Flexibility of Cox rings of del Pezzo surfaces

Thm / Def / Motivation (Arzhantsev, Flenner, Kaliman, Kutzschebauch, Zaidenberg)

X affine variety, $\dim X \geq 2$,

$\mathrm{SAut}(X) \subseteq \mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-actions.

Then X is called flexible if the following equivalent conditions hold:

i) $\mathrm{SAut}(X)$ acts transitively on X_{reg}

ii) $\mathrm{SAut}(X)$ acts infinitely transitively on X_{reg}.

iii) For $x \in X_{reg}$ $T_x X$ is spanned by directions, which are tangent to $\mathbb{G}_a$-orbits.

Affine cones over del Pezzo surfaces

Facts

- degree ≥ 4 all cones are flexible
- degree ≤ 3 pluri-anticanonical cones are not flexible.

Cox rings

Examples:

- $\mathbb{A}^n$ $n \geq 2$
- toric varieties without torus factors

affine varieties from projective ones

i) affine cones

ii) affine charts

iii) Cox ring / total coord. space

$$R(X) = \bigoplus_{[D] \in Cl(X)} H^0(X, \mathcal{O}(D))$$

Main theorem (Michałek, Perepechko, —)

The Cox ring of a smooth del Pezzo surface is flexible.

Remark: degree 1: - 242 generators

- 10800 relations

Flexibility of cones vs flexibility of affine charts

$\boxed{\text{Thm}}$ I (Michałek Perepechko, —)

given $X \subset \mathbb{P}^N$ and a flexible covering $\{U_i\}_i$ with $U_i = X \setminus \{x_i\}$. Then the corresponding cone Y is flexible.

$\pi : Y \setminus \{0\} \to X$

proof:

- For U_i consider $\mathbb{G}_a$-actions $\lambda_{i1}, \dots \lambda_{i e_i}$ carrying $(U_i)_{reg}$
- lift λ_{ij} to $\hat{\lambda}_{ij}$ on $\pi^{-1}(U_i) \cong U_i \times k^*$
- Then after rescaling $\tilde{\lambda}_{ij}(a) := \hat{\lambda}_{ij}(x_i \cdot a)$

$\tilde{\lambda}_{ij}$ extends to Y.

- the subgroups generated by all $\tilde{\lambda}_{ij}$ acts transitively on Y_{reg} $\square$

Cox rings of del Pezzo surfaces (after Batyrev–Popov)

- $R(X)_{[D]} = H^0(X, \mathcal{O}(D))$

$(s)_0 \in |D|$

- given $D \geq 0$ there is $\chi^D \in R(X)_{[D]}$ with $(\chi^D)_0 = D$.
- we have $\chi^{D+D'} \sim \chi^D \chi^{D'}$

$S_8 \to \cdots \to S_1 \xrightarrow{\varphi} S_0 = \mathbb{P}^2$

$\boxed{\text{Prop}}$

$$R(S_{i+1})_{\chi^E} \cong R(S_i)[t^{\pm 1}]$$

proof:

$$\varphi^* : R(S_i)[t^{\pm 1}] \longrightarrow R(S_{i+1})_{\chi^E}$$
$$\chi^D \longmapsto \varphi^*(\chi^D) = \chi^{\varphi^* D}$$
$$t \longmapsto \chi^E$$

consider $D' \geq 0 \Rightarrow D' + aE = \varphi^* D + bE$ $a, b \geq 0$

$\Rightarrow \chi^{D'} \cdot (\chi^E)^a \sim \varphi^*(\chi^D) \cdot (\chi^E)^b$

$\Rightarrow \chi^D \sim \varphi^*(\chi^D \cdot t^{b-a})$ $\square$

Prop Consider $\mathbb{Z}$-grading on $R(S_i)$ via $\deg(\chi^D) = D \cdot (-K_{S_i})$

$\boxed{\text{Thm}}$ ~~For~~ II

- For $3 \leq i \leq 7$ $R(S_i)$ is generated in degree 1 by χ^{E_i} $i = 1, \dots, N$ with $(E_i)^2 = -1$
- For $i = 8$ there are two additional generators f_1, f_2 with $f_1^2, f_2^2 \in (\chi^{E_1}, \dots, \chi^{E_N})$

Set $X_i = \operatorname{Proj} R(S_i)$

$Y_i = \operatorname{Spec} R(S_i)$

Proof of the Main theorem (i.e. Y_i is flexible)

induction on i:

For $i=1,2,3$ S_i is toric $\Rightarrow$ $Y_i \cong \mathbb{A}^{i+3}$

. For $4 \le i \le 8$ X_i is covered by

$U_j := X_i \setminus \{z^{E_j} = 0\}$ (by Thm II)

. ~~Now~~ $\cancel{X_i \setminus \{z}$ Now $k[U_j] = \left(R(S_{i+1})_{z^{E_j}}\right)_0$

$\underset{\text{Prop.}}{\cong} \left(R(S_i)[t^{\pm 1}]\right)_0$

$\cong R(S_i)$

$= k[Y_i]$

by induction hypothesis U_j is flexible for all j

$\underset{\text{Thm I}}{\Rightarrow}$ Y_{i+1} is flexible $\square$