My aim today, with affection and deference towards Edge's wonderfull use of projective geometry, is to show you some applications of projective geometry to the vast and active area that surrounds "tensor decompositions".

$\boxed{\begin{array}{c} h = k \\ \text{check} = 0 \end{array}}$

The questions I'll address can be phrased as follows:

Let $V_1, \dots, V_a$ be k vector spaces and $T \in V_1 \otimes \cdots \otimes V_a$ a tensor. I want to express T as a linear combination of "simpler" tensors

$$T = \lambda_1 U_1 + \cdots + \lambda_h U_h \qquad \left(\text{for instance } U_i = v_1^i \otimes \cdots \otimes v_a^i \quad v_j^i \in V_j \text{ or even } v_j^i \text{ in some special subset of } V_j \right)$$

I'd like to know what is the ~~itself~~ minimum possible h and if this expression is unique up to obvious operations. Let me stress that I'm interested in a specific tensor and not in the general tensor.

Beside its intrinsic interest these kind of questions have applications to algebraic statistic (hidden variables problems), phylogenetic (trees reconstructions), BSS and many more (chemistry,

Given $T \in V_1 \otimes \cdots \otimes V_k$ we let $\operatorname{rank}^X(T) = \min_h \{ \exists\, T = \lambda_1 U_1 + \cdots + \lambda_h U_h \}_{U_i \in X_i \subseteq V_i}$

~~we may stratify $V_1 \otimes \cdots \otimes V_k$ via rank letting~~

~~$W_b = \mathrm{rank}^{-1}(b) = \{T \mid \mathrm{rank}^X(T) = b\}$~~ This has an immediate counterpart in projective geometry

$$[T] \in \underbrace{\mathbb{P}(V_1 \otimes \cdots \otimes V_d)}_{=\mathbb{P}^N} \supset X \quad \text{locus of simple tensors}$$

$$W_b^X := \overline{\mathrm{rank}^{X-1}(b)} = \overline{\{[T] \mid \mathrm{rank}^X(T) = b\}} = \overline{\{[T] \mid [T] \in \langle x_1, \dots, x_b \rangle,\ x_i \in X\}}$$

thus we have $W_b^X = \mathrm{Sec}_b(X)$ as long as $\mathrm{Sec}_b(X) \subsetneq \mathbb{P}^N$. Then we have a notion of generic rank g ~~that is the first~~ where $\mathrm{Sec}_g(X) = \mathbb{P}^N$ and $\mathrm{Sec}_{g-1}(X) \subsetneq \mathbb{P}^N$

that is W_g^X contains a dense open set of $\mathbb{P}^N$ (note that here it is crucial that $k = \bar{k}$; over $\mathbb{R}$ in the euclidean topology there may be different values on various open euclidean balls)

and $m = \max_h \{\exists [T] \text{ with } \mathrm{rank}(T) = h\}$.

Via this description is clear that $W_{j-1} \subset W_j$ if $j \leq g$; it is less clear what is W_j for $j > g$.

The expected behavior ~~is that~~ for general $[T] \in W_b^X$ is that the decomposition is unique when $b < g$ and it is not for $b \geq g$.

but as you may imagine both have special cases, frequently associated to special subvarieties of X.

The first question I'd like to address is called effective identifiability.
That is given a decomposition $T = \lambda_1 v_1 + \dots + \lambda_h v_h$ understand if it is the only one with h-fold
(This is different from generic identifiability this time we really want to cut out a ~~a~~ dense open subset of
$Sec_h(X)$ where the decomposition is unique).
The main weapon ~~here~~ till now was ~~is~~ the so called reshaped Kruskal i.e. A way to slice a tensor in
$V_1 \otimes \dots \otimes V_d$ to a tensor in $A \otimes B \otimes C$ and then apply ~~an~~ effective algorithm produced in the '70
by Kruskal for these tensors [COV].
Together with A. Massarenti and G. Staglianò we realised that a simpler and computational
easier construction based on projectivity often gives better results.
For the sake of this seminar I'll only do it for symmetric tensors (i.e. hom. ~~poly~~ polynomials)

$F \in k[x_0, \dots, x_n]_d \qquad [F] \in \mathbb{P}^N = \mathbb{P}(k[x_0, \dots, x_n]_d)$

$$\cup$$

$X = V_{n,d}$ the Veronese variety

we are looking for $F = \sum_1^h \lambda_i L_i^d \quad L_i \in k[x_0, \dots, x_n]_d$ clearly for any partial derivative
$\partial_j F = \sum_1^h \mu_i^j L_i^{d-1}$ therefore $\langle [L_1^{d-1}], \dots, [L_h^{d-1}] \rangle$ simultaneously decompose $\partial_j F$

$$\langle \partial_j F \rangle =: \partial^1 F$$

similarly for any s $\partial^s F \subseteq \langle [L_1^{d-s}], \dots, [L_h^{d-s}] \rangle$ as soon as $\binom{n+s}{n} \geq h$ this is a meaningful ~~and~~ constrain and one expects (it is not difficult to show that) $\partial^s F = \langle [L_1^{d-s}], \dots, [L_h^{d-s}] \rangle$ ~~generically~~.

If this is the case then $\partial^s F \cap V_{n,d-s}$ ~~then~~ contains all points corresponding to all possible decompositions of F. indeed the locus where $\dim \partial^s F \leq h-2$ is closed in Sec_h

Prop. $F \in k[x_0, \dots, x_n]_d$ assume $F = \sum_1^h \lambda_i L_i^d$ and $\exists$ s s.t. $\binom{n+s}{n} \geq h > \binom{n+s-1}{n}$

if
- $\partial^s F$ has dimension $h-1$
- $\dim \partial^s F \cap V_{n,d-s} = 0$
- $\deg \partial^s F \cap V_{n,d-s} = h$

the decomposition is unique.

Rmk For tensors one has to use flattenings instead of partial derivatives and the appropriate Segre-Veronese, Segre-Grassmann ... for $V_{n,d-s}$.

As for Kruskal this rarely gives a full set of criteria for all h available, but our range is often better than the other one.

Computationally is very quick only linear spaces and intersection with varieties generated by quadrics.

The second aspect I want to talk about today is related to

m_X the maximal rank.

For this I introduce an additive notation for joins

$X, Y \subset \mathbb{P}^N$ $\quad X+Y = \{p \mid p \in \langle x, y\rangle\ x \in X, y \in Y\}$ $\quad Sec_a(X) = aX$

It is quite embarassing but very few is known about m

$X \subset \mathbb{P}^N$ irreducible non degenerate $\cancel{\text{rank}_X(p) = \min_h \{p \in \langle x_1, \dots, x_h\rangle, x_i \in X\}}$

g_X the general rank $\quad$ clearly $\quad m_X \leq 2g_X$ $\quad$ (this has been only recently noticed by [BT])

m_X maximal rank

two general pts their linear span does at most rank $2g_X$

With similar easy arguments it is easy to get a sharp bound for curves

Prop $X \subset \mathbb{P}^N$ a non degenerate curve. Then $m \leq 2g-1$

and if $Sec_{g-1}(X)$ is a hypersurface $m \leq 2g-2$.

Rmk This is exactly the behavior of RNC. For those there is a complete understanding of all possible ranks.

~~Let $W_k^X = \text{rank}^{-1}(k) = \{p \in \mathbb{P}^N \mid \text{rank}_X(p) = k\}$~~

X RNC of degree d $\quad W_m = W_d = \tau(X)$ tangential variety of X

$$W_h = \tau(X) + (d-h)X \qquad g < h \leq m$$

also based on this example together with J. Buczyński, K. Han, Z. Teitler we tried to understand the W_k^* for $k > g$. Again via projective techniques.

Our starting point is the following observation

Let $W \subset W_m$ be an irr. component

a gen. pt in $J(W, X) = W + X$ has either rank m or rank $m-1$

if the rank is m $\quad W \subseteq W + X \subseteq W \Rightarrow X \subset \mathrm{Vert}(W) \Rightarrow X$ degenerate

$W+X$ is irreducible

$\Rightarrow W_m + X \subseteq W_{m-1}$ $\quad$ inductively using the m. deg. of $X, 2X, \ldots$

one proves $W_h + X \subseteq W_{h-1}$ $\quad m > h \geq g+1$

In particular we have the following behaviour

$$X = W_1 \subset \underset{2X}{\underset{\parallel}{W_2}} \subset \cdots \subset \underset{\mathbb{P}^N}{\underset{\parallel}{W_g}} \supset W_{g+1} \supset \cdots \supset W_{m-1} \supset W_m$$

playing with these curvature and standard properties of Jacobians

Corollary - cod $W_{g,k} \ge 2k-1$ ~~we have examples where~~ for $k < c$ equality is attained

. If $m = 2g$ then for $1 \le k \le g-1$ $kX \not\subset W_{m-k+1}$

$(\mathbb{P}^N = gX = kX + (g-k)X \subseteq W_{2g-k+1} + (g-k)X \subseteq W_{g+1} \subsetneq \mathbb{P}^N)$

Theorem If G is a connected group, V an irr. rep. of G and $X = G/P \subset \mathbb{P}(V)$ a proj. homog. variety. Then $m \le 2g-1$

(for instance this is true for all Veronese, Segre ... Grassm. dec. ass. varieties).

In the opposite direction we proved a lower bound on $\dim W_m$ (for Veronese)

Th. $X = V_{n,d} \subset \mathbb{P}^N$ $n \ge 3$. Then $\dim W_m \ge \binom{n+2}{2} - 1$ when equality is fulfilled

we can characterize the component.

But we do not know m even for Veronese (arbitrary n, d)