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Adelic lifts of geometry and arithmetic of arithm. surfaces and their BSD interaction

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talk notes
(4 pages)

Usually, alg geometry studies codim 1 or dim 1 structures.

The main motif of this talk is codim 1 & codim 2 structures study, together.

The role of adelic structures: they are topological objects, with self-duality property at the additive level. Their symmetries underly key properties of discrete classical groups such as Pic, H^i , etc.

Example. S smooth projective surface over perfect field k , $K = k(S)$.

For every irreducible curve $y \subset S \rightsquigarrow \mathcal{O}_y \rightsquigarrow$ completion $\hat{\mathcal{O}}_y \rightarrow K_y$ fraction f.

closed point $x \in S \rightsquigarrow \mathcal{O}_x \rightsquigarrow$ completion $\hat{\mathcal{O}}_x \rightarrow K_x = K \otimes \hat{\mathcal{O}}_x$

$x \in y \rightsquigarrow (\hat{\mathcal{O}}_x)_y \rightsquigarrow$ 2d local ring $\mathcal{O}_{x,y} \rightarrow K_{x,y}$ 2d fraction ring

$$K_{x,y} = \prod_{\text{local branches } z \text{ of } y \text{ at } x} K_{x,z}, \quad K_{x,z} \text{ 2d local field}$$

if $k = \mathbb{C}$, then

$$\begin{array}{ccccc}
 K_{x,z} & > & \mathcal{O}_{x,z} & > & \mathcal{O}_{x,z} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{C}((t))((u)) & > & \mathbb{C}((t))[[u]] & > & \mathbb{C}[[t]] + u\mathbb{C}((t))[[u]] \\
 & & \downarrow \text{mod } u & & \downarrow \text{mod } u \\
 & & \mathbb{C}((t)) & > & \mathbb{C}[[t]]
 \end{array}$$

rk 1 integral structure rk 2 integral str.

Question: "refined" alg. geometry taking into account higher codimension data? $\mathcal{O}_y \rightarrow \mathcal{O}_{x,y}$, etc.?

2d adèles on surfaces $\begin{cases} \rightarrow \text{geometric rk1 } A \\ \rightarrow \text{arithmetic rk2 } A \end{cases}$

Geometric adèles: define first

$$y \in S \rightsquigarrow \begin{matrix} A_y & \supset & \mathcal{O}A_y & \supset & \mathcal{O}A_y \\ \parallel & & \parallel & & \parallel \\ A_{k(y)}^{(t_y)} & \supset & A_{k(y)}[[t_y]] & \supset & \mathcal{O}A_{k(y)} + t_y A_{k(y)}[[t_y]] \end{matrix}$$

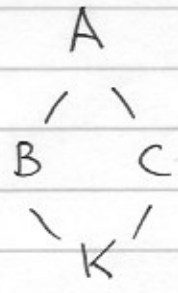
where $A_{k(y)}$ is the usual 1d adelic ring $= \prod'_{x \in y} \widehat{k(y)}_x =$
 the restricted product wrt integral rk1 structure $\widehat{\mathcal{O}}_{x \in y}$, where
 $\widehat{k(y)}_x$ is the fraction ring of $\widehat{\mathcal{O}}_{x \in y}$, $\mathcal{O}A_{k(y)} = \prod \widehat{\mathcal{O}}_{x \in y}$.

Now $A := \prod' A_y$ wrt $\mathcal{O}A_y$

$$B := \prod K_y \cap A \quad (\text{inside } \prod K_{x,y})$$

$$C := \prod K_x \cap A$$

Get



$B^x / K^x \cdot \text{units} \simeq \text{Pic}(S)$

A has appropriate topology and is self-dual $A \simeq \text{Hom}_{\text{cont}}(A, \mathbb{C})$

For a divisor $D \in \text{Div}(S)$, $D = \sum n_y [y] \rightsquigarrow A(D) = \prod t_y^{n_y} A_y \cap A$

Adelic complex $K \oplus B(D) \oplus C(D) \rightarrow B \oplus C \oplus A(D) \rightarrow A \rightarrow A_S(D)$

$$B(D) = B \cap A(D), C(D) = C \cap A(D)$$

Then $H_{\text{Zar}}^i(S, \mathcal{O}(D)) \cong H^i(A_S(D))$

H^i are \mathbb{C} -linearly compact & discrete $\Rightarrow \dim_{\mathbb{C}} H^i < \infty$, so
can define $\chi_{A_S}(D) = \dim_{\mathbb{C}} H^0(A_S(D)) - \dim_{\mathbb{C}} H^1(A_S(D)) + \dim_{\mathbb{C}} H^2(A_S(D))$

Then for $E_1, E_2 \in \text{Div}(S)$ the intersection

$$(E_1, E_2) = \chi_{A_S}(E_1 + E_2) - \chi_{A_S}(E_1) - \chi_{A_S}(E_2) + \chi_{A_S}(0)$$

and the R-R thm follows from self-duality of A and discreteness of K in A .

Arithmetic adeles $A = \prod_{y \in S} \mathcal{O}_{A_y}$ w.r.t \mathcal{O}_{A_y}
vert.

Codim 2 structure: closed pts of S .

Assume, from now on, that S is an arithm. surface
(including surfaces over finite fields).

Then each closed pt x of S has finite residue field
 $k(x) = \mathcal{O}_x / \mathcal{M}_x$, let $|k(x)|$ be its cardinality.

The ζ -function $\zeta_S(s) = \prod (1 - |k(x)|^{-s})^{-1}$, $s \in \mathbb{C}$

Examples: $\zeta_{\text{pt}}(s) = \frac{1}{1 - |k(x)|^{-s}}$, $\zeta_{\mathbb{P}_k^1}(s) = \frac{1}{(1 - |k(x)|^{-s})} \frac{1}{(1 - |k(x)|^{-s})}$

Blow up at a point replaces $\frac{1}{1-q^{-s}}$ with $\frac{1}{1-q^{-s}} \frac{1}{1-q^{1-s}}$

so one new pole/zero: pole at $s=1$ of multiplicity 1.

2d adelic theory: $\zeta_S(s)^2 = m(s) \cdot \zeta_E(f, s) = \int f(z) |z|^s d\mu(z)$
 if $S = E$ elliptic surface

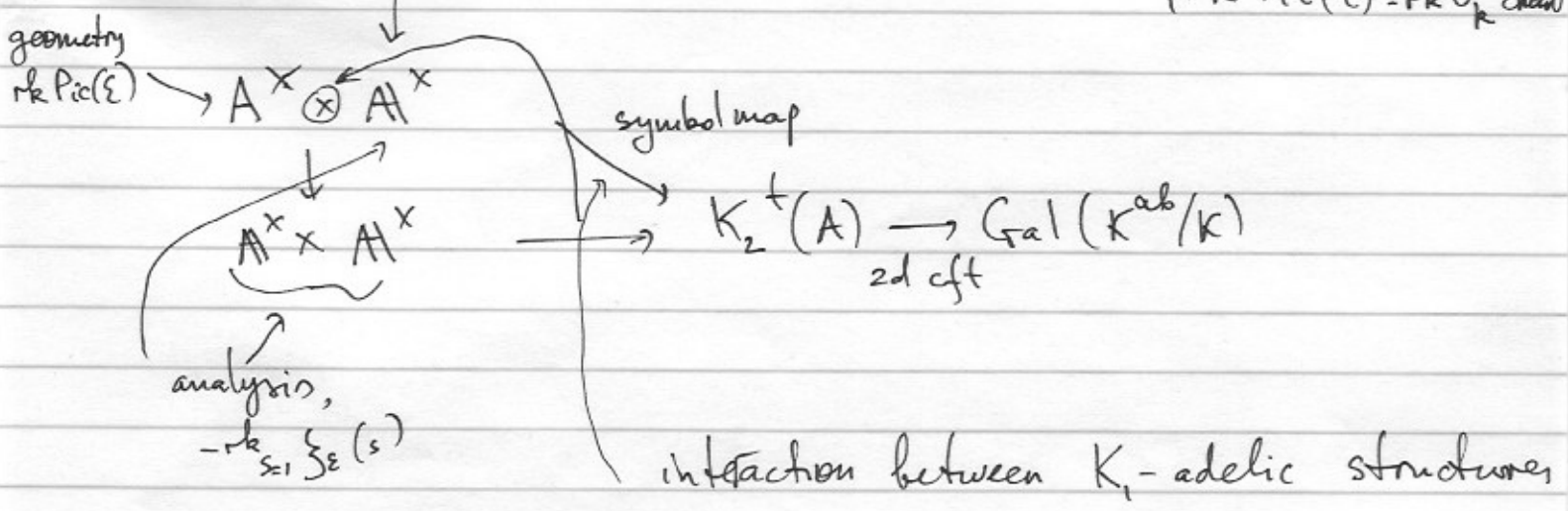
easy factor \uparrow
 $A^{\times} \times A^{\times}$
 2d ζ -integral against
 2d Haar measure μ

equality of $A \rightsquigarrow \zeta_E(f, s) = \underbrace{\zeta(s)}_{\text{entire}} + \zeta(2-s) + w(s),$

boundary/edge term

$$w(s) = \int_{A^{\times} \times A^{\times} / \mathbb{B}^{\times} \times \mathbb{B}^{\times}} \int_{\partial(\mathbb{B}^{\times} \times \mathbb{B}^{\times})} (f - \hat{f})$$

Adelic interpretation of BSD $\longrightarrow -\text{rk}_{s=1} \zeta_E(s) = \begin{cases} \text{rk Pic}(E) & \text{char } \neq 0 \\ \text{rk Pic}(E) - \text{rk } \mathcal{O}_k^{\times} & \text{char } = 0 \end{cases}$



Bridge between the geometric and arithmetic adelic structures gives the rank part of the BSD equality.