1,$	$1,$	$1,$	$-1,$	-1	2
56	$1,$	$\omega,$	ω^2	$1,$	$\omega^2,$	$\omega,$	$1,$	$\omega^2,$	ω	3
42	$1,$	$i,$	$-i$	$1,$	$-1,$	$-1,$	$1,$	$-i,$	i	4
24	$\varepsilon,$	$\varepsilon^2,$	ε^4	$\varepsilon^2,$	$\varepsilon^4,$	$\varepsilon,$	$\varepsilon^6,$	$\varepsilon^5,$	ε^3	7
24	$\varepsilon^3,$	$\varepsilon^5,$	ε^6	$\varepsilon^6,$	$\varepsilon^3,$	$\varepsilon^5,$	$\varepsilon^4,$	$\varepsilon^2,$	ε	7

From this it follows that

$$168\,\Psi(x) \equiv \frac{1}{(1-x)^6} + \frac{21}{(1-x^2)^2(1-x)^2} + \frac{56}{(1-x^3)^2} + \frac{42}{(1-x^2)(1-x^4)} + \frac{48(1-x)}{1-x^7},$$

$$\Psi(x) \equiv 1 + x^2 + 2x^3 + 3x^4 + 4x^5 + 8x^6 + 10x^7 + \cdots$$

54. The presence of the term x^2 in $\Psi(x)$ indicates that **J** has a quadratic invariant I^2; the primal $I^2 = 0$ must of course be Ω, and we may take $I^2 = y_1 y_4 + y_2 y_5 + y_3 y_6$.

55. The presence of the term $2x^3$ in $\Psi(x)$ indicates that **J** has two linearly independent invariants of degree 3. Now two cubic primals invariantly related to the figure in Σ are very prominent; namely those generated by the chords of the respective Veronese surfaces v and w. The left-hand sides of the equations of these two primals may therefore be taken as the two linearly independent cubic invariants of **J**; the parametric forms 49.2 for v and 50.1 for w show at once that the cubic primals are, respectively, $I^3 = 0$ and $J^3 = 0$ where

$$I^3 \equiv \tau^{-1} \begin{vmatrix} \tau y_3 & y_5 & y_6 \\ y_5 & \tau y_2 & y_4 \\ y_6 & y_4 & \tau y_1 \end{vmatrix} \equiv 2 y_1 y_2 y_3 + \tau y_4 y_5 y_6 - y_1 y_5^2 - y_2 y_6^2 - y_3 y_4^2,$$

$$J^3 \equiv \tau^{-1}\begin{vmatrix} \tau y_6 & y_2 & y_3 \\ y_2 & \tau y_5 & y_1 \\ y_3 & y_1 & \tau y_4 \end{vmatrix} \equiv \tau y_1 y_2 y_3 + 2 y_4 y_5 y_6 - y_1^2 y_6 - y_2^2 y_4 - y_3^2 y_5.$$

The loci in which $I^3 = 0$ and $J^3 = 0$ meet Ω must correspond to two cubic complexes that are covariant for **G**. Now in Note III three cubic complexes covariant for a net of quadrics were signalised: they were

(i) the complex of transversals of canonical sets of generators of R^8;

(ii) the complex of transversals of canonical sets of generators of ϱ^8, that is, for the net **N**, of H^8;

(iii) the complex of generators of quadrics of the net.

When the net of quadrics is given by 3.1 there is associated with it the net of quadric envelopes 4.2, so that there arises also

(iv) the complex of generators of quadrics which touch all the base planes.

It was also explained in Note III that the complex (i) is represented by the section of Ω by the cubic primal generated by the chords of w while the complex (ii) is represented by the section of Ω by the cubic primal generated by the chords of v. The complexes (iii) and (iv) must, since **J** has only two linearly independent cubic invariants, be linear combinations of (i) and (ii), and this dependence will now be verified by obtaining their equations.

Since (i) and (ii) are reciprocals of one another with respect to any quadric Q_{ij} aud so, in particular, with respect to Q_{07}, their equations must be derivable from each other by interchanging dashed and undashed Plücker coordinates; this is in agreement with the fact that I^3 and J^3 are interchanged when the suffixes of the y_i are subjected to the permutation (14) (25) (36). The same relation holds between (iii) and (iv).

56. The equation of (iii) is at once obtainable from the fact that any line of (iii) is cut in involution by the quadrics 3.1, so that the join of (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) belongs to (iii) if

$$\begin{vmatrix} y_1^2 - \tau t_1 x_1 & \tau y_1 y_2 - (t_1 x_2 + t_2 x_1) & y_2^2 - \tau t_2 x_2 \\ z_1^2 - \tau t_1 y_1 & \tau z_1 z_2 - (t_1 y_2 + t_2 y_1) & z_2^2 - \tau t_2 y_2 \\ x_1^2 - \tau t_1 z_1 & \tau x_1 x_2 - (t_1 z_2 + t_2 z_1) & x_2^2 - \tau t_2 z_2 \end{vmatrix} = 0.$$

This determinant may be evaluated by remarking that the matrix within it is the product obtained on premultiplying the transposed matrix of

$$\begin{bmatrix} x_1^2 & y_1^2 & z_1^2 & t_1^2 & 2y_1z_1 & 2z_1x_1 & 2x_1y_1 & 2x_1t_1 & 2y_1t_1 & 2z_1t_1 \\ x_1x_2 & y_1y_2 & z_1z_2 & t_1t_2 & y_1z_2+y_2z_1 & z_1x_2+z_2x_1 & x_1y_2+x_2y_1 & x_1t_2+x_2t_1 & y_1t_2+y_2t_1 & z_1t_2+z_2t_1 \\ x_2^2 & y_2^2 & z_2^2 & t_2^2 & 2y_2z_2 & 2z_2x_2 & 2x_2y_2 & 2x_2t_2 & 2y_2t_2 & 2z_2t_2 \end{bmatrix}$$

by the matrix

$$\begin{bmatrix} . & 1 & . & . & . & . & . & -\tau^{-1} & . & . \\ . & . & 1 & . & . & . & . & . & -\tau^{-1} & . \\ 1 & . & . & . & . & . & . & . & . & -\tau^{-1} \end{bmatrix}$$

so that the value of the determinant is the sum of all the products of corresponding three-rowed determinants selected from the two matrices[1]. Only eight such determinants of the latter matrix do not vanish, and the one formed from the last three columns is to be multiplied by a vanishing determinant, namely that formed from the last three columns of the former matrix. There remain only seven products which do not vanish and these, because of relations like

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_1x_2 & y_1y_2 & z_1z_2 \\ x_2^2 & y_2^2 & z_2^2 \end{vmatrix} = -(y_1z_2-y_2z_1)(z_1x_2-z_2x_1)(x_1y_2-x_2y_1) = -\lambda\mu\nu,$$

$$\begin{vmatrix} y_1^2 & z_1^2 & 2z_1t_1 \\ y_1y_2 & z_1z_2 & z_2t_1+z_1t_2 \\ y_2^2 & z_2^2 & 2z_2t_2 \end{vmatrix} = (y_1z_2-y_2z_1)^2(z_1t_2-z_2t_1) = \lambda^2\nu',$$

$$\begin{vmatrix} x_1^2 & 2x_1t_1 & 2y_1t_1 \\ x_1x_2 & x_1t_2+x_2t_1 & y_1t_2+y_2t_1 \\ x_2^2 & 2x_2t_2 & 2y_2t_2 \end{vmatrix} = 2(x_1y_2-x_2y_1)(x_1t_2-x_2t_1)^2 = 2\nu\lambda'^2,$$

give, for the equation of the cubic complex,

$$\lambda\mu\nu + \tau^{-1}(\lambda^2\nu' + \mu^2\lambda' + \nu^2\mu') = \nu\lambda'^2 + \lambda\mu'^2 + \mu\nu'^2.$$

The lines of this complex (iii) are therefore represented by the intersection of Ω with the primal

$$y_1y_2y_3 + \tau^{-1}(y_1^2y_6 + y_2^2y_4 + y_3^2y_5) = y_3y_4^2 + y_1y_5^2 + y_2y_6^2,$$

and this is simply the equation $J^3 = \tau I^3$.

[1] This is of course the Binet-Cauchy theorem on the multiplication of determinantal arrays: see SALMON: *Lessons on Higher Algebra*: 4th edition (Dublin 1885), 22.

Similarly the complex (iv) is associated with the equation $I^3 = \tau J^3$, and the anticipated linear dependence is confirmed.

57. The complexes given by $\alpha I^3 + \beta J^3 = 0$, where α and β are any numerical multipliers, are polar reciprocals of one another in pairs with respect to the quadrics Q; of this pencil of cubic complexes there are two, namely $I^3 \pm J^3 = 0$, which are self-reciprocal.

58. The presence of the term $3x^4$ in $\Psi(x)$ indicates that $\mathbf{J}$ has three linearly independent quartic invariants; one of these must of course be $(I^2)^2$. Either all these invariants are unaffected when the suffixes of the y_i are subjected to the permutation $(14)(25)(36)$ or else there is a quartic invariant I^4 which is changed by this permutation into a different invariant J^4. It is the latter contingency which is, in fact, realised, and all quartic invariants must be linear combinations of $(I^2)^2$, I^4, J^4. I^4 may be taken to be any quartic polynomial in the y_i provided that it is invariant for the group and is not unchanged by the permutation $(14)(25)(36)$.

The geometrical approach leads readily to such polynomials. For example: the coordinates of the polar prime of a point A with respect to a cubic primal are quadratic in the coordinates of A while the condition for a prime to touch a quadric is quadratic in the coordinates of the prime. Hence the locus of a point which has the property that its polar prime with respect to either $I^3 = 0$ or $J^3 = 0$ touches Ω is a quartic primal which affords an invariant of $\mathbf{J}$. If the quartic primals thus arising from $I^3 = 0$ and $J^3 = 0$ are different, as in fact they prove to be, their equations are, like I^3 and J^3 themselves, obtainable from each other by the permutation $(14)(25)(36)$ and so give two invariants I^4 and J^4 on which, together with $(I^2)^2$, the whole set of quartic invariants can be based.

The symbol S will now, to save space, be used to denote the sum of three terms; these are obtained by applying the permutation $(123)(456)$ to the term to which S is prefixed.

The polar prime of the point $y = \eta$ with respect to $I^3 = 0$ is

$$S\, y_1 (2\,\eta_2 \eta_3 - \eta_5^2) + S\, y_4 (\tau \eta_5 \eta_6 - 2\,\eta_3 \eta_4) = 0;$$

this touches Ω, whose prime equation is $v_1 v_4 + v_2 v_5 + v_3 v_6 = 0$, if η lies on the quartic primal

$$I^4 \equiv S\,(2\,y_2 y_3 - y_5^2)(y_5 y_6 - \tau\, y_3 y_4) = 0.$$

Hence we take

$$I^4 \equiv S\{2\,y_2 y_3 y_5 y_6 + \tau y_1 y_5 y_6^2 - 2\,\tau y_2 y_3^2 y_4 - y_5^3 y_6\},$$
$$J^4 \equiv S\{2\,y_2 y_3 y_5 y_6 + \tau y_2 y_3^2 y_4 - 2\,\tau y_1 y_5 y_6^2 - y_2^3 y_3\},$$
$$(I^2)^2 \equiv S\{y_1^2 y_4^2 + 2\,y_2 y_3 y_5 y_6\}.$$

Since I^4 is a quadratic form in the quadrics which contain v this surface is a double surface on $I^4 = 0$. Each secant plane of v that contains a point of $I^4 = 0$ in addition to the conic, nodal on $I^4 = 0$, in which the plane meets v therefore lies entirely on the primal. The three-dimensional locus $I^4 = I^3 = 0$ thus consists of a singly-infinite set of planes secant to v; it is easily verified, by substitution of coordinates in I^4, that a secant plane of v lies on $I^4 = 0$ when, and only when, its four intersections with Γ constitute an equianharmonic tetrad. Similar relations hold between $J^4 = 0$ and w.

59. The primals $\alpha I^4 + \beta J^4 + \gamma (I^2)^2 = 0$ obtained by equating to zero the various quartic invariants constitute a net of which $(I^2)^2$ is a member; on Ω itself they cut a pencil of loci all containing the surface $I^4 = J^4 = I^2 = 0$. This points to the existence of a pencil of quartic complexes covariant for the three-dimensional figure.

One covariant complex is afforded by the lines which cut the surface $F^4 = 0$ equianharmonically. If the equation of a surface is $a_x^4 = 0$ its equianharmonic complex is

$$(a\,b\,u\,v)^4 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{vmatrix}^4 = 0,$$

$$\{(a_2 b_3 - a_3 b_2)\lambda + (a_3 b_1 - a_1 b_3)\mu + (a_1 b_2 - a_2 b_1)\nu + (a_1 b_4 - a_4 b_1)\lambda' + (a_2 b_4 - a_4 b_2)\mu' + (a_3 b_4 - a_4 b_3)\nu'\}^4 = 0.$$

Now

$$F^4 \equiv t^4 + 6\tau x y z t + 2(y^3 z + z^3 x + x^3 y),$$

so that every expression of the fourth degree in either Aronhold symbol a or b vanishes except for

$$a_4^4 = b_4^4 = 1, \quad a_1 a_2 a_3 a_4 = b_1 b_2 b_3 b_4 = \frac{1}{4}\tau$$
$$a_2^3 a_3 = b_2^3 b_3 = a_3^3 a_1 = b_3^3 b_1 = a_1^3 a_2 = b_1^3 b_2 = \frac{1}{2}.$$

Thus, for this particular quartic surface,

$$
\begin{aligned}
(a\,b\,u\,v)^4 = {} & 24\, a_1 a_2 a_3 a_4 b_1 b_2 b_3 b_4 \{\lambda^2\lambda'^2 + \mu^2\mu'^2 + \nu^2\nu'^2 \\
& \qquad - 2(\mu\mu'\nu\nu' + \nu\nu'\lambda\lambda' + \lambda\lambda'\mu\mu')\} \\
& + 24\, a_1 a_2 a_3 a_4 (b_2^3 b_3 \lambda^2\mu'\nu + b_3^3 b_1 \mu^2\nu'\lambda + b_1^3 b_2 \nu^2\lambda'\mu) \\
& + 24\, b_1 b_2 b_3 b_4 (a_2^3 a_3 \lambda^2\mu'\nu + a_3^3 a_1 \mu^2\nu'\lambda + a_1^3 a_2 \nu^2\lambda'\mu) \\
& + 4\, a_4^4 (b_2^3 b_3 \mu'^3\nu' + b_3^3 b_1 \nu'^3\lambda' + a_3^3 a_1 \lambda'^3\mu') \\
& \qquad + 4\, b_4^4 (a_2^3 a_3 \mu'^3\nu' + a_3^3 a_1 \nu'^3\lambda' + a_1^3 a_2 \lambda'^3\mu') \\
& + 4(a_1^3 a_2 b_3^3 b_1 \mu^3\nu + a_2^3 a_3 b_1^3 b_2 \nu^3\lambda + a_3^3 a_1 b_2^3 b_3 \lambda^3\mu) \\
& + 4(b_1^3 b_2 a_3^3 a_1 \mu^3\nu + b_2^3 b_3 a_1^3 a_2 \nu^3\lambda + b_3^3 b_1 a_2^3 a_3 \lambda^3\mu) \\
= {} & 3\{\lambda^2\lambda'^2 + \mu^2\mu'^2 + \nu^2\nu'^2 - 2(\mu\mu'\nu\nu' + \nu\nu'\lambda\lambda' + \lambda\lambda'\mu\mu')\} \\
& + 6\tau(\lambda^2\mu'\nu + \mu^2\nu'\lambda + \nu^2\lambda'\mu) + 4(\mu'^3\nu' + \nu'^3\lambda' + \lambda'^3\mu') + 2(\mu^3\nu + \nu^3\lambda + \lambda^3\mu).
\end{aligned}
$$

If this is subtracted from the vanishing expression $3(\lambda\lambda' + \mu\mu' + \nu\nu')^2$ and the result divided by 2 we obtain, for the equation of the equianharmonic complex of $F^4 = 0$,

$$
\begin{aligned}
& 6(\mu\mu'\nu\nu' + \nu\nu'\lambda\lambda' + \lambda\lambda'\mu\mu') \\
& \qquad = 3\tau(\lambda^2\mu'\nu + \mu^2\nu'\lambda + \nu^2\lambda'\mu) + 2(\mu'^3\nu' + \nu'^3\lambda' + \lambda'^3\mu') + \mu^3\nu + \nu^3\lambda + \lambda^3\mu.
\end{aligned}
$$

The lines of this complex are represented by the intersection of Ω with the primal

$$S\{6\, y_2 y_3 y_5 y_6 - 3\tau y_2 y_3^2 y_4 - 2 y_5^3 y_6 - y_2^3 y_3\} = 0,$$

$$2 I^4 + J^4 = 0.$$

The interchange of dashed and undashed Plücker coordinates gives a second covariant quartic complex, namely that formed by lines which have the property that the four planes which pass through them and belong to Gundelfinger's contravariant envelope form an equianharmonic set. This complex is represented by the intersection of Ω with the primal

$$I^4 + 2 J^4 = 0.$$

These results tell us that the points of Ω which lie on the surface $I^2 = I^4 = J^4 = 0$ represent those lines having the double property that the four points of F^4 which lie on them form an equianharmonic range and, at the same time, the four planes of ϕ^4 which pass through them form an equianharmonic pencil.

Such lines make up a congruence, and the generators both of R^8 and of H^8 are (at least) double lines of it.

As for the pencil of covariant cubic complexes, so for the pencil of covariant quartic complexes; the complexes of the pencil are polar reciprocals of one another with respect to the quadrics Q, while two members of the pencil, namely $I^4 \pm J^4 = 0$, are self-reciprocal.