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Adelic lifts of geometry and arithmetic of arithm. surfaces and their BSD interaction

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(4 pages)

Usually, alg geometry studies codim 1 or dim 1 structures.
The main motif of this talk is codim 1 & codim 2 structures study, together.
The role of adelic structures: they are topological objects, with self-duality property at the additive level. Their symmetries underly key properties of discrete classical groups such as Pic, H^i, etc.

Example. S smooth projective surface over perfect field k, $K = k(S)$.

For every irreducible curve $y \subset S \rightsquigarrow \mathcal{O}_y \rightsquigarrow$ completion $\hat{\mathcal{O}}_y \to K_y$ fraction f.
closed point $x \in S \rightsquigarrow \mathcal{O}_x \to$ completion $\hat{\mathcal{O}}_x \to K_x = K \otimes \hat{\mathcal{O}}_x$
$x \in y \rightsquigarrow (\hat{\mathcal{O}}_x)_y \rightsquigarrow$ 2d local ring $\mathcal{O}_{x,y} \to K_{x,y}$ 2d fraction ring

$$K_{x,y} = \prod_{\substack{\text{local branches } z\\ \text{of } y \text{ at } x}} K_{x,z}, \qquad K_{x,z} \text{ 2d local field}$$

rk 1 integral structure, rk 2 integral str.

$$\text{if } k=\mathbb{C}, \text{ then } \quad \begin{array}{ccccc} K_{x,z} & \supset & \mathcal{O}_{x,z} & \supset & O_{x,z} \\ \| & & \| & & \| \\ \mathbb{C}((t))((u)) & \supset & \mathbb{C}((t))[[u]] & \supset & \mathbb{C}[[t]] + u\,\mathbb{C}((t))[[u]] \\ & & \downarrow{\scriptstyle \bmod u} & & \downarrow{\scriptstyle \bmod u} \\ & & \mathbb{C}((t)) & \supset & \mathbb{C}[[t]] \end{array}$$

Question: "refined" alg. geometry taking into account higher codimension data? $\mathcal{O}_y \to \mathcal{O}_{x,y}$, etc.?

2d adeles on surfaces $\longrightarrow$ geometric rk1 A

2d adeles on surfaces $\longrightarrow$ arithmetic rk2 $\mathbb{A}$

<u>Geometric adeles</u>: define first

$$y \subset S \rightsquigarrow \begin{array}{ccccc} A_y & \supset & \mathcal{O}A_y & \supset & OA_y \\ \| & & \| & & \| \\ \mathbb{A}_{k(y)}((t_y)) & \supset & \mathbb{A}_{k(y)}[[t_y]] & \supset & O\mathbb{A}_{k(y)} + t_y \mathbb{A}_{k(y)}[[t_y]] \end{array}$$

where $\mathbb{A}_{k(y)}$ is the usual 1d adelic ring $= \prod'_{x \in y} \widehat{k(y)}_x =$ the restricted product wrt integral rk1 structure $\hat{\mathcal{O}}_{x\in y}$, where $\widehat{k(y)}_x$ is the fraction ring of $\hat{\mathcal{O}}_{x\in y}$, $O\mathbb{A}_{k(y)} = \prod \hat{\mathcal{O}}_{x,y}$.

Now $A := \prod' A_y$ wrt $\mathcal{O}A_y$

$B := \prod K_y \cap A$ (inside $\prod K_{x,y}$)

$C := \prod K_x \cap A$

Get

$$\begin{tikzcd}[column sep=small] & A & \\ B \arrow[ur, dash] & & C \arrow[ul, dash] \\ & K \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

<u>$B^\times / K^\times \cdot \text{units} \simeq \mathrm{Pic}(S)$</u>

A has appropriate topology and is self-dual $A \simeq \mathrm{Hom}_{\mathrm{cont}}(A, \mathbb{C})$

For a divisor $D \in \mathrm{Div}(S)$, $D = \sum n_y [y] \rightsquigarrow A(D) = \prod t_y^{n_y} A_y \cap A$

Adelic complex $K \oplus B(D) \oplus C(D) \longrightarrow B \oplus C \oplus A(D) \longrightarrow A$ $\quad A_S(D)$

$B(D) = B \cap A(D)$, $C(D) = C \cap A(D)$

Then $H^i_{Zar}(S, \mathcal{O}(D)) \simeq H^i(\mathcal{A}_S(D))$

H^i are <u>$\mathbb{C}$-linearly compact & discrete</u> $\Rightarrow \dim_{\mathbb{C}} H^i < \infty$, so can define $\chi_{\mathcal{A}_S}(D) = \dim_{\mathbb{C}} H^0(\mathcal{A}_S(D)) - \dim_{\mathbb{C}} H^1(\mathcal{A}_S(D)) + \dim_{\mathbb{C}} H^2(\mathcal{A}_S(D))$

Then for $E_1, E_2 \in \mathrm{Div}(S)$ <u>the intersection</u>

$$(E_1, E_2) = \chi_{\mathcal{A}_S}(E_1+E_2) - \chi_{\mathcal{A}_S}(E_1) - \chi_{\mathcal{A}_S}(E_2) + \chi_{\mathcal{A}_S}(0)$$

and the <u>R-R thm</u> follows from <u>self-duality</u> of A and <u>discreteness</u> of K in A.

<u>Arithmetic adeles</u> $\mathbb{A} = \prod_{\substack{y \in S \\ \text{vert.}}} \mathcal{O}A_y$ wrt $\mathcal{O}A_y$

Codim 2 structure: closed pts of S.

Assume, from now on, that S is an arithm. surface (including surfaces over finite fields).

Then each closed pt x of S has finite residue field $k(x) = \mathcal{O}_x/\mathcal{M}_x$, let $|k(x)|$ be its cardinality.

The ζ-function $\zeta_S(s) = \prod (1 - |k(x)|^{-s})^{-1}$, $s \in \mathbb{C}$

Examples: $\zeta_{pt}(s) = \dfrac{1}{1-|k(x)|^{-s}}$, $\zeta_{\mathbb{P}^1_k}(s) = \dfrac{1}{(1-|k(x)|^{-s})} \dfrac{1}{(1-|k(x)|^{1-s})}$

Blow up at a point replaces $\frac{1}{1-q^{-s}}$ with $\frac{1}{1-q^{-s}}\,\frac{1}{1-q^{1-s}}$,

so one new pole/zero: pole at $s=1$ of multiplicity 1.

2d adelic theory: $\zeta_S(s)^2 = m(s)\cdot \zeta_{\mathcal{E}}(f,s) = \int f(\alpha)|\alpha|^s d\mu(\alpha)$

if $S=\mathcal{E}$ elliptic surface

easy factor

$\mathbb{A}^\times \times \mathbb{A}^\times$

2d ζ-integral against 2d Haar measure μ

duality of $\mathbb{A}$ $\rightsquigarrow$ $\zeta_{\mathcal{E}}(f,s) = \xi(s) + \xi(2-s) + \omega(s)$,

entire

boundary/edge term

$$\omega(s) = \int_{\mathbb{A}^\times\times\mathbb{A}^\times/\mathbb{B}^\times\times\mathbb{B}^\times} \int_{\partial(\mathbb{B}^\times\times\mathbb{B}^\times)} (f-\hat{f})$$

Adelic interpretation of BSD $\longrightarrow$ $-\mathrm{rk}_{s=1}\zeta_{\mathcal{E}}(s) = \begin{cases} \mathrm{rk}\,\mathrm{Pic}(\mathcal{E}) & \mathrm{char}\neq 0 \\ \mathrm{rk}\,\mathrm{Pic}(\mathcal{E}) - \mathrm{rk}\,\mathcal{O}_k^\times & \mathrm{char}\,0 \end{cases}$

geometry $\mathrm{rk}\,\mathrm{Pic}(\mathcal{E}) \longrightarrow A^\times \otimes \mathbb{A}^\times$

symbol map

$\mathbb{A}^\times \times \mathbb{A}^\times \longrightarrow K_2^t(A) \xrightarrow[\text{2d cft}]{} \mathrm{Gal}(K^{ab}/K)$

analysis, $-\mathrm{rk}_{s=1}\zeta_{\mathcal{E}}(s)$

interaction between K_1-adelic structures

Bridge between the geometric and arithmetic adelic structures gives the rank part of the BSD equality.