Finitely generated simple subgroups of the plane Cremona group

$/\mathbb{C}$

Thanks
Observation

$Cr_2 := \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$

shares properties with linear groups

Thm: A finitely generated simple subgroup of Cr_2 is finite.
The only finitely generated simple subgroups of Cr_2 are

$\mathbb{Z}/p\mathbb{Z}$, A_4, A_5, $PSL_2(7)$.

Ex: Tarski Monsters, Higman group, Thompson group T and V, ...

Def: A group G satisfies the property of Malcev if every finitely generated subgroup $\Gamma \subset G$ is residually finite, i.e. $\forall \gamma \in \Gamma \setminus \{id\}$ $\exists$ $\varphi_\gamma : \Gamma \to H$ s.t. $\gamma \notin \ker \varphi_\gamma$.

finite

<u>Remark</u>: If G satisfies the property of Malcev, all f.g. simple subgrps of G are ~~sim~~ finite.

<u>Question (Cantat)</u>: Does Cr_2 satisfy the property of Malcev?

<u>Thm (Malcev)</u>: Linear grps satisfy the property of Malcev.

<u>Idea of proof</u>: $\Gamma \subset GL_n(\mathbb{C})$

$\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$, $g \in \Gamma \setminus \{id\}$

finite set S consisting of all the entries of the matrices γ_i, g.

reduction modulo p for suitable prime p

$\rightsquigarrow \Gamma \longrightarrow GL_n(\mathbb{F}_p)$.

<u>Thm</u> (Bass, Lubotsky)

X alg. variety

$Aut(X)$ satisfies the property of Malcev.

Example $\Gamma \subseteq Aut(\mathbb{A}^d)$, $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$

$\gamma_i = (f_1, \dots, f_d)$

Thm (Gizatullin; Diller, Favre; Cantat)

$f \in Cr_2$, then one of the following is true:

(1) $\{\deg(f^n)\}$ is bounded and $\exists\, \varphi : \mathbb{P}^2 \dashrightarrow X$, $n > 0$ s.t. $\varphi f^n \varphi^{-1} \in \mathrm{Aut}^0(X)$.

(2a) $\deg(f^n) \sim c \cdot n$ and f preserves a rat'l fibration.

(2b) $\deg(f^n) \sim c \cdot n^2$ and f preserves a fibration of genus 1 curves.

(3) $\deg(f^n) \sim c \cdot \lambda(f)^n$, f does not preserve any fibration.

The Picard-Manin space

Cr_2 acts by isometries on a ∞-dimensional hyperbolic space $\mathbb{H}^\infty$

Idea of construction:

again only finitely many coefficients $\leq$

reduction mod p

$\rightsquigarrow \varphi : \Gamma \longrightarrow \mathrm{Aut}(\mathbb{A}^d_{\mathbb{F}_p}) \curvearrowright (\overline{\mathbb{F}}_p)^d$

$g \notin \ker \varphi$

similarly one ~~can~~ shows:

Lemma: Let $\Gamma \subset \mathrm{Cr}_2$ be finitely gen.

$g \in \Gamma \setminus \{id\}$. Then there exists

a hom. $\varphi_g : \Gamma \longrightarrow \mathrm{Bir}(\mathbb{P}^2_{\overline{\mathbb{F}}_p})$ for some prime p

s.t. $g \notin \ker \varphi_g$.

and such that $\deg(\varphi_g(f)) \leq \deg(f)$.

$f = [f_0 : f_1 : f_2]$ $f_i \in \mathbb{C}[x, y, z]$ homog. no common factor

$\deg f = \deg f_i$

Types of plane Cremona transformations

Ex: $f \in \mathrm{Aut}(\mathbb{P}^2) \simeq \mathrm{PGL}_3$, e.g. $f = (x+y, y)$

$\deg(f) = 1$

and $\deg(f^n) = 1 \quad \forall n$

Ex: $f = (xy, xy)$, $\deg(f) = 2$

$\deg(f^n) = n+1 \quad \forall n \geq 0$

Ex: $f = (y, x + y^2)$, $\deg(f) = 2$

$\deg(f^n) = 2^n$

proj. surface $X \rightsquigarrow NS(X)$

$X \to Y$ blow-up of proj. surfaces

$\rightsquigarrow NS(Y) \hookrightarrow NS(X)$.

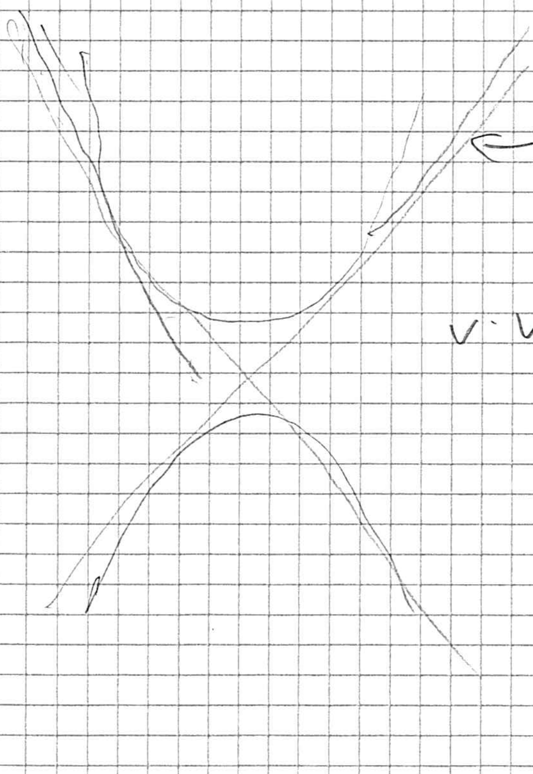
$$\begin{tikzcd} & Y_3 \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

$\lim\limits_{\pi : X \to \mathbb{P}^2} NS(X) =: Z(\mathbb{P}^2)$

∘ intersection form with signature $(1, \infty)$.

$Z(\mathbb{P}^2)$ completion of $Z(\mathbb{P}^2) \otimes \mathbb{R}$

$Cr_2 \curvearrowright Z(\mathbb{P}^2)$ preserving the intersection form



$\leftarrow \mathbb{H}^\infty$

$Cr_2 \curvearrowright \mathbb{H}^\infty$ by isometries

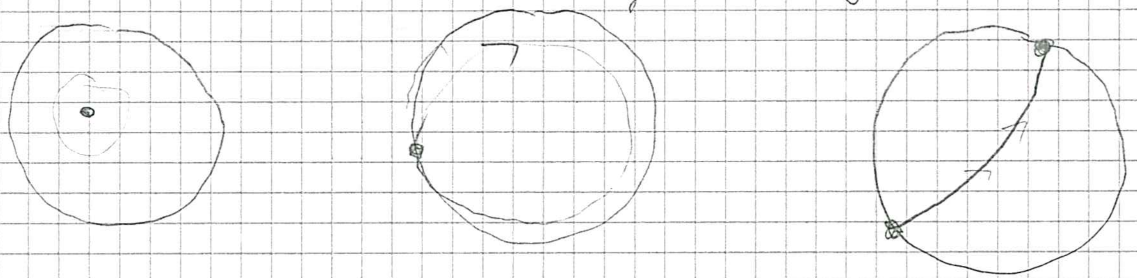
$v \cdot v = 1$

$\partial \mathbb{H}^\infty$ 1-dim subspaces in the isotropic cone

Three types of isometries on $\mathbb{H}^\infty$:

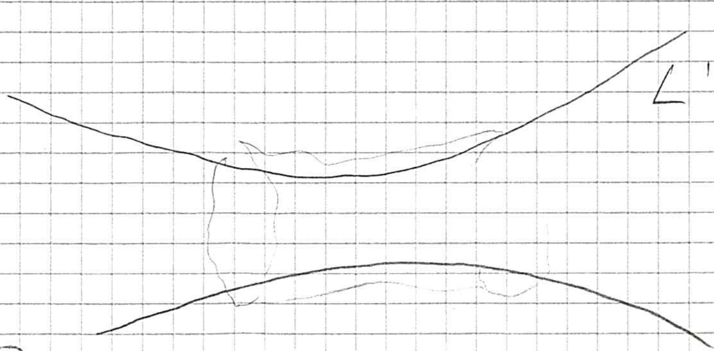
an isometry f of $\mathbb{H}^\infty$ is

- *elliptic* if it has a fixed pt in $\mathbb{H}^\infty$
- *parabolic* if it has no fixed pt in $\mathbb{H}^\infty$ but exactly one fixed pt on $\partial\mathbb{H}^\infty$
- *loxodromic* if it has no fixed pt in $\mathbb{H}^\infty$, but two fixed pts in $\mathbb{H}^\infty$. The action of f is given by translation along an axis $Ax(f)$ by translation length $\lambda(f)$



- interpretation in terms of alg. geometry.
- analogy with MCG, $Out(\mathbb{F}_n)$, .

Small cancellation



$L, L' \subset \mathbb{H}^\infty$ geodesic lines are (ε, B)-close if the diameter

$\varepsilon, B > 0$

of the following set is at least B:

$$\{ x \in L \mid d(x, L') \leq \varepsilon \}$$

<u>Def</u>: $G \subset Cr_2$ a subgroup.

a loxodromic element $g \in G$ is <u>rigid in G</u> if $\exists\, \varepsilon, B > 0$ s.t.

$\forall h \in G$ we have:

$h(Ax(g))$ is (ε, B)-close to $Ax(g)$

$\Leftrightarrow$ $h(Ax(g)) = Ax(g)$.

<u>Def</u>: $G \subset Cr_2$, $g \in G$ loxodromic is <u>tight in G</u> if g is rigid in G and if $\forall h \in G$, $h(Ax(g)) = A(g)$ implies $hgh^{-1} = g$ or $hgh^{-1} = g^{-1}$.

<u>Rem</u>: g tight $\Rightarrow$ g^n tight $\forall n$

<u>Thm (Cantat, Lamy)</u>

$G \subset Cr_2$ subgroup, $g \in G$ tight in G.

~~For all~~ $h \in \langle\langle g \rangle\rangle$ then

either $h = id$

or ~~h is conjugate to g~~

conjugation length of h is $\geq$ ~~translation~~ length of g.

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$\big(\quad \langle\langle g^2 \rangle\rangle \not\ni g \big)$.

We have an embedding

$GL_2(\mathbb{Z}) \longrightarrow Cr_2$

$(a_{ij}) \longmapsto (x^{a_{11}} y^{a_{12}}, x^{a_{21}} y^{a_{22}})$

maps of this form normalize the subgroup $D_2 \subset PGL_3$ of diagonal automorphisms. In fact $\mathrm{Norm}_{Cr_2} D_2 = GL_2(\mathbb{Z}) \ltimes D_2$.

$GL_2(\mathbb{Z})$ contains loxodromic elements that normalize a big subgroup of elliptic elements.

<u>Thm</u> (Shepherd-Barron):

$G \subset Cr_2$, $g \in G$ loxodromic

$\Rightarrow$ then g is rigid in G and the following are equivalent:

(1) no power of g is tight

(2) $\exists$ subgroup $\Delta_2 \subset G$ s.t. g normalizes Δ_2 and $\exists f \in Cr_2$ s.t.

$f \Delta_2 f^{-1} \subset D_2$ is dense and

$f g f^{-1} \in GL_2(\mathbb{Z}) \ltimes D_2$

Proof of Thm about simple subgroups:

$\Gamma \subset Cr_2$ f.g.

case 1) $\exists g \in \Gamma$ loxodromic $\nearrow$ g^n tight in Γ $\leadsto$ not simple

$\searrow$ g^n not tight in Γ $\forall n$

$\Rightarrow \exists \Delta_2 \subset Cr_2$

may assume $\Delta_2 \subset D_2$ dense

in particular, Δ_2 is infinite.

reduction mod p

$\rightarrow \varphi : \Gamma \longrightarrow Bir(\mathbb{P}^2_{\overline{\mathbb{F}}_p})$

non-trivial homomorphism

$\rightarrow \varphi(\Delta_2) \subseteq D_2(\overline{\mathbb{F}}_p)$

$\supset$ finite

$\Rightarrow \varphi$ is not injective

$\Rightarrow \Gamma$ not simple.

case 2: • Γ contains no loxodromic elements but a parabolic element

$\Rightarrow$ either Γ preserves a rational fibration, then ~~Γ is conjugate~~

$\Gamma \subset PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C}(t))$

$\Rightarrow \Gamma$ finite by [illegible] or Γ preserves ~~fib~~ genus 1 - fibration

$\Rightarrow \Gamma \subset \mathrm{Aut}(X)$ $\quad X$ Halphen surface

~~Aut(X)~~ Automorphism groups of Halphen surfaces contain finite index normal abelian subgroups

$\Rightarrow \Gamma$ is not simple.

- all elements in Γ are elliptic

$\Rightarrow \Gamma$ is contained in an alg. subgroup of Cr_2, these are linear

Γ

theorem of Cantat $\quad \Rightarrow \Gamma$ finite by [illegible]

or Γ is de Jonquières

$\Rightarrow \Gamma$ finite.