NOTES ON A NET OF QUADRIC SURFACES: (V) THE PENTAHEDRAL NET

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Introduction.

The four preceding Notes of this series* have, save for a few paragraphs at the end of Note IV, been concerned with a general net of quadric surfaces; in them enough properties of the general net have been found to justify an examination of special nets, and it may be hoped that, from this standpoint, these special nets will be seen in their proper perspective, more of their properties discovered, and those of their properties which are already known better appreciated. It would be a long undertaking to consider all the different specialisations of a net of quadrics, and the importance of some of these would be small and out of all proportion to the labour involved in investigating them thoroughly. Some few of the special nets, however, would well repay study, and of these the pentahedral net is outstanding.

Contrary to what was first believed, a net of quadric surfaces does not, in general, possess any self-conjugate pentahedron. Reye showed that the existence of such a pentahedron demanded a particularisation of the net†, and that the condition thus imposed was poristic, the existence of one such pentahedron involving that of a singly-infinite set; the resulting net of quadrics may therefore fittingly be described as *pentahedral*. The faces of the pentahedra all belong to a developable ω of the third class, and so osculate a twisted cubic γ; each plane of ω belongs to one and

* (I) "The Cremona transformation", *Proc. London Math. Soc.* (2), 43 (1937), 302–315. (II) "Anharmonic covariants", *Journal London Math. Soc.*, 12 (1937), 276–280. (III) "The scroll of trisecants of the Jacobian curve", *Proc. London Math. Soc.* (2), 44 (1938), 466–480. (IV) "Combinantal covariants of low order", *Proc. London Math. Soc.* (2), 47 (1941), 123–141.

† *Journal für Math.*, 82 (1877), 75. Reye proved that a net of quadric envelopes does not, in general, possess a self-conjugate pentad of points.

only one of the pentahedra, which therefore constitute the sets of a linear series g_5^1 among the planes of ω or, what is the same thing, the sets of osculating planes of γ at the pentads of a g_5^1 on the curve.

It was suggested a few years ago* that it is preferable to define the pentahedral net of quadrics in this way rather than in the way in which it originally presented itself—as the net of polar quadrics of the points of a plane with respect to a cubic surface. It is now proposed to pursue this suggestion further, and use it, in conjunction with the results obtained in the preceding Notes of the series, to give an exposition of the properties of the pentahedral net. This net will be denoted throughout by the symbol $\mathfrak{P}$, and the linear complex to which the tangents of γ belong by the symbol $\mathfrak{L}$.

Among the advantages of this approach to the study of the pentahedral net is one which is exploited in the third section (§§ 13–22) of this Note. A g_5^1 on a rational curve is given algebraically by a pencil of binary quintics. Any condition which is imposed on this pencil must be reflected in some corresponding particularisation of the net of quadrics; in some further particularisation, that is, in addition to the one to which it has already been subjected in order to be pentahedral. For our purpose any particularisation of the g_5^1 is admissible so long as there is no member common to all its sets; *i.e.* so long as the quintics of the pencil do not have a common factor. One particular pencil of binary quintics, in some ways the simplest of all, is that constituted by those quintics which are linearly dependent on two fifth powers; and the special pentahedral net which arises from this pencil is studied in detail. It will be found, *inter alia*, that, while this net does not contain any plane-pairs, it has only two distinct base points, each of which must be counted four times to make up the eight which a net of quadrics in general possesses; that the Jacobian curve breaks up into two twisted cubics and its trisecant scroll into two quartic scrolls. The simplification of the algebra consequent upon the specialisation of the g_5^1 allows forms for some concomitants to be obtained which do not yield themselves so readily for a general pentahedral net; in particular an explicit expression is found for Gundelfinger's contravariant.

The first section of the Note (§§ 1–8) is concerned to set up the configuration and obtain equations for some of the loci which play principal parts in it. In § 1 and § 2 equations are given for the quadrics of $\mathfrak{P}$ and for their Jacobian curve ϑ, these being the particular instances in three

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* *Proc. Edinburgh Math. Soc.* (2), 4 (1936), 185. Some references to papers concerned with the pentahedral net are given there.

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dimensions of equations that have previously been given for n dimensions. In § 3 the plane equation of the scroll generated by the trisecants of ϑ is obtained. In § 4 certain apolarity relations between sets of points on γ are described; these are used, in § 5, to obtain a quartic surface combinantally covariant for $\mathfrak{P}$ (the equation of another such surface is given in passing), and, in § 6, to establish a (1, 1) correspondence between the quadrics of $\mathfrak{P}$ and the axes of ω. At the end of § 6 a further quartic covariant appears. § 7 is concerned with the base points of the net, while the geometry in § 8 is preparatory to the second section as well as an ending for the first.

In the second section (§§ 9–12) the net $\mathfrak{P}$ is considered in the light of some of the results of Notes III and IV. It will be remembered that, in Note III, a five-dimensional configuration consisting of a quadric and two Veronese surfaces was encountered; this configuration is markedly particularised for the net $\mathfrak{P}$, and is described in outline in § 9 and § 10. In § 11 the net of quadric envelopes is identified which is the special instance, for $\mathfrak{P}$, of the contravariant net discovered by W. P. Milne, and the geometrical identification which we give is corroborated by using Milne's canonical form. In § 12 the special forms assumed, for the pentahedral net, by one or two of its covariants are given.

The classical results in the geometry of the twisted cubic, and of the linear complex associated with it, are assumed throughout.

I.

1. The plane whose equation is

$$P \equiv x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3 = 0,$$

varying with the parameter θ, belongs to a developable ω of class 3, and osculates a twisted cubic γ.

Take any two sets of five of these planes; the parameters of the five planes of the first set are the roots of a quintic

$$f(\theta) \equiv a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 = 0,$$

while the second set of planes is similarly determined by a quintic equation

$$g(\theta) \equiv b_0 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4 + b_5\theta^5 = 0.$$

Then, as the ratio $\lambda : \mu$ varies, the equation

$$\lambda f(\theta) + \mu g(\theta) = 0 \qquad (1)$$

determines an involution, or $g_5{}^1$, of pentahedra whose faces are planes of ω.

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These pentahedra are known to be self-conjugate for the quadrics of a net $\mathfrak{P}$; if the roots of $f(\theta)=0$ are $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$, then $\mathfrak{P}$ is the net determined by the three quadrics

$$Q_0 \equiv \sum_{i=1}^{5} \frac{P_i^2}{f'(\theta_i)\,g(\theta_i)}=0, \quad Q_1 \equiv \sum_{i=1}^{5} \frac{\theta_i P_i^2}{f'(\theta_i)\,g(\theta_i)}=0, \quad Q_2 \equiv \sum_{i=1}^{5} \frac{\theta_i^2 P_i^2}{f'(\theta_i)\,g(\theta_i)}=0, \quad (2)$$

where $$P_i \equiv x_0+\theta_i x_1+\theta_i^2 x_2+\theta_i^3 x_3.$$

The left-hand sides of the equations of these quadrics are functions of the roots of $f(\theta)=0$, these roots being the parameters of the faces of a definite pentahedron. But if the corresponding functions of the roots of any other of the quintic equations (1) are taken, the three quadrics so obtained are always* the quadrics (2). Every quadric of $\mathfrak{P}$ is linearly dependent on the quadrics (2), and is outpolar to every quadric inscribed in ω.

2. Each pentahedron is self-conjugate for each quadric of $\mathfrak{P}$, so that the polar plane of the vertex of a pentahedron with respect to any quadric of the net passes through the opposite edge of this pentahedron. Whence the Jacobian curve ϑ of $\mathfrak{P}$ is the locus of vertices of the pentahedra, while the scroll R^8 of trisecants of ϑ is generated by their edges.

Suppose then that x is a point of ϑ; the three planes of ω which pass through x all belong to the same pentahedron; hence there exist constants $\lambda, \mu, \alpha, \beta, \gamma$ such that

$$\lambda f(\theta)+\mu g(\theta) \equiv (x_0+\theta x_1+\theta^2 x_2+\theta^3 x_3)(\alpha+\beta\theta+\gamma\theta^2).$$

Equating the different powers of θ on the two sides of this identity, and eliminating the five constants from the six linear equations so arising, we have, for the equations of ϑ,

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ x_0 & x_1 & x_2 & x_3 & \cdot & \cdot \\ \cdot & x_0 & x_1 & x_2 & x_3 & \cdot \\ \cdot & \cdot & x_0 & x_1 & x_2 & x_3 \end{matrix} \right\| = 0.$$

* These results are included in the more general results for space of any number of dimensions which are obtained in the paper, already referred to, in the *Proc. Edinburgh Math. Soc.*

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An alternative form for the equations of ϑ, obtained directly as the Jacobian of Q_0, Q_1, Q_2, is

$$\left\| \begin{matrix} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \end{matrix} \right\| = 0,$$

where
$$\sigma_p = \sum_{i=1}^{5} \frac{\theta_i^p P_i}{f'(\theta_i)\, g(\theta_i)}.$$

3. It is also easy to obtain a form for the plane equation of R^8. For if the plane
$$l_0 x_0 + l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$
touches R^8, it contains an edge of one of the pentahedra, and so has an equation of the form

$$\kappa_i(x_0 + \phi_i x_1 + \phi_i^2 x_2 + \phi_i^3 x_3) + \kappa_j(x_0 + \phi_j x_1 + \phi_j^2 x_2 + \phi_j^3 x_3) = 0,$$

where both ϕ_i and ϕ_j are roots of an equation $\lambda f(\theta) + \mu g(\theta) = 0$. Thus, since

$$l_0 = \kappa_i + \kappa_j, \quad l_1 = \kappa_i \phi_i + \kappa_j \phi_j, \quad l_2 = \kappa_i \phi_i^2 + \kappa_j \phi_j^2, \quad l_3 = \kappa_i \phi_i^3 + \kappa_j \phi_j^3,$$

we have
$$1 : \phi_i + \phi_j : \phi_i \phi_j = L_0 : -L_1 : L_2,$$

where
$$L_0 \equiv l_0 l_2 - l_1^2, \quad L_1 \equiv l_1 l_2 - l_0 l_3, \quad L_2 \equiv l_1 l_3 - l_2^2.$$

Hence there must be constants λ, μ, α, β, γ, δ such that

$$\lambda f(\theta) + \mu g(\theta) \equiv (L_2 + L_1 \theta + L_0 \theta^2)(\alpha + \beta\theta + \gamma\theta^2 + \delta\theta^3).$$

Equating the coefficients of the different powers of θ on the two sides of this identity, and eliminating the six constants from the six equations which arise, we find

$$\begin{vmatrix} a_0 & b_0 & L_2 & . & . & . \\ a_1 & b_1 & L_1 & L_2 & . & . \\ a_2 & b_2 & L_0 & L_1 & L_2 & . \\ a_3 & b_3 & . & L_0 & L_1 & L_2 \\ a_4 & b_4 & . & . & L_0 & L_1 \\ a_5 & b_5 & . & . & . & L_0 \end{vmatrix} = 0.$$

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This form of the plane equation of R^8 shows most clearly a particular feature of the pentahedral net, namely, that every plane of ω is a quadritangent plane of R^8. For the planes of ω satisfy the equations

$$L_0 = L_1 = L_2 = 0,$$

and, when these are simultaneously satisfied, the determinant has rank two only.

4. The quadrics of $\mathfrak{P}$ cut out a g_6^2 on γ, the six points of the set cut out by the quadric $\xi Q_0 + \eta Q_1 + \zeta Q_2 = 0$ being given by

$$\sum_{i=1}^{5} \frac{\xi + \eta\theta_i + \zeta\theta_i^2}{f'(\theta_i)\, g(\theta_i)} (\theta - \theta_i)^6 = 0.$$

This sextic polynomial, being a linear combination of the five sixth powers $(\theta-\theta_i)^6$, is apolar to the product of the five factors $\theta-\theta_i$; it follows similarly, from the other forms for the equations of the quadrics, that it is apolar to every quintic giving a set of the g_5^1. Thus the g_6^2 is particularised since, in general, there is no quintic apolar to all the sets of a g_6^2. Now those binary quintics which are apolar to the binary quintics of a given pencil constitute a triply infinite set, so that the g_5^1 has an apolar g_5^3. Moreover the sets of any g_5^3 consist of the third polars of a definite octavic, the only exception to this being when the g_5^3 includes a set which consists of a single element counted five times over*. But this exception cannot arise here, since such an element would belong to every set of the original g_5^1, and it is supposed always that these sets do not have a common element; thus this unique octavic always exists for a pentahedral net. Its second polars are then apolar to g_5^1, and constitute the g_6^2 cut out by the quadrics of $\mathfrak{P}$ on γ. The particularisation of the g_6^2 therefore consists in its property, not possessed by a general g_6^2, of being the set of second polars of a binary octavic. This has recently been pointed out by B. Ramamurti†, and we will call this octavic by his name. If Ramamurti's octavic, written with the usual binomial coefficients, is

$$(A, B, C, D, E, F, G, H, K \between \theta, 1)^8,$$

* Grace, *Proc. London Math. Soc.* (2), 28 (1928), 423–425.

† *Proc. Indian Academy of Sciences*, 9 (1939), 316.

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then the conditions of apolarity are expressed by

$$\begin{pmatrix} a_0 & -a_1 & a_2 & -a_3 & a_4 & -a_5 \\ b_0 & -b_1 & b_2 & -b_3 & b_4 & -b_5 \end{pmatrix} \begin{pmatrix} A & B & C & D \\ B & C & D & E \\ C & D & E & F \\ D & E & F & G \\ E & F & G & H \\ F & G & H & K \end{pmatrix} = 0,$$

the mutual ratios of the nine coefficients of the octavic being uniquely determined by this set of eight homogeneous linear equations.

5. The g_6^2 on γ, with the corresponding g_6^2 of planes of ω, is covariant for $\mathfrak{P}$, and loci which are covariantly related to it must themselves be covariants of $\mathfrak{P}$; for instance, the g_6^2 of planes of ω yields a set of hexahedra whose vertices lie on a surface, while, reciprocating with respect to $\mathfrak{L}$, the hexads formed by the sets of the g_6^2 on γ give, as the envelope of the planes joining groups of three points in the same set, a contravariant of $\mathfrak{P}$.

The equations of this surface and envelope are obtainable forthwith. If a point lies on the surface, then, since it is a vertex of a hexahedron, those three planes of ω which pass through it must all belong to the same set of g_6^2; hence there must exist constants λ, μ, ν, α, β, γ, δ such that

$$\lambda\phi(\theta)+\mu\psi(\theta)+\nu\chi(\theta) \equiv (x_0+\theta x_1+\theta^2 x_2+\theta^3 x_3)(\alpha+\beta\theta+\gamma\theta^2+\delta\theta^3),$$

where $\phi(\theta)$, $\psi(\theta)$, $\chi(\theta)$ are any three linearly independent sextics belonging to the g_6^2. Equating the coefficients of the different powers of θ on the two sides of this identity, and eliminating the seven constants from the resulting seven equations, we obtain

$$\begin{vmatrix} A & B & C & . & . & . & x_3 \\ B & C & D & . & . & x_3 & x_2 \\ C & D & E & . & x_3 & x_2 & x_1 \\ D & E & F & x_3 & x_2 & x_1 & x_0 \\ E & F & G & x_2 & x_1 & x_0 & . \\ F & G & H & x_1 & x_0 & . & . \\ G & H & K & x_0 & . & . & . \end{vmatrix} = 0.$$

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This is, then, a quartic covariant of $\mathcal{P}$. Reciprocating with respect to $\mathcal{L}$ we obtain a contravariant, of class four, whose equation arises from the above determinant on replacing x_0, x_1, x_2, x_3 respectively by $-l_3, 3l_2, -3l_1, l_0$.

Another quartic surface which, as its definition shows, is a combinantal covariant of $\mathcal{P}$ is the locus of a point O such that the intersection, O', of its polar planes with respect to all the quadrics of $\mathcal{P}$ lies in its polar plane with respect to $\mathcal{L}$. This surface must contain the Jacobian curve and the eight base points of $\mathcal{P}$; its equation is

$$\begin{vmatrix} 3x_3 & -x_2 & x_1 & -3x_0 \\ \dfrac{\partial Q_0}{\partial x_0} & \dfrac{\partial Q_0}{\partial x_1} & \dfrac{\partial Q_0}{\partial x_2} & \dfrac{\partial Q_0}{\partial x_3} \\ \dfrac{\partial Q_1}{\partial x_0} & \dfrac{\partial Q_1}{\partial x_1} & \dfrac{\partial Q_1}{\partial x_2} & \dfrac{\partial Q_1}{\partial x_3} \\ \dfrac{\partial Q_2}{\partial x_0} & \dfrac{\partial Q_2}{\partial x_1} & \dfrac{\partial Q_2}{\partial x_2} & \dfrac{\partial Q_2}{\partial x_3} \end{vmatrix} = 0.$$

6. Each quadric of $\mathcal{P}$ cuts out on γ a set of six points which is the polar set of some pair of points on γ with respect to Ramamurti's octavic; the two planes which osculate γ at this pair of points intersect in an axis of ω. Conversely: given an axis of ω, the two planes of ω which pass through it determine a pair of points on γ; if Ramamurti's octavic is polarised by this pair of points a set of six points is obtained in which γ is met by a quadric of $\mathcal{P}$. Thus there is* a (1, 1) correspondence between the quadrics of $\mathcal{P}$ and the axes of ω.

Suppose that, to speak the language more appropriate to binary forms, we take for the moment the ratio $\theta:\theta'$ in place of the parameter θ. Then the sets of points in which γ is met by the three quadrics $Q_0 = 0$, $Q_1 = 0$, $Q_2 = 0$ respectively are, on referring to (2), seen to be given by

$$\sum_{i=1}^{5} a_i \theta_i'^2 (\theta\theta_i' - \theta'\theta_i)^6 = 0, \quad \sum_{i=1}^{5} a_i \theta_i \theta_i' (\theta\theta_i' - \theta'\theta_i)^6 = 0, \quad \sum_{i=1}^{5} a_i \theta_i^2 (\theta\theta_i' - \theta'\theta_i)^6 = 0,$$

* This correspondence has, under another aspect, been noticed before. When $\mathcal{P}$ is regarded as a net of polar quadrics of a cubic surface, the point of which a given quadric of $\mathcal{P}$ is the polar must lie on a definite line, namely that axis of ω which corresponds to the quadric. See Turnbull (who refers also to Töplitz), *Proc. Cambridge Phil. Soc.*, 31 (1935), 177–179. The argument used by Ramamurti on 318 of his paper is also relevant.

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where the a_i's are certain constants, the same in each of the three equations. The sextics on the left of these equations are second polars of an octavic r_θ^8, and their form shows that they must respectively be

$$\frac{\partial^2 r}{\partial \theta^2}, \quad -\frac{\partial^2 r}{\partial \theta \, \partial \theta'}, \quad \frac{\partial^2 r}{\partial \theta'^2}.$$

The sextic which gives the six points cut out on γ by the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ is therefore

$$\xi \frac{\partial^2 r}{\partial \theta^2}-\eta \frac{\partial^2 r}{\partial \theta \, \partial \theta'}+\zeta \frac{\partial^2 r}{\partial \theta'^2},$$

and this is obtained by polarising r with the pair of points whose parameters are the roots of the quadratic $\xi\theta'^2+\eta\theta\theta'+\zeta\theta^2=0$.

So, returning to the non-homogeneous parameter, the (1, 1) correspondence between the quadrics of $\mathfrak{P}$ and the axes of ω is given by the fact that to the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ there corresponds that axis of ω which is the line common to those two planes whose parameters satisfy $\xi+\eta\theta+\zeta\theta^2=0$.

To this result two corollaries may be appended.

Take an edge e of one of the pentahedra; it is an axis of ω, and as such corresponds to a quadric of $\mathfrak{P}$. The intersections of this quadric with γ are obtained on polarising Ramamurti's octavic by means of two points belonging to the same set of g_5^1. It may be supposed that the quintic $f(\theta)$, whose five roots θ_i appear in (2), *is the one which gives this set, so that the group of intersections is obtained by polarisation of the octavic with two of these roots, say θ_1 and θ_2. These are the roots of the quadratic $\theta_1\theta_2-\theta(\theta_1+\theta_2)+\theta^2=0$, so that the quadric of $\mathfrak{P}$ which corresponds to e is $\theta_1\theta_2 Q_0-(\theta_1+\theta_2)Q_1+Q_2=0$; this is a linear combination of P_3^2, P_4^2, P_5^2, being without the terms in P_1^2 and P_2^2; hence it is that cone of $\mathfrak{P}$ whose vertex is that vertex of the pentahedron opposite to e. The correspondence between quadrics of $\mathfrak{P}$ and axes of ω is therefore such that to the cones of $\mathfrak{P}$ there correspond those trisecants of ϑ which are respectively conjugate to the vertices of the cones.*

Secondly: *those quadrics of $\mathfrak{P}$ which correspond to tangents of γ are such that the quadratic $\xi+\eta\theta+\zeta\theta^2=0$ has equal roots; hence they are those quadrics $\xi Q_0+\eta Q_1+\zeta Q_2=0$ for which $\eta^2=4\zeta\xi$. The envelope of these*

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quadrics is the octadic surface* $Q_1^2 = Q_0 Q_2$; *it is a covariant of the net of quadrics, and is the third covariant quartic surface that we have obtained.*

7. No mention seems ever to have been made of the base points of the pentahedral net; they may be found as follows.

The identity $\sum_{i=1}^{5} \frac{\theta_i^p}{f'(\theta_i)} \equiv 0$ is true for $p = 0, 1, 2, 3$. Hence, if it is possible to find constants, κ and ρ say, such that

$$P_i^2 = (\kappa + \rho\theta_i)\, g(\theta_i)$$

for each of the five values of i, the equations (2) will all be satisfied. But this demands that the equation

$$(x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3)^2 = (\kappa + \rho\theta)\, g(\theta)$$

should be satisfied by each of the five roots of $f(\theta) = 0$, and hence there will be, corresponding to each base point, an identity

$$(x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3)^2 \equiv (\kappa + \rho\theta)\, g(\theta) + (\sigma + \tau\theta) f(\theta), \tag{3}$$

where σ, τ are two more constants. The form of this identity shows that its existence or non-existence depends on the g_5^1 as a whole, and not on the choice of some particular set belonging to it.

The existence of such identities is best established by consideration of linear series. The planes of the pentahedra osculate γ at points forming sets of a g_5^1; those sets of six points which consists of a set of this g_5^1 and a point of γ constitute a (non-linear) doubly-infinite aggregate which belongs to that g_6^3 whose different sets are found by equating the right-hand side of (3) to zero and varying the constants κ, ρ, σ, τ. If it is possible to choose these constants so that the right-hand side of (3) becomes a perfect square, then the g_6^3 contains a corresponding set of points which consists of three points of γ, taken twice over. The corresponding base point (x_0, x_1, x_2, x_3) of $\mathfrak{P}$ is then the intersection of the osculating planes of γ at these three points. Thus it is to be expected that the g_6^3 contains eight such double sets of three points, and this expectation is soon confirmed. If γ is transformed, by means of the g_6^3,

* See the paper "Octadic surfaces and plane quartic curves", *Proc. London Math. Soc.* (2), 34 (1932) 492–525.

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into a rational twisted sextic s, the sets of the g_6^3 become the sections of s by the planes of the space in which it lies, and the number of double sets of these points is therefore the number of planes which are tritangent to s. Known formulae show, however, that s has precisely eight tritangent planes. It is true that s is not a general twisted sextic, since the g_6^3 is not itself general; through every point of s there pass the planes of a pencil cutting out a g_5^1, each set of which may be obtained in the same way from every point of s; thus s lies on a quadric, meeting the lines of one regulus each in one point and the lines of the complementary regulus each in five points. But this specialisation of s does not affect the validity of the formula for the number of its tritangent planes.

The g_6^3 is that constituted by those sextics which are apolar to all the sextics of the g_6^2 cut out on γ by the quadrics of $\mathfrak{P}$; for each member of the g_6^2 is apolar to any quintic of the g_5^1, and so to every sextic of which this quintic is **a** factor.

8. Consider now the relation between the surface R^8, generated by the edges of the pentahedra, and the developable surface F^4 generated by the tangents of γ.

A pentahedron consists of those five planes which osculate γ at the points of a set of the g_5^1. Were two of these planes to meet in a tangent of γ, the corresponding set of g_5^1 would have a double point at the point of contact of this tangent. Conversely: if a set of g_5^1 has a double point, the osculating plane of γ at this point counts for two among the faces of the pentahedron to which it belongs, and the tangent of γ there is an edge of this pentahedron and so a generator of R^8. Thus R^8 and F^4 have eight common generators, namely the tangents of γ at the points of the Jacobian set J of g_5^1.

Having shown that R^8 and F^4 have these eight common generators, let us consider the residual curve in which they meet. Any generator of R^8 is common to two planes of ω, and an intersection of this generator with F^4 lies on a tangent of γ. But through a point on a tangent of γ there passes only one plane of ω other than that in which the tangent itself lies; hence the only tangents of γ which the generator of R^8 can meet are those which lie in the two planes of ω intersecting in the generator. Thus the generators of R^8 are bitangents of F^4. On the other hand, let t be a tangent of γ at a point not belonging to J. Through a point of t there passes only one plane of ω other than that in which t itself lies; hence, since a generator of R^8 is the intersection of two planes of ω which belong to the same set of g_5^1, R^8 cannot meet t except in the four points

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where t is met by those planes of ω which are the four remaining planes of that set of g_5^1 to which the plane of ω through t belongs. Conversely, each of these four points does lie on R^8. Hence t is a quadritangent line of R^8.

Thus, apart from their eight common generators, R^8 and F^4 touch wherever they meet.

Those four generators of R^8 on which lie its points of contact with a generator of F^4 are, as has just been shown, the four lines in which one face of a pentahedron is met by its remaining four faces; they are conjugate to the vertices of the tetrahedron formed by these remaining four faces. Now the equations (2) show clearly that the tetrahedron which is obtained by omitting any face of one of the pentahedra is self-conjugate for a pencil of quadrics of $\mathfrak{P}$, and so its vertices form a canonical set on ϑ. Thus those four generators of R^8 on which lie its points of contact with any generator of F^4 form a canonical set of generators of R^8. Now it was shown in Note III that, of the ∞^2 canonical sets of generators of R^8, ∞^1 are such that one of their singly-infinite set of transversal lines is a quadritangent line of R^8; and it was further shown that these quadritangent lines of R^8 generate a scroll ρ^8. Hence, for the net $\mathfrak{P}$, the covariant ρ^8 must include the scroll F^4; it will indeed be seen that ρ^8 is simply F^4 itself, counted twice over.

II.

9. It was shown in Note III that, when the lines of [3] are represented by the points of a quadric Ω in [5], the generators of R^8 are represented by the curve in which Ω is met by a Veronese surface.

For the net $\mathfrak{P}$ it is seen immediately that the curve on Ω which represents R^8 lies on a Veronese surface v; for the generators of R^8 are all axes of ω, and it is known* that the axes of a cubic developable are represented on Ω by the points of a Veronese surface. But now, since *all* points of this surface v represent lines of [3], *v lies entirely on* Ω. Here, then, is a property which sharply distinguishes the pentahedral net $\mathfrak{P}$ from a general net; the expression $\Xi' M' A M \Xi$ of Note III must, in consequence of the form of the matrix M which corresponds to a pentahedral net, vanish identically. The fact that v lies entirely on Ω, however, in no way prevents it from being reciprocated with respect to Ω,

* Baker, *Principles of geometry*, 4 (Cambridge, 1925), 52.

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and so a second Veronese surface w, the locus of points whose polar primes with respect to Ω touch v along conics, is obtained as in Note III.

10. An axis of ω is the intersection of two planes

$$x_0+\phi_1 x_1+\phi_1^2 x_2+\phi_1^3 x_3=0,$$

$$x_0+\phi_2 x_1+\phi_2^2 x_2+\phi_2^3 x_3=0,$$

so that its line coordinates are

$$\{\phi_1\phi_2,\ -(\phi_1+\phi_2),\ 1,\ -(\phi_1^2+\phi_1\phi_2+\phi_2^2),\ -\phi_1\phi_2(\phi_1+\phi_2),\ -\phi_1^2\phi_2^2\}. \quad (4)$$

If the quadratic of which ϕ_1 and ϕ_2 are roots is $\xi+\eta\theta+\zeta\theta^2=0$, these are proportional to

$$\{\zeta\xi,\quad \eta\zeta,\quad \zeta^2,\quad \zeta\xi-\eta^2,\quad \xi\eta,\quad -\xi^2\}, \quad (5)$$

which are therefore the coordinates of that axis of ω which, in the manner described in § 6, corresponds to the quadric whose equation is

$$\xi Q_0+\eta Q_1+\zeta Q_2=0.$$

Since they are homogeneous quadratic polynomials in ξ, η, ζ the points in [5] which represent the axes of ω lie on a Veronese surface v. When ξ, η, ζ are so restricted that the quadric is a cone the axis of ω, as was proved in § 6, is a generator of R^8, and so the representative point in [5] describes, on v, the curve Γ which represents this scroll.

It was proved in Note III that the coordinates of a point in [5] which represents a generator of R^8 are given by

$$y=M[\xi^2,\ \eta^2,\ \zeta^2,\ \tau\eta\zeta,\ \tau\zeta\xi,\ \tau\xi\eta]'=M\Xi,$$

where M is a non-singular matrix and $\tau^2=2$. The set of coordinates (5), when written as a matrix of one column, is in agreement with this, and has

$$M=\begin{pmatrix} . & . & . & . & \tau^{-1} & . \\ . & . & . & \tau^{-1} & . & . \\ . & . & 1 & . & . & . \\ . & -1 & . & . & \tau^{-1} & . \\ . & . & . & . & . & \tau^{-1} \\ -1 & . & . & . & . & . \end{pmatrix}.$$

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The equation of Ω, the quadric in [5] whose points represent the lines of [3], is, here, $y_0 y_3 + y_1 y_4 + y_2 y_5 = 0$, which may be written $y'Ay = 0$ with

$$A = \begin{pmatrix} . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \end{pmatrix}.$$

These forms of M and A give

$$M'AM = \begin{pmatrix} . & . & -1 & . & . & . \\ . & . & . & . & -\tau^{-1} & . \\ -1 & . & . & . & . & . \\ . & . & . & . & . & \frac{1}{2} \\ . & -\tau^{-1} & . & . & 1 & . \\ . & . & . & \frac{1}{2} & . & . \end{pmatrix},$$

and it is now easily verified that $\Xi' M'AM \Xi$ vanishes identically, in conformity with the fact that v lies entirely on Ω.

It was also shown in Note III that the points of the Veronese surface w, being the poles with respect to Ω of those primes which touch v along conics, are given parametrically by

$$y = (M'A)^{-1} [l^2, m^2, n^2, \tau mn, \tau nl, \tau lm]'$$

$$= \begin{pmatrix} . & -1 & . & . & . & . \\ . & . & . & . & . & \tau \\ -1 & . & . & . & . & . \\ . & 1 & . & . & \tau & . \\ . & . & . & \tau & . & . \\ . & . & 1 & . & . & . \end{pmatrix} \begin{pmatrix} l^2 \\ m^2 \\ n^2 \\ \tau mn \\ \tau nl \\ \tau lm \end{pmatrix}$$

$$= [-m^2, 2lm, -l^2, m^2+2nl, 2mn, n^2]'.$$

This surface does not lie entirely on Ω; those of its points which do lie on Ω satisfy

$$-m^2(m^2+2nl) + (2lm)(2mn) - l^2 n^2 = 0,$$

$$(m^2-nl)^2 = 0.$$

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Thus Ω meets w twice in that curve Δ on w for which $m^2 = nl$; this is the section of w by the prime L whose equation is $3y_0+y_3 = 0$.

The intersection of L with v is seen, on referring to (5), to be given by taking $\eta^2 = 4\zeta\xi$. Indeed L meets both v and w in the same curve Δ, as is seen by taking

$$\xi:\eta:\zeta = n:-2m:l = \psi^2:2\psi:1;$$

for this gives the parametric form

$$(\psi^2,\ 2\psi,\ 1,\ -3\psi^2,\ 2\psi^3,\ -\psi^4)$$

for the coordinates of a point of Δ. Any such point represents a tangent of γ, for it is obtained by putting $\phi_1 = \phi_2 = -\psi$ in (4).

It was seen in Note III that the curve of intersection of w and Ω was, for a general net of quadrics, of the eighth order, and that it represented the scroll ρ^8 generated by quadritangent lines of R^8. Thus, as was anticipated at the end of § 8, ρ^8 becomes, for the pentahedral net, F^4 taken twice.

11. Let us now consider the net $\mathcal{P}$ from the standpoint adopted in Note IV. It was there pointed out that, for a general net of quadrics, R^8 has eight tritangent planes; these are associated, and so determine a net ν of quadric envelopes. For $\mathcal{P}$, however, R^8 has not merely eight, but an infinite number of tritangent planes, and, indeed, of quadritangent planes, namely the planes of ω. Thus, in order to identify for $\mathcal{P}$ those eight planes which correspond to the tritangent planes of R^8 for a general net of quadrics, some further property would have to be appealed to. This has been done previously*; but it is not necessary to know the result in order to identify the net which, for $\mathcal{P}$, plays the part of the net ν. For every tritangent plane of R^8 is now (a quadritangent plane and) a plane of ω, so that every quadric envelope to which any eight tritangent planes of R^8 belong is inscribed in ω. The net ν is therefore now the net ϖ of quadrics which are inscribed in ω.

The existence of the net ν, as acknowledged in Note IV, was established by W. P. Milne†; moreover Milne gives a set of equations for three linearly independent envelopes $Q = 0$, $R = 0$, $S = 0$ of ν. The condition for $\kappa Q+\rho R+\sigma S = 0$ to consist of planes which pass through the tangents

* The eight planes are those which constitute the Jacobian set of the $g_5{}^1$ given by the pentahedra; see *Proc. London Math. Soc.* (2), 34 (1932), 521–522.

† *Journal London Math. Soc.*, 8 (1933), 211–216.

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of a conic is that the discriminant D of $\kappa Q+\rho R+\sigma S$ should vanish. The sets of values of $\kappa:\rho:\sigma$ for which this happens are in (1, 1) correspondence with the planes of the conics; these planes form, in general, a developable of the sixth class. When the coefficients, in the equations of the quadrics of ν, are so specialised that ν becomes a net ϖ of quadrics inscribed in a cubic developable, the envelope of the planes of the conics, instead of being of the sixth class, is the cubic developable taken twice over; D must become a perfect square. This can be verified for Milne's canonical form. For this canonical form is derived from a canonical form for a net of quadric loci, and, from what has been said above, when the coefficients in the equations of these loci are so chosen that they yield a pentahedral net, ν must thereby become a net ϖ. Now the condition for Milne's equations (1) to represent a pentahedral net is easily shown to be $a_2 b_3 c_1 = a_3 b_1 c_2$; and, *when this is satisfied*, the discriminant of the quadratic form $\kappa Q+\rho R+\sigma S$ is found, in Milne's notation, to be the square of the expression

$$k[\rho\sigma\{ka_2 a_3(c_2 r-b_3 s)+b_1 c_1(a_2^2 c_2-a_3^2 b_3)\}$$
$$+\sigma\kappa\{kb_3 b_1(a_3 s-c_1 q)+c_2 a_2(b_3^2 a_3-b_1^2 c_1)\}$$
$$+\kappa\rho\{kc_1 c_2(b_1 q-a_2 r)+a_3 b_3(c_1^2 b_1-c_2^2 a_2)\}].$$

12. After what has been written in Note IV, some of the concomitants of $\mathfrak{P}$ may be readily identified.

The surface S^8, generated by the conics belonging to ν, is here the surface generated by the conics belonging to ϖ; it is known, from the geometry of the twisted cubic, that these conics all lie on F^4. Hence, for the pentahedral net, S^8 is the surface F^4 taken twice.

The quartic surface G^4, dual to Gundelfinger's contravariant, contains the eight curves along which S^8 is touched by the base planes of ν. Thus G^4 now contains eight quartic curves in which F^4 is met by planes of ω, and must therefore be the same surface as F^4. The property, possessed by G^4, of having all the twenty-eight lines of intersection of pairs of base planes of ν for bitangents, is quickly verified. For the curve in which F^4 is met by a plane of ω consists of a tangent of γ and a conic, the tangent of γ counting twice because the plane touches F^4 all along it. The remaining planes of ω meet this one in the tangents of the conic, which are all bitangents of the composite quartic curve in which the plane meets F^4.

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The identification of Gundelfinger's contravariant itself is not so simple; but it is obtained below for the specialised form of $\mathfrak{P}$ to the consideration of which we now proceed.

III.

13. A pentahedral net is determined when a twisted cubic γ and a g_5^1 thereon have been assigned; and numerous specialisations of $\mathfrak{P}$ arise corresponding to the different special types of g_5^1. Let us then take what is, in one way, the extreme specialisation; namely, a g_5^1 two of whose sets consist of single points counted five times; each other set of g_5^1 is linearly dependent on these two. Using the notation and coordinate system of the earlier part of this Note, and choosing the parameter θ to have the values 0 and ∞ at the two points, we take

$$f(\theta) \equiv \theta^5, \qquad g(\theta) \equiv 1.$$

The expressions appearing in the equations (2) of § 1 then become, with ϵ denoting a primitive fifth root of unity,

$$\left.\begin{aligned} Q_0 &\equiv \sum_{i=1}^{5} \frac{P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{4i}} \equiv x_2^2+2x_1x_3; \\ Q_1 &\equiv \sum_{i=1}^{5} \frac{\theta_i P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{3i}} \equiv 2x_0x_3+2x_1x_2; \\ Q_2 &\equiv \sum_{i=1}^{5} \frac{\theta_i^2 P_i^2}{5\theta_i^4} \equiv \sum_{i=1}^{5} \frac{(x_0+\epsilon^i x_1+\epsilon^{2i}x_2+\epsilon^{3i}x_3)^2}{5\epsilon^{2i}} \equiv x_1^2+2x_0x_2. \end{aligned}\right\} \qquad (6)$$

Here the five roots θ_i may be the parameters of any set of the g_5^1 which consists of five distinct points, and so the roots of any equation $\lambda\theta^5+\mu = 0$ which has five distinct roots; we have therefore taken them to be the roots of $\theta^5 = 1$. The net determined by (6) will be denoted by the symbol $\wp$.

If the coordinates of a point on γ had been given the homogeneous form $(\theta^3, -3\theta^2\theta', 3\theta\theta'^2, -\theta'^3)$, then the g_5^1 would be determined by the two binary quintics

$$f(\theta, \theta') \equiv \theta^5, \quad g(\theta, \theta') \equiv \theta'^5.$$

Ramamurti's octavic, apolar both to $f(\theta, \theta')$ and $g(\theta, \theta')$, is then*

$$r_\theta^8 \equiv \theta^4\theta'^4.$$

* For this special pencil of binary quintics, Ramamurti's octavic is the same as the Jacobian; this is due to the specialisation of the pencil, and is not true in general.

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The $g_6{}^2$ cut out by the quadrics on γ must be the set of second polars of r, and, from the forms obtained above for Q_0, Q_1, Q_2 it is seen that a point of γ lies on the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ provided that its parameter satisfies

$$\theta^2\theta'^2(15\xi\theta'^2-20\eta\theta\theta'+15\zeta\theta^2)=0.$$

And this sextic is obtainable by polarisation from r, being in fact

$$\tfrac{5}{4}\left(\xi\frac{\partial^2 r}{\partial\theta'^2}-\eta\frac{\partial^2 r}{\partial\theta\,\partial\theta'}+\zeta\frac{\partial^2 r}{\partial\theta^2}\right),$$

in agreement with § 6. Thus, for the net $\wp$, the $g_6{}^2$ cut out by the quadrics on γ has four fixed points, namely the two vertices X_0 and X_3 of the tetrahedron of reference each counted twice; and the sets of $g_6{}^2$ are got by adding the pairs of points of γ to this fixed group of four.

14. That X_0 and X_3 are both base points of $\wp$ is obvious from (6). It may be confirmed, and more completely explained, by the considerations of § 7. For the $g_6{}^3$ which is apolar to the $g_6{}^2$ cut out on γ by the quadrics is that one of which four sets are determined by the sextics θ^6, $\theta^5\theta'$, $\theta\theta'^5$, θ'^6. The only linear combinations of these which are perfect squares are θ^6 and θ'^6; there are no others, and these two must count each for four of the eight which would arise, as explained in § 7, for a more general $g_6{}^3$. Hence *the base points of $\wp$ coincide four at each of two points*, namely X_0 and X_3.

Conversely: the argument of § 7 shows that if $\mathfrak{P}$ has a base point on γ this point, when counted six times, makes up a set of the $g_6{}^3$; it then counts for four among the eight sets of repeated triads which a general $g_6{}^3$ possesses, and for four of the eight base points of $\mathfrak{P}$.

15. Let us now address ourselves to the study of the Jacobian curve ϑ and of the scroll R^8 which is generated by its trisecants. These break up, each into two component parts. The plane equation of R^8 is, in the notation of § 3,

$$\begin{vmatrix} L_1 & L_2 & . & . \\ L_0 & L_1 & L_2 & . \\ . & L_0 & L_1 & L_2 \\ . & . & L_0 & L_1 \end{vmatrix}=0,$$

that is $$L_1{}^4-3L_0L_2L_1{}^2+L_0{}^2L_2{}^2=0,$$

and so it factorises. Thus, for the net $\wp$, R^8 consists of two quartic scrolls,

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Since the quadric envelopes

$$L_0 = 0, \quad L_1 = 0, \quad L_2 = 0,$$

are all inscribed in ω, each quartic scroll has ω for its bitangent developable; this is in accordance with the fact that every plane of ω is a quadritangent plane of R^8. Moreover, since the simultaneous interchange of l_0 with l_3 and l_1 with l_2 interchanges L_0 with L_2 and leaves L_1 unaltered, it is seen that each quartic scroll is symmetrically related to the two base points.

It is now to be expected that the sextic curve ϑ, whose trisecants generate R^8, will also break up; the presumption is that ϑ will consist of two twisted cubics, those chords of either which meet the other being trisecants of the composite curve and generating a quartic scroll. In order that these two scrolls should each be of the fourth order, it will be necessary for the two cubics to have four intersections.

These expectations are easily confirmed. The equations of ϑ are

$$\left\| \begin{matrix} 0 & x_3 & x_2 & x_1 \\ x_3 & x_2 & x_1 & x_0 \\ x_2 & x_1 & x_0 & 0 \end{matrix} \right\| = 0;$$

it is thus the intersection, apart from the twisted cubic $x_0/x_1 = x_1/x_2 = x_2/x_3$, of the two surfaces

$$2x_0 x_1 x_2 = x_1^3 + x_0^2 x_3 \quad \text{and} \quad 2x_1 x_2 x_3 = x_2^3 + x_3^2 x_0;$$

these are Cayley scrolls, their nodal lines being, respectively, $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$, the tangents of γ at X_3 and X_0. They intersect, apart from the twisted cubic which has already been excluded, in the two twisted cubics

$$\frac{qx_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{qx_3} \quad \text{and} \quad \frac{x_0}{qx_1} = -\frac{x_1}{x_2} = \frac{qx_2}{x_3},$$

where q is written for $\frac{1}{2}(1+\sqrt{5})$, so that $q^2 = q+1$; the two curves are obtained, each from the other, on replacing q by* $-q^{-1} = \frac{1}{2}(1-\sqrt{5})$. They will be called γ_1 and γ_2 respectively; the coordinates of their points are given parametrically by

$$(1,\, q\theta,\, q\theta^2,\, \theta^3) \quad \text{and} \quad (1,\, -q^{-1}\theta,\, -q^{-1}\theta^2,\, \theta^3),$$

* The primitive fifth root ϵ may be chosen so that $\epsilon + \epsilon^4 = q^{-1}$ and $\epsilon^2 + \epsilon^3 = -q$.

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so that they both pass through X_0 and X_3, and have the same tangents and osculating planes there as does γ. Incidentally it is confirmed that γ_1 and γ_2 have four intersections, these being X_0 and X_3, each taken twice.

16. Those chords of γ_2 which meet γ_1 generate a quartic scroll Γ_2; two of them pass through each point of γ_2, which is therefore the nodal curve of Γ_2. Hence the equation of Γ_2 is obtained by equating to zero some homogeneous quadratic polynomial in the three expressions

$$x_0 x_2 + q x_1^2, \quad x_0 x_3 - q^2 x_1 x_2, \quad x_1 x_3 + q x_2^2;$$

for these, when equated to zero, give the linearly independent quadrics through γ_2. The quadratic polynomial is identified by the fact that it must vanish at every point of γ_1, and thus the equation of Γ_2 is found to be

$$(x_0 x_3 - q^2 x_1 x_2)^2 = (x_0 x_2 + q x_1^2)(x_1 x_3 + q x_2^2).$$

The equation of the scroll Γ_1, generated by those chords of γ_1 which meet γ_2, is obtained from this on replacing q by $-q^{-1}$, and so is

$$(x_0 x_3 - q^{-2} x_1 x_2)^2 = (x_0 x_2 - q^{-1} x_1^2)(x_1 x_3 - q^{-1} x_2^2).$$

The equation of R^8 is found by taking the product of the equations of Γ_1 and Γ_2. The factorised form of the *plane* equation, found above, for R^8 is

$$(L_1^2 - q^2 L_0 L_2)(L_1^2 - q^{-2} L_0 L_2) = 0.$$

17. We now have a Jacobian curve consisting of two twisted cubics γ_1 and γ_2, and its trisecant scroll consisting of two quartic scrolls Γ_1 and Γ_2; the generators of Γ_1 are chords of γ_1 and secants of γ_2, while the generators of Γ_2 are chords of γ_2 and secants of γ_1.

For a general net of quadrics, any point of the Jacobian curve ϑ is conjugate to a trisecant of ϑ; thus the points of γ_1 are conjugate to the generators of one of the two scrolls Γ_1, Γ_2, while the points of γ_2 are conjugate to the generators of the other scroll. On examination it is seen that the scroll whose generators are conjugate to the points of γ_1 is Γ_1, while the points of γ_2 are conjugate to the generators of Γ_2. For consider any pentahedron which is self-polar for $\wp$; each face has three intersections with γ_1 and three with γ_2, the ten vertices lying five on γ_1 and five on γ_2. An edge of the pentahedron is a chord of one of the two curves and a secant of the other. Let an edge e be a chord of γ_1, and so meet γ_2 in one

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point. Any plane through e has one further intersection with γ_1 and two further intersections with γ_2; this applies, in particular, to either of the two faces of the pentahedron which intersect in e, so that these two faces together account for four points of γ_1 and five points of γ_2. The remaining vertex of the pentahedron must therefore be a point of γ_1. Now this vertex is the one opposite to e; thus an edge of a pentahedron is a chord of that twisted cubic, γ_1 or γ_2, as the case may be, which passes through its opposite vertex. This proves that the lines conjugate to the points of γ_1 are the generators of Γ_1, while the lines conjugate to the points of γ_2 are the generators of Γ_2.

It is easily verified that the points of γ_1 whose parameters are θ_1 and θ_2 are conjugate with respect to every quadric of $\wp$ provided that

$$\theta_1^2+q\theta_1\theta_2+\theta_2^2=0,$$

the corresponding relation between conjugate points of γ_2 being

$$\theta_1^2-q^{-1}\theta_1\theta_2+\theta_2^2=0.$$

18. The breaking up of the cones belonging to $\wp$ into two families can be seen otherwise; for the discriminant of the quadratic form

$$\xi Q_0+\eta Q_1+\zeta Q_2\equiv\xi(x_2^2+2x_1x_3)+2\eta(x_0x_3+x_1x_2)+\zeta(x_1^2+2x_0x_2)$$

is

$$\begin{vmatrix} 0 & 0 & \zeta & \eta \\ 0 & \zeta & \eta & \xi \\ \zeta & \eta & \xi & 0 \\ \eta & \xi & 0 & 0 \end{vmatrix} \equiv \zeta^2\xi^2-3\zeta\xi\eta^2+\eta^4\equiv(\zeta\xi-q^{-2}\eta^2)(\zeta\xi-q^2\eta^2).$$

Thus the quadric $\xi Q_0+\eta Q_1+\zeta Q_2=0$ is a cone either when $\zeta\xi=q^{-2}\eta^2$ or when $\zeta\xi=q^2\eta^2$, the existence of two families of cones being thus demonstrated. When the quadrics of $\wp$ are represented by the points (ξ, η, ζ) of a plane the quartic curve whose points represent the cones of the net breaks up into two conics having double contact, the points of contact of the conics representing the cones, $Q_0=0$ and $Q_2=0$, which are common to both families.

The locus of vertices of the cones of one family will be the twisted cubic γ_1, while the cones of the other family will have their vertices on γ_2. It is seen that those cones which are represented by the points of the

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conic $\zeta\xi = q^{-2}\eta^2$ have their vertices on γ_1. For the coordinates of a point on this conic are expressible parametrically as $\xi:\eta:\zeta = 1:-q\theta:\theta^2$, and the vertex of the cone

$$x_2^2+2x_1x_3-2q\theta(x_0x_3+x_1x_2)+\theta^2(x_1^2+2x_0x_2) = 0 \tag{7}$$

is found to be the point $(1, q\theta, q\theta^2, \theta^3)$, which is on γ_1. Similarly, those cones which are represented by the points of the conic $\zeta\xi = q^2\eta^2$ have their vertices on γ_2.

19. We now proceed to obtain, for the net $\wp$, the form of the identity

$$\pi^8 \equiv (\phi^4)^2+\lambda\sigma^8$$

which occurred in Note IV, and was there denoted by (G). This identity was not obtained for $\mathfrak{P}$; but the specialisation that has now been imposed on the pentahedral net renders its derivation more practicable.

First let us obtain the contravariant σ^8 which, when equated to zero, gives the envelope, of class eight, of the cones belonging to $\wp$. Here σ^8 will be the product of two factors Σ_1 and Σ_2, where $\Sigma_1 = 0$ is the envelope of those cones whose vertices are on γ_1 while $\Sigma_2 = 0$ arises similarly from those cones whose vertices are on γ_2. Both Σ_1 and Σ_2 will be homogeneous quartic polynomials in the plane coordinates l_0, l_1, l_2, l_3, either being obtained from the other by changing q into $-q^{-1}$ wherever it occurs in the coefficients.

The condition for a plane to touch a quadric whose point equation is given is obtained, in all the text-books on three-dimensional analytical geometry, as the vanishing of the determinant that arises when the matrix of the coefficients in the point equation of the quadric is bordered by a row and column of plane coordinates. When a plane touches a quadric, it passes through its own pole; there thus arises a system of five homogeneous linear equations having a non-zero solution, and so the matrix of these equations, which is precisely the matrix of the above bordered determinant, must have rank 4 instead of its full rank 5.

Now, though the books generally omit to say so*, this argument can be carried further. For if a plane touches a quadric at every point of a line, then there is a singly-infinite set of points, linearly dependent on any two of them, which are poles of the plane with respect to the quadric

* Sommerville, in particular, in his *Analytical geometry of three dimensions* (Cambridge, 1934), seems strangely to miss the opportunity on 217–218.

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and which also lie on the plane itself. The above set of five homogeneous equations must now have two independent solutions, and so the rank of the matrix of the bordered determinant must reduce to 3. This gives a necessary and sufficient set of conditions for a plane to touch a quadric (then necessarily a cone) along a generator*.

This being so, the plane

$$l_0 x_0 + l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$

is a tangent plane of the cone (7) when and only when the matrix

$$\begin{pmatrix} 0 & 0 & \theta^2 & -q\theta & l_0 \\ 0 & \theta^2 & -q\theta & 1 & l_1 \\ \theta^2 & -q\theta & 1 & 0 & l_2 \\ -q\theta & 1 & 0 & 0 & l_3 \\ l_0 & l_1 & l_2 & l_3 & 0 \end{pmatrix}$$

has rank 3. The determinant of the matrix obtained by omitting the last row and column vanishes identically, since it is the discriminant of a cone. The determinant of the whole matrix is $(l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3)^2$, and its vanishing is the condition for the plane to pass through the vertex of the cone. If the last column and any row other than the last, or the last row and any column other than the last, are omitted, the determinant of the resulting matrix is a multiple of $l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3$. But, if the last row and column are both retained, and some other row and column omitted, the resulting determinant is always quadratic in l_0, l_1, l_2, l_3 and, after taking account of the vanishing of $l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3$ and perhaps dividing through by some power of θ, in θ. The vanishing of such a determinant causes the plane, already constrained to pass through the vertex of the cone, to touch the cone along a generator. For example: omitting the first row and column a determinant is obtained whose value is $l_2^2+2l_3 l_1+2l_2 l_3 q\theta+l_3^2 q\theta^2$; omitting the central row and fourth column a determinant is obtained whose value is

$$l_0\theta(l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3)-q\theta\{ql_0^2+2ql_0 l_1\theta+(l_1^2+2l_0 l_2)\theta^2\};$$

and so on. The envelope of all planes which touch cones whose vertices

* An alternative form of these conditions, but not so suitable for our present purpose, is given by Bertini: *Geometria proiettiva degli iperspazi* (Messina, 1923), 148.

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are on γ_1 is found by the elimination of θ between the cubic

$$l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3=0$$

and any one of the quadratics. When this elimination is carried out, for example by the dialytic method (the routine algebra may be passed over), the result is found always to be

$$\Sigma_1 \equiv 2l_0 l_1 l_2 l_3+l_1^2 l_2^2+2(l_0 l_2^3+l_1^3 l_3)+q^5 l_0^2 l_3^2=0.$$

Let us write $\qquad \Sigma_1 \equiv \Lambda+q^5 l_0^2 l_3^2.$

Then $\qquad \Sigma_2 \equiv \Lambda-q^{-5} l_0^2 l_3^2,$

and so $\qquad \sigma^8 \equiv \Sigma_1 \Sigma_2 \equiv \Lambda^2+11\Lambda l_0^2 l_3^2-l_0^4 l_3^4.$

20. Having calculated σ^8, we proceed to calculate ϕ^4. This contravariant, as explained in Note IV, is the envelope of those planes which cut $\wp$ in nets of conics for which Sylvester's invariant T vanishes. Now the plane

$$l_0 x_0+l_1 x_1+l_2 x_2+l_3 x_3=0$$

meets the quadrics

$$x_2^2+2x_3 x_1=0, \quad 2x_0 x_3+2x_1 x_2=0, \quad x_1^2+2x_0 x_2=0,$$

in conics which are projected from X_0 by the cones

$$\begin{aligned} x_2^2 \qquad\qquad +2x_3 x_1 \qquad\qquad &=0,\\ 2l_3 x_3^2+2l_2 x_2 x_3+2l_1 x_3 x_1-2l_0 x_1 x_2&=0,\\ -l_0 x_1^2+2l_2 x_2^2 \qquad +2l_3 x_2 x_3 \qquad\qquad +2l_1 x_1 x_2&=0.\end{aligned}$$

The value of the invariant T is now obtained forthwith from the expression given in § 389 of Salmon's *Conic sections*; omitting again a few lines of routine algebra, we find that

$$T=-16(2\Lambda+11l_0^2 l_3^2),$$

where Λ is as before. So we may take

$$\phi^4 \equiv 2\Lambda+11 l_0^2 l_3^2,$$

Gundelfinger's contravariant for $\wp$.

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21. The identity (G) satisfied by the contravariants of a net of quadrics tells us that there must be some linear combination of σ^8 and of the square of ϕ^4 which splits up into eight linear factors, these factors, when equated to zero, giving the base points of the net. But the eight base points of $\wp$ consist of the two points X_0 and X_3, each counted four times; it is therefore to be expected that the expression which, for a general net, is the product of eight linear factors is here a numerical multiple of $l_0^4 l_3^4$. The forms found for σ^8 and ϕ^4 clearly admit this, since

$$(2\Lambda+11 l_0^2 l_3^2)^2-4(\Lambda^2+11\Lambda l_0^2 l_3^2-l_0^4 l_3^4)\equiv 125 l_0^4 l_3^4.$$

22. An alternative derivation of the forms Σ_1 and Σ_2 is of sufficient interest to be described before the Note ends.

The cones of $\wp$ whose vertices lie on γ_1 are those quadrics

$$\xi Q_0+\eta Q_1+\zeta Q_2=0$$

for which $\zeta\xi=q^{-2}\eta^2$; since the conic $\zeta\xi=q^{-2}\eta^2$ has the line equation $4\nu\lambda=q^2\mu^2$, the envelope of these cones is the quartic surface*

$$4Q_0 Q_2=q^2 Q_1^2.$$

This is then the surface whose plane equation is $\Sigma_1=0$, and is thus of the fourth order as well as of the fourth class. It may therefore be suspected that it is a scroll, and this suspicion is confirmed by noticing the identity

$$q^3\Sigma_1\equiv(q^4 l_0 l_3-l_1 l_2)^2+2q(l_1^2+q^2 l_0 l_2)(l_2^2+q^2 l_1 l_3),$$

which shows that $\Sigma_1=0$ has the bitangent developable, of the third class, $q^2 l_0/l_1=-l_1/l_2=l_2/q^2 l_3$.

The nodal curve of the surface is put in evidence by observing that

$$\tfrac{1}{4}(q^2 Q_1^2-4Q_0 Q_2)\equiv q^2(x_0x_3+x_1x_2)^2-(x_2^2+2x_3x_1)(x_1^2+2x_0x_2)$$
$$\equiv q^2\{(x_0x_3-q^{-2}x_1x_2)^2+2q^{-1}(x_0x_2-q^{-1}x_1^2)(x_1x_3-q^{-1}x_2^2)\},$$

which shows that the nodal curve is, as will be expected, γ_1. The cones of $\wp$ whose vertices lie on γ_1 are the tangent cones to this scroll from the points of its nodal curve.

The plane equation, $\Sigma_1=0$, of the scroll can now be deduced from its point equation. For it is found that the chord which joins the two

* See the paper, "Octadic surfaces and plane quartic curves", quoted in § 6.

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points of γ_1 whose parameters are θ_1 and θ_2 lies on the scroll provided that $(\theta_1+\theta_2)^2+2q\theta_1\theta_2=0$. Hence we require the envelope of a plane, of coordinates l_0, l_1, l_2, l_3, such that the cubic equation

$$l_0+l_1 q\theta+l_2 q\theta^2+l_3\theta^3=0$$

has two of its three roots connected by the above relation. Thus, if the three roots of the cubic are called θ_1, θ_2, θ_3, the symmetric function

$$\{(\theta_2+\theta_3)^2+2q\theta_2\theta_3\}\{(\theta_3+\theta_1)^2+2q\theta_3\theta_1\}\{(\theta_1+\theta_2)^2+2q\theta_1\theta_2\}$$

must vanish. This symmetric function is found, save for a factor which is a numerical multiple of a power of l_3, to be identical with the form found for Σ_1 in § 19.

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