Rationality problems

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(R) rational: if $X \sim \mathbb{P}^n$ for some n

(S) stably rational: if $X \times \mathbb{P}^n$ is rational, for some n

(U) unirational: if $\mathbb{P}^n \dashrightarrow X$, for some n

Classical results

In dimensions ≤ 2, over $\mathbb{C}$,

$$\text{rationality} = \text{stable rationality}=\text{unirationality}$$

- Curves: Lüroth
- Surfaces: Castelnuovo, Enriques

This can fail over nonclosed ground-fields k.

Del Pezzo surfaces over nonclosed fields

Theorem

Let X be a smooth del Pezzo surface over a field k.

- $\deg(X) \geq 5$*: If* $X(k) \neq \emptyset$ *then* X *is* k*-rational.*
- $\deg(X) = 4, 3$*: If* $X(k) \neq \emptyset$ *then* X *is* k*-unirational.*
- $\deg(X) = 2$*: same, with three omissions (Salgado, Testa, Varilly-Alvarado 2013)*

Open problems

- $k = \mathbb{F}_3$, is X/k given by

$$-w^3 = x^4 + y^3 z - y z^3$$

unirational?

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unirational?
- If $\deg(X) = 1$ then $X(k) \neq \emptyset$. Is X unirational? Are k-rational points Zariski dense? (Some results by Salgado and van Luijk, 2014.)

Cohomology

Let

$$\mathrm{H}^i(G, M)$$

be the i-cohomology group of a finite or profinite group G, with coefficients in a G-module M. Recall:

- $\mathrm{H}^0(G, M) = M^G$, the submodule of G-invariants

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Obstruction to rationality

$$\mathrm{Br}(X) = \mathrm{H}^2_{et}(X, \mathbb{G}_m).$$

For Del Pezzo surfaces,

$$\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{H}^1(G_k, \mathrm{Pic}(\bar{X})).$$

Computing the obstruction group

Let $X \subset \mathbb{P}^4$ be a smooth DP4. The Galois action on the 16 lines factors through the Weyl group $\mathsf{W}(\mathsf{D}_5)$ (a group of order 1920).

Bright, Bruin, Flynn, Logan 2007

- If the degree of the splitting field over $\mathbb{Q}$ is > 96 then

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{Pic}(\bar{X})) = 0.$$

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- If the degree of the splitting field over $\mathbb{Q}$ is > 96 then
$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{Pic}(\bar{X})) = 0.$$
- In all other cases, the obstruction group is either
$$1, \mathbb{Z}/2\mathbb{Z}, \text{ or } (\mathbb{Z}/2\mathbb{Z})^2.$$

The obstruction group

This obstruction is effectively computable for all Del Pezzo surfaces over number fields.

Obstruction to stable rationality

If X is stably rational then $\mathrm{H}^1(G_{k'}, \mathrm{Pic}(\bar{X})) = 0$, for all k'/k.

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Conjecture (Colliot-Thélène–Sansuc)

If $X(k) \neq \emptyset$ and this obstruction vanishes then X is stably rational.

The only known case:

Example

Let X be a conic bundle over $\mathbb{P}^1$, over a field k, given by

$$x^2 - ay^2 = f(s), \quad \deg(f) = 3, \quad \mathrm{disc}(f) = a,$$

with f irreducible over k. Then X is nonrational over k, but

$$\mathrm{H}^1(G_{k'}, \mathrm{Pic}(\bar{X})) = 0, \qquad \text{for all } k'/k.$$

Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer 1985:

X is stably rational

Candidates, DP4:

- I_1: $y^2 - xz^2 = (x-3)(x+3)(x^3+9)$
- I_2: $y^2 - xz^2 = -(x^3 + 2apx^2 + a^2p^2x - a^3q^3)(x^2 - 2rx + s)$, such that
  - a is not a cube,
  - $g(x) := x^3 + px + q$ is irreducible,
  - $\mathrm{disc}(g)/(r^2 - s)$, $s/(r^2 - s)$, and $a/\mathrm{disc}(g)$ are squares
- I_3: $y^2 - xz^2 = -(x^2 - 3)(x^3 + 3)$

Higher dimensions: invariant theory

Data:

- G/k linear algebraic group (e.g., finite group)
- $\rho : G \to V$ faithful representation

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Noether's problem

Is $X := V/G$ rational?

More generally, G acting on a variety Y, is $X := Y/G$, resp. $k(Y)^G$, rational?

Why interesting? Applications to the inverse problem of Galois theory - realizing a finite group G as the Galois group of a field extension (via Hilbert's irreducibility).

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Why difficult? $\mathrm{Gr}(2,n) = \mathrm{SL}_2 \backslash \mathrm{Mat}_{2\times n}$ is rational. The ring of invariants has $\binom{n}{2}$ generators and $\binom{n}{4}$ relations.

Noether's problem

- If G is SL_n, Sp_n, SO_n, ... then V/G is stably rational.
- If $G = \mathrm{PGL}_3$ then V/G is rational (Böhning–von Bothmer 2008)

Noether's problem: counterexamples

Nonlinear actions: Saltman (1984)

Let

- $G = (\mathbb{Z}/p)^3$, p prime,
- $M := \operatorname{Ker}(\mathbb{Z}[G \times G] \to \mathbb{Z}[G])$,
- $X = \operatorname{Spec}(k[M])$,

Then X/G is not rational.

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Linear actions: Bogomolov (1988)

Nontriviality of the unramified Brauer group of the function field $k(V)^G$, for some group of order p^6. In particular, V/G is not stably rational.

Unramified cohomology and the Brauer group

Let $K = k(X)$ be a function field over $k = \bar{k}$, $G_K := \mathrm{Gal}(\bar{K}/K)$ its Galois group, and

$$H^i(K) := H^i(G_K, \mathbb{Z}/n)$$

its i-th Galois cohomology. For every divisorial valuation ν of K we have a natural homomorphism

$$H^i(K) \xrightarrow{\partial_\nu} H^{i-1}(\kappa(\nu))$$

The group

$$H^i_{nr}(K) := \cap_\nu \mathrm{Ker}(\partial_\nu)$$

is a birational invariant; it vanishes for rational K. For smooth X we have

$$H^2_{nr}(K) = \mathrm{Br}(X)[n]$$

Theorem (Bogomolov–T. 2015)

Let X be a variety of dimension ≥ 2 over $k = \bar{\mathbb{F}}_p$, $K = k(X)$, and $\ell \neq p$. Every $\alpha \in H^i_{nr}(K, \mathbb{Z}/\ell))$ is induced from an unramified class in the cohomology of a quotient

$$(\prod_j \mathbb{P}(V_j))/G^a,$$

for some finite ℓ-group G^a.

Counterexamples to Lüroth's problem

Major developments in 1971-72:

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Major developments in 1971-72:

- Iskovskikh-Manin: quartic in $\mathbb{P}^4$ via birational rigidity
- Clemens-Griffiths: cubic in $\mathbb{P}^4$ via intermediate Jacobians
- Artin-Mumford: conic bundles via Brauer groups

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Birational rigidity

This approach stimulated major developments in algebraic geometry.

- Reid, Pukhlikov, Cheltsov: birational rigidity of many smooth and singular (high degree) Fano hypersurfaces in weighted projective spaces
- Some of these are known to be unirational. **Guess:** a (very general) birationally rigid threefold is not stably rational.

Intermediate Jacobians

Theorem

If the intermediate Jacobian $\mathrm{IJ}(X)$ *of a complex threefold* X *is not a product of Jacobians of curves then* X *is nonrational.*

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Implementation:

- Cubic threefolds (Clemens–Griffiths)
- Intersection of 3 quadrics and conic bundles (Beauville)
- Certain del Pezzo surface fibrations over $\mathbb{P}^1$ (Alexeev, Grinenko, Cheltsov)

Idea (Clemens 1974): Let

$$\phi : \mathcal{X} \to B$$

be a family of Fano threefolds, with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has at most rational double points

(O) Obstruction: the intermediate Jacobian $\mathrm{IJ}(\tilde{\mathcal{X}}_0)$ (of the resolution of singularities $\tilde{\mathcal{X}}_0$) is not a product of Jacobians of curves.

Then a general fiber $\mathcal{X}_b$ is not rational.

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Implementation (Beauville 1977): nonrationality of quartic and sextic double solids

Theorem (Artin-Mumford)

Let $X \to S$ be a conic bundle over a smooth projective rational surface S with discriminant a smooth curve

$$D = \sqcup_{j=1}^{r} D_j \subset S,$$

and with $g(D_j) \geq 1$ for all j. Then

$$\mathrm{Br}(X) = (\mathbb{Z}/2)^{r-1}.$$

Cycle-theoretic tools: CH_0

$\mathrm{CH}_0(X_k)$ is the abelian group generated by zero-dimensional subvarieties $x \in X$ (e.g., points $x \in X(k)$), modulo k-rational equivalence.

Assuming $X(k) \neq \emptyset$, there is a surjective degree homomorphism

$$\mathrm{CH}_0(X_k) \to \mathbb{Z}.$$

For which X is this an isomorphism?

Example

- X a unirational or rationally-connected variety over $k = \mathbb{C}$.

CH_0-triviality

A projective X/k is universally CH_0-trivial if for all k'/k

$$\mathrm{CH}_0(X_{k'}) \xrightarrow{\sim} \mathbb{Z}$$

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For example, smooth k-rational varieties are universally CH_0-trivial. Unirational or rationally-connected varieties are not necessarily universally CH_0-trivial. Smooth projective X/k with $\mathrm{Br}(X) \neq \mathrm{Br}(k)$, or more generally, with nontrivial higher unramified cohomology, are not universally CH_0-trivial.

CH_0-triviality

This condition is difficult to check, in general. Here is a sample of results: Universal CH_0-triviality holds for

- For cubic threefolds parametrized by a countable union of subvarieties of codimension ≥ 3 of the moduli space (Voisin 2014); these should be dense in moduli
- For special cubic fourfolds with discriminant not divisible by 4 (Voisin 2014)
- For cubic fourfolds (of discriminant 8) containing a plane (Auel–Colliot-Thélène–Parimala, 2015)

CH_0-TRIVIALITY

A projective morphism

$$\beta : \tilde{X} \to X$$

of k-varieties is universally CH_0-trivial if for all k'/k

$$\beta_* : \mathrm{CH}_0(\tilde{X}_{k'}) \xrightarrow{\sim} \mathrm{CH}_0(X_{k'}).$$

CH_0-triviality

Theorem (Colliot-Thélène–Pirutka, 2015)

Let

$$\beta : \tilde{X} \to X$$

be a projective morphism such that for every scheme point x of X, the fiber $\beta^{-1}(x)$, considered as a variety over the residue field $\kappa(x)$, is universally CH_0-trivial. Then β is universally CH_0-trivial.

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For example,

$$\beta : \mathrm{Bl}_Z(X) \to X,$$

the blowup of a smooth variety X in a smooth subvariety Z, is universally CH_0-trivial.

Specialization method
Voisin 2014, Colliot-Thélène–Pirutka 2015

Let

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X admits a desingularization

$$\beta : \tilde{X} \to X$$

such that the morphism β is universally CH_0-trivial;

(O) Obstruction: the group $H^2_{nr}(\mathbb{C}(X), \mathbb{Z}/2)$ is nontrivial.

Then a very general fiber of ϕ is not stably rational.

Specialization method: First applications

Very general varieties below are not stably rational:

- Quartic double solids $X \to \mathbb{P}^3$ with ≤ 7 double points (Voisin 2014)
- Quartic threefolds (Colliot-Thélène–Pirutka 2014)
- Sextic double solids $X \to \mathbb{P}^3$ (Beauville 2014)
- Fano hypersurfaces of high degree (Totaro 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of prime degree (Colliot-Thélène–Pirutka 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of arbitrary degree (Okada 2016)

Conic bundles over rational surfaces

Theorem (Hassett-Kresch-T. 2015)

Let S be a smooth projective rational surface over k, an uncountable algebraically closed field of characteristic $\neq 2$. Let $\mathcal{L}$ be a linear system of effective divisors on S whose general member is smooth and irreducible. Let $\mathcal{M}$ be an irreducible component of the space of reduced nodal curves in $\mathcal{L}$ together with degree 2 étale covering. Assume that $\mathcal{M}$ contains a cover, nontrivial over every irreducible component of a reducible curve with smooth irreducible components. Then the conic bundle over S corresponding to a very general point of $\mathcal{M}$ is not stably rational.

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Example: A very general conic bundle $X \to \mathbb{P}^2$, with discriminant a curve of degree ≥ 6, is not stably rational.

Conic bundles over rational surfaces

Theorem (Böhning–von Bothmer 2016)

A very general hypersurface $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ *of bi-degree* $(2, d)$, $d \geq 2$, *is not stably rational.*

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Via explicit degeneration of Prym curves.

Conic bundles over higher-dimensional bases

Stable rationality fails for general varieties in the following families:

- Certain conic bundles over $\mathbb{P}^3$, e.g.,

  $$X \subset \mathbb{P}^2 \times \mathbb{P}^3$$

  of bi-degree $(2,2)$ (Auel–Böhning–von Bothmer–Pirutka 2016)
- Conic bundles over $\mathbb{P}^{n-1}$: smooth $X \subset \mathbb{P}(\mathcal{E})$, for $\mathcal{E}$ direct sum of three line bundles, if $-K_X$ is not ample. In particular

  $$X \subset \mathbb{P}^2 \times \mathbb{P}^{n-1}$$

  of bi-degree $(2,d)$, $d \geq n \geq 3$ (Ahmadinezhad–Okada 2017)

Conic bundles over rational surfaces

Let $X \to S$ be a very general conic bundle over a del Pezzo surface of degree 1, with discriminant $C \in |-2K_S|$. Then

- X is not birationally rigid
- $\mathrm{IJ}(X)$ is an elliptic curve
- X has trivial Brauer group

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- X is not birationally rigid
- $\mathrm{IJ}(X)$ is an elliptic curve
- X has trivial Brauer group
- X is not stably rational

Del Pezzo fibrations

Theorem (Hassett-T. 2016)

A very general fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ *in quartic del Pezzo surfaces which is not rational and not birational to a cubic threefold is not stably rational.*

Del Pezzo fibrations

Theorem (Krylov-Okada 2017)

A very general nonrational del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ of degree 1, 2, or 3 which is not birational to a cubic threefold is not stably rational.

Similar results over higher-dimensional bases.

Fano threefolds

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Generalizations by Okada to certain singular Fano varieties.

Fano threefolds: idea and implementation

Find suitable degenerations with mild singularities and birational to conic bundles.

Nonrational Fano threefolds with

$$\mathrm{Pic}(V) = -K_V\mathbb{Z} \quad \text{and} \quad d = d(V) = -K_V^3 :$$

- $d = 2$ sextic double solid
- $d = 4$ quartic
- $d = 6$ intersection of a quadric and a cubic
- $d = 8$ intersection of three quadrics
- $d = 10$ section of $\mathrm{Gr}(2,5)$ by two linear forms and a quadric
- $d = 14$ birational to a cubic threefold

Nonrational Fano threefolds of index 2:

- $d = 1 \cdot 8$ double cover of $\mathbb{P}(1,1,1,2)$ ramified in a cubic
- $d = 2 \cdot 8$ quartic double solid
- $d = 3 \cdot 8$ cubic threefold

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Nonrational Fano threefolds of higher Picard rank:

- double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ ramified in D of bi-degree $(2,4)$
- divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(2,2)$
- double cover of $\mathrm{Bl}_p(\mathbb{P}^3)$
- double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified in D of degree $(2,2,2)$

From general quartic del Pezzo $\mathcal{X} \to \mathbb{P}^1$ to Fano threefolds V:

- $d = 2$: $h(\mathcal{X}) = 22 \Rightarrow$ sextic double solid V with 32+4 nodes
- $d = 4$: $h(\mathcal{X}) = 20 \Rightarrow$ quartic threefold with 16 nodes
- $d = 6$: $h(\mathcal{X}) = 18 \Rightarrow$ quadric $\cap$ cubic with 8 nodes
- $d = 8$: $h(\mathcal{X}) = 16 \Rightarrow$ intersection of three quadrics with 4 nodes
- $d = 10$: $h(\mathcal{X}) = 14 \Rightarrow$ specialization of a V with 2 nodes

Consider the intersection of two $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^4$:

$$sP_1 + tQ_1 = sP_2 + tQ_2 = 0.$$

Let $v_1, \ldots, v_{16} \in \mathbb{P}^4$ denote the solutions to

$$P_1 = Q_2 = P_2 = Q_2 = 0$$

Fano threefolds and del Pezzo fibrations

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$$P_1 = Q_2 = P_2 = Q_2 = 0$$

- Projection onto the first factor gives a degree 4 del Pezzo fibration over $\mathbb{P}^1$ (with 16 constant sections)
- Projection onto the second factor gives a quartic threefold

$$V := \{P_1 Q_2 - Q_1 P_2 = 0\} \subset \mathbb{P}^4$$

with 16 nodes $v_1, \ldots, v_{16}$.

The other families of Fano threefolds are conic bundles, but not very general, as in the theorem above. Additional work is needed.

Example

$X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, double cover ramified in a $(2, 2, 2)$ hypersurface; conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with discriminant of bi-degree $(4, 4)$ – not generic in its linear series!

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$X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, double cover ramified in a $(2, 2, 2)$ hypersurface; conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with discriminant of bi-degree $(4, 4)$ – not generic in its linear series! The corresponding K3 double cover $S \to \mathbb{P}^1 \times \mathbb{P}^1$ has Picard rank 3 and not 2.

Rationality in families

Let $\pi : \mathcal{X} \to B$ be a family of rationally connected varieties and put

$$\mathrm{Rat}(\pi) := \{\, b \in B \mid \mathcal{X}_b \quad \text{is rational} \,\}.$$

de Fernex–Fusi 2013

In dimension 3, $\mathrm{Rat}(\pi)$ is a countable union of closed subsets of B.

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What about higher dimensions? E.g., moduli spaces of Fano varieties?

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Remark

Over number fields, $\mathrm{Rat}(\pi)$ has been studied, in connection with specializations in Brauer-Severi fibrations (Serre's problem).

Rat(π): Hassett-Pirutka-T. 2016

Rat(π) and its complement can be dense on the base.

There exist smooth families of projective rationally connected fourfolds $\mathcal{X} \to B$ over $k = \mathbb{C}$ such that:

- For every $b \in B$ the fiber X_b is a quadric surface bundle over a rational surface S;
- For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;
- The set of $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B.

Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)

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- Construction of special X satisfying (**O**) and (**S**)
- Rationality constructions

Rationality in families: idea

Consider a quadric surface bundle

$$\pi : \mathcal{Q} \to \mathbb{P}^2,$$

with smooth generic fiber. Let $D \subset \mathbb{P}^2$ be the degeneration curve; assume that D is smooth. Then $\mathcal{Q}$ is characterized by:

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When $\deg(D) \geq 6$, $\mathrm{Pic}(T)$ and $\mathrm{Br}(T)$ can change as we vary D.

Rationality in families: implementation

We consider bi-degree $(2,2)$ hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

Projection onto the first factor gives a quadric bundle over $\mathbb{P}^2$, its degeneration divisor $D \subset \mathbb{P}^2$ is an octic curve.

Note: Cubic fourfolds containing a plane give rise to quadric surface bundles with degeneration curve of degree 6.

Special fiber

Let

$$X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[s:t:u:v]}$$

be a bi-degree $(2, 2)$ hypersurface given by

$$yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

where

$$F(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2yz - 2xz.$$

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The discriminant curve for the projection $X \to \mathbb{P}^2$ is given by

$$x^2y^2z^2F(x,y,z) = 0.$$

Special fiber

- Computing $H^2_{nr}(X, \mathbb{Z}/2)$: general approach by Pirutka (2016)

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- Computing $H^2_{nr}(X, \mathbb{Z}/2)$: general approach by Pirutka (2016)
- Desingularization: by hand; the singular locus is a union of 6 conics, intersecting transversally

Rationality

It suffices to produce Hodge classes in $H^{2,2}(X,\mathbb{Z})$ intersecting the class of the fiber of $\pi : X \to \mathbb{P}^2$ in odd degree. Then the quadric over the function field $\mathbb{C}(\mathbb{P}^2)$ has a section, and X is rational.

The corresponding Noether-Lefschetz locus is dense in the usual topology of the moduli space.

Idea: Make the function field of $\mathcal{X}_b$ the groundfield of a stably rational but not rational DP4 (conic bundle with 4 degenerate fibes).

- If $k(\mathcal{X}_b)$ fails universal CH_0-triviality, then the total space fails stable rationality.
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- When $k(\mathcal{X}_b)$ is rational, the total space is stably rational.

How to show that it is not rational?

Other applications

Theorem (Hassett–Pirutka–T. 2017)

Let $X \subset \mathbb{P}^7$ be a very general intersection of three quadrics. Then X is not stably rational. Rational X are dense in moduli.

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Let $X \subset \mathbb{P}^7$ be a very general intersection of three quadrics. Then X is not stably rational. Rational X are dense in moduli.

Idea: Such X admit a fibration $X \to \mathbb{P}^2$, with generic fiber a quadric surface and octic discriminant.

Smooth cubic hypersurfaces $X_3 \subset \mathbb{P}^n$

- $\dim = 1$ - nonrational

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- dim = 1 - nonrational
- dim = 2 - rational
- dim = 3 - nonrational, are there any stably rational examples?
- dim = 4 - periodicity??

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Unirational parametrizations:

- all admit unirational parametrizations of degree 2
- (Hassett-T. 2001) Cubic fourfolds with an odd degree unirational parametrization are dense in moduli

Special cubic fourfolds

Addington–Hassett–T.–Várilly-Alvarado 2016

The locus of rational cubic fourfolds in $\mathcal{C}_{18}$ – special cubic fourfolds of discriminant 18 – is dense.

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The locus of rational cubic fourfolds in $\mathcal{C}_{18}$ – special cubic fourfolds of discriminant 18 – is dense.

Idea: Every $X \in \mathcal{C}_{18}$ admits a fibration $X \to \mathbb{P}^2$ with general fiber a degree 6 Del Pezzo surface. A multisection of degree coprime to 3 forces rationality. The locus of such cubics is dense in $\mathcal{C}_{18}$.

Remark

Something like this should work for 6-dimensional cubics.

Summary

- There are many instances of fascinating interactions between arithmetic and geometric properties of higher-dimensional algebraic varieties.

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- There are many instances of fascinating interactions between arithmetic and geometric properties of higher-dimensional algebraic varieties.
- Rationality and stable rationality of cubic hypersurfaces remain a major challenge.