Cylinders in del Pezzo (fibrations ..) (1)
Edinburgh 30/06/17

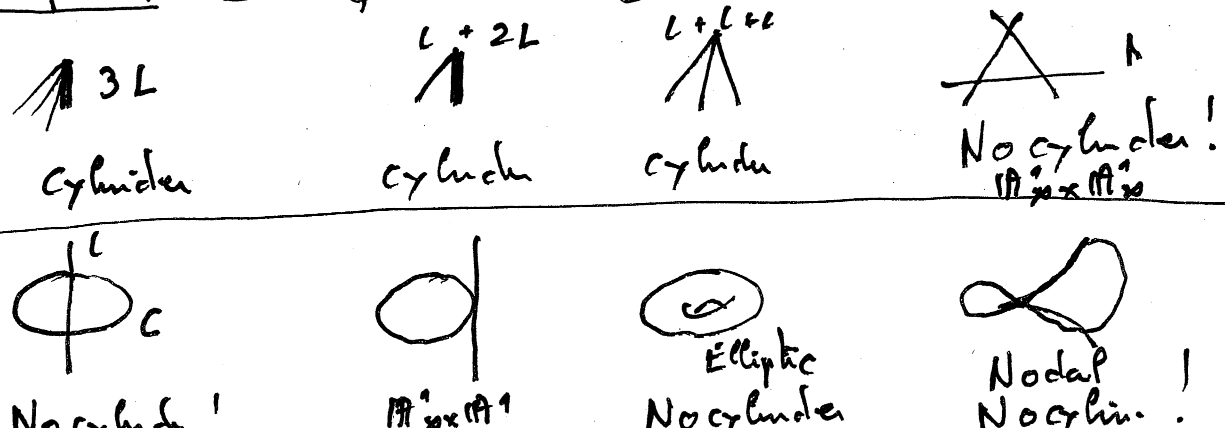
Def : X q-projective variety$/k$
A k-cylinder in X is an open subset $U \subset X$ iso to $Z \times \mathbb{A}^1_k$ for some k-variety Z.

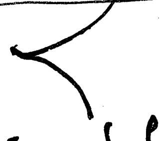
Questions :
1) Which varieties admit k-cylinders
2) X smooth projective, $L \in \mathrm{Pic}(X)$ fixed familiar line bundle
Which $D \in H^0(X, L)$ have the property that $X \setminus D$ contains / is a cylinder

Ex : • $X = \mathbb{P}^n_k$, D = line Hyperplane
• S smooth rational surface $/k = \bar{k}$ then $\forall x \in S \ \exists\ U \ni x$ open neighborhood iso to $\mathbb{A}^2_k$
• X smooth projective non k-unirulled: then X does not contain a cylinder : X = Smooth conic without k-rational pt

Example for 2): $S = \mathbb{P}^2_{\mathbb{C}}$ $L = -K_S$

D:





Proof for (*) :

$$\begin{tikzcd}
 & & & \tilde{\mathbb{P}}^2 \arrow[dl, "\sigma"'] \arrow[ddl, "\tilde{\varphi}"] \\
Z \times \mathbb{A}^1 = U \arrow[r, hook] \arrow[d, "pr_2"] & \mathbb{P}^2 \setminus E \arrow[r, hook] \arrow[d, "\psi \text{ well define}"] & \mathbb{P}^2 \arrow[d, dashed, "\varphi"] & \\
Z \arrow[r, hook] & \bar{Z} \arrow[r, Rightarrow] & \bar{Z} &
\end{tikzcd}$$

- φ has ~~a~~ at most a proper basept ~~..~~ at the ~~closure~~ common closure of the general fibre of pr_2 (which are rational curves with a unique pt at infinity
- p is not in $\mathbb{P}^2 \setminus E$ since the latter is affine
- p exist bcau $\bar{Z} \simeq \mathbb{P}^1$ and not regular ~~..~~ $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

• Take resolution (minimal) (2)

$\tilde{\varphi} : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with a section $C \cong \mathbb{P}^1$

$$\begin{tikzcd} \tilde{\mathbb{P}}^2 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d, equal] \\ \mathbb{F} \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

sequence of blow-ups

• Fibers of $\tilde{\varphi}$ are trees of $\mathbb{P}^1$.

• $p \in E \implies E$ contained in a fiber !!

⇝ Need criteria / obstruction for existence of cylinders.

For complex surfaces: Kodaira dimension κ:

X non complete smooth + definition

$\kappa(U) = \kappa(Z \times \mathbb{A}^1) = -\infty \geq \kappa(X)$ $\quad$ X contains a cylinder $\Rightarrow \kappa(X) = -\infty$

Theorem (Miyanishi-Fujita): X smooth qp non complete with connected boundary

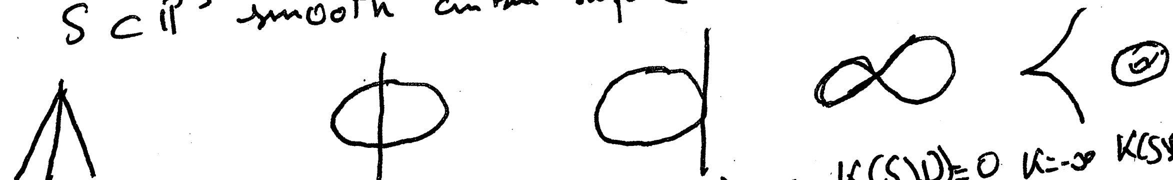
$\kappa(X) = -\infty \iff X$ is $\mathbb{A}^1$-ruled $\iff X$ contains a cylinder.

Rq: Fails without connectedness hypothesis:

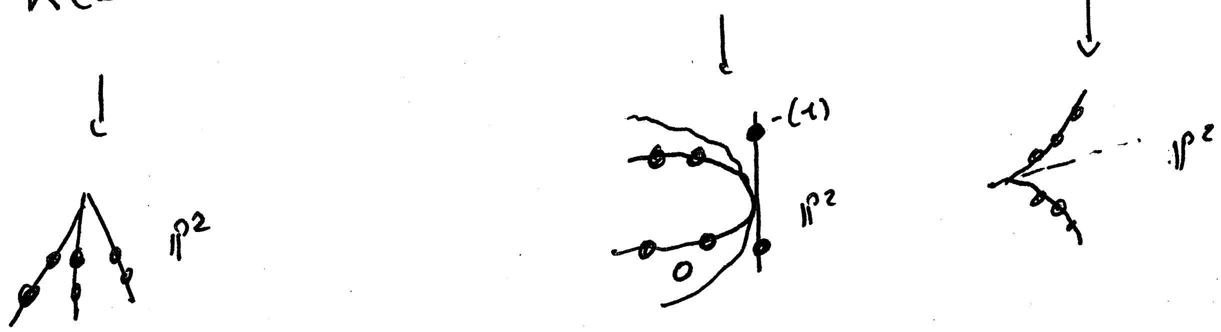
$X = \{x^2 + y^3 + z^3 = 0\} \setminus \{(0,0,0)\}$ $\quad$ $\kappa(X) = -\infty$ but no cylinders.

Example: $S \subset \mathbb{P}^3$ smooth cubic surface $\quad L = -K_S$

D:



$\kappa(S \setminus D) = -\infty$ $\quad$ $\kappa(S \setminus D) = 0$ $\quad$ $\kappa(S \setminus D) = -\infty$ $\quad$ $\kappa(S \setminus D) = 0$ $\quad$ $\kappa = -\infty$ $\quad$ $\kappa(S \setminus D) = 0$



Problem: Criterion fail:

– In higher dimension: $X = \mathbb{P}^3 \setminus$ Smooth cubic surface
$\kappa(X) = -\infty$ but no cylinder !!

– For surfaces over non closed fields
$S = \mathbb{P}^2_k \setminus$ Smooth conic without rational pt $\quad \kappa(S) = -\infty$
but non cylinder.

Proof: Suppose

$$\begin{tikzcd} Z\times\mathbb{A}^1_k = U \arrow[r, hook] \arrow[d] & \mathbb{P}^2_k\setminus C \arrow[r, hook] \arrow[dr] & \mathbb{P}^2_k \arrow[d, dashed, "\varphi"] \\ Z \arrow[rr, hook] & & \overline{Z}\simeq\mathbb{P}^1_k \end{tikzcd}$$

φ has a base pt p on $\varphi_{\bar k}: \mathbb{P}^2_{\bar k} \dashrightarrow \overline{Z}_{\bar k}$ has an unique proper base pt $\Rightarrow$ p is a k-rational pt, supported on C, absurd!

§ Cylinders in minimal del Pezzo surfaces.

Theorem: Let S be a smooth minimal dP surface$/k$ of degree $d = K_S^2$ $(\rho(S)=1)$
If $d\leq 4$ Then S does not contain any cylinder
If $d\geq 5$ then S contains a cylinder iff S has a k-rational pt.

Idea of proofs:

$$\begin{tikzcd} Z\times\mathbb{A}^1 = U \arrow[r, hook] \arrow[d, "pr_2"'] & S \arrow[d, dashed, "\varphi"] \\ Z \arrow[r, hook] & \overline{Z} \end{tikzcd}$$

- Minimality $\Rightarrow$ φ is not a morphism: otherwise: $\mathbb{P}^1$-fibration with a section C $\quad \mathrm{Pic}(\overline{Z})\oplus\mathbb{Z}C \hookrightarrow \mathrm{Pic}(S)$!
- φ has a unique proper base pt p, and p is k-rational $\Rightarrow$ S is rational: $C\simeq\mathbb{P}^1_k$ $\Rightarrow$ $\overline{Z}=\mathbb{P}^1_k$ $\cup$ k-rational (*)

(*) Minimality is again crucial:

$C\times\mathbb{P}^1_k$
C smooth conic without rational pt: del Pezzo degree 8 with $\rho=2$ contains a cylinder and no rational...

- $d\leq 3$: ~~En~~ Non-rationality of S by Segre–Manin.
- $d\geq 5$ $\quad (-K_S)^2 > 4 \qquad (-nK_S)^2 > 4n^2$ $\rightsquigarrow$ Birational rigidity.

④

$$\begin{tikzcd} & & \tilde{S} \supset C \text{ section} \arrow[dl, "\sigma"'] \arrow[ddl, "\tilde{\varphi}"] \\ U \arrow[r, hook] \arrow[d] & (S,p) \arrow[d, dashed, "\varphi"] & \\ Z \arrow[r, hook] & \bar{Z} = \mathbb{P}^1_k & \end{tikzcd}$$

$\varphi \rightsquigarrow$ Mobile linear system $\mathcal{C} \subset |-nK_S|$ $\quad n \geq 1$

$\tilde{\varphi} \rightsquigarrow \tilde{\mathcal{C}}$ proper transform of $\mathcal{C}$

$$\left(K_{\tilde{S}} + \frac{1}{n}\tilde{\mathcal{C}}\right) = \sigma^*\left(K_S + \frac{1}{n}\mathcal{C}\right) + aC + R$$

$\cap$ general fiber of $\tilde{\varphi}$: $\quad -2 = 0 + a$

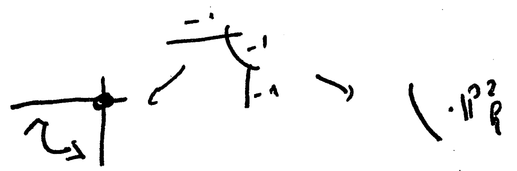
$\Rightarrow \quad (S, \frac{1}{n}\mathcal{C})$ not lc at p.

$$n^2K_S^2 = (\mathcal{C}_1 \cdot \mathcal{C}_2) \geq \underbrace{(\mathcal{C}_1 \cdot \mathcal{C}_2)_p \geq}_{\text{[Corti's inequality]}} 4n^2 \quad !$$

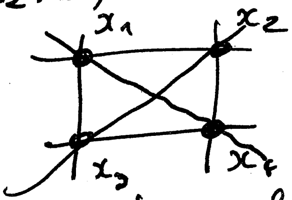
[Slogan: If a mildly singular variety with $\rho = 1$ contains a cylinder it contains a divisor with bad discrepancy corresponding to a section.]

Existence: Follow the proof of rationality keeping track of cylinders.

- $d = 9$: $\mathbb{P}^2_k$
- $d = 8$: $Q \leftarrow \sim \rightarrow \mathbb{P}^2_k$



- $d = 5$: $S_{\bar{k}}$: blow-up of $\mathbb{P}^2_k$ in 4 pts forming an orbit of $Gal(\bar{k}/k)$



p k-rational pt. Not on the union of (-1)-curves.
[otherwise one of these curves would be defined over k and S would not be minimal ...]

Blow-up to get 5 new (-1)-curves ~~all defined~~ disjoint
what curve is defined over k: $\ell_{x_i p}$ or $C_{x_1 \dots x_4 p}$
because they are the only ones intersecting the exceptional divisor E_0

$$\begin{tikzcd} & \tilde{S} \arrow[dl] \arrow[dr, "\text{contract}"] & \\ S & & \mathbb{P}^2_k \end{tikzcd}$$

Image of E_0 is a smooth conic with a rational pt!
$\rightsquigarrow$ Cylinder!

§ Del Pezzo fibrations :

Prop : Let $\pi : V \xrightarrow{\curvearrowleft} W$ be a del Pezzo fibration (relative dim 2 proper dominant morphism with connected fibers)

V: $\mathbb{Q}$-factorial terminal sing; W: Normal

$-K_{V/W}$ ample, $\rho(V/W) = 1$

- If $(K_{V/W})^2 \geq 5$ and π has a rational section then V contains a cylinder.
- Otherwise: if $(K_{V/W})^2 \leq 4$ or π has no rational section then V does not contain any <u>VERTICAL</u> cylinder

Def : • Recall what "Vertical" means
• Equivalent to existence of a $\mathrm{Frac}(\mathcal{O}_W)$-cylinder in the generic fiber of π which is a normal dP surface of degree $(K_{V/W})^2$ over $\mathrm{Frac}(\mathcal{O}_W)$.

But what about "twisted" cylinders ?

Prop : For every $d \geq 1$ $\exists$ $V \xrightarrow{\pi} \mathbb{P}^1$ del Pezzo fibration containing a cylinder : actually V is a compactification of $\mathbb{A}^3$!

Note : dP fibration of degree $d \leq 3$ are usually non rational (when smooth).

If time permits !

Idea of the construction : (degree 3 and 4 ; $d = 1, 2$ are similar)

- Start from $S \hookrightarrow \mathbb{P}^3$ smooth cubic surface
- $H \subset \mathbb{P}^3$ hyperplane and consider the pencil $\langle S ; 3H \rangle : \mathbb{P}^3 \overset{\varphi}{\dashrightarrow} \mathbb{P}^1$
- Take a good resolution

$$\begin{array}{ccc} \mathbb{P}^3 & \xleftarrow{\sigma} & \tilde{\mathbb{P}}^3 \\ \varphi \vdots & & \downarrow \tilde{\varphi} \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}$$

of φ

- σ is made of blow-ups,
- which to iso on general fiber of φ
- $\mathbb{P}^3 \setminus H \xleftarrow{\sigma} \tilde{\mathbb{P}}^3 \setminus \sigma^{-1}(H)$ iso

: It exists !

- Run a MMP over $\mathbb{P}^1$ and cross your fingers so that

$$\begin{tikzcd} \tilde{\mathbb{P}}^3 \arrow[r, dashed, "\tau"] \arrow[d, "\hat{\varphi}"'] & \tilde{\mathbb{P}}' \arrow[d, "\varphi'"] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

- φ' is a del Pezzo fibration
- τ induces an iso between $\mathbb{A}^3 = \tilde{\mathbb{P}}^3 \setminus \sigma^{-1}(H)$ and its image.

$\sigma^{-1}(H) = \{$ Horizontal components $\} \cup \{$ Vertical components $\}$

Horizontal components: Exactly as many as irreducible component of $H \cap S$ and each intersect a general fibre S_c along $S_c \cap H$.

Vertical components: $\tilde{\varphi}^{-1}(\varphi(H))$

- Facts:
  - $3H$ is the unique non integral member of the pencil $\Longrightarrow$ Relative MMP contracts either irreducible component of $\sigma^{-1}(H)$ or Horizontal components not contained in $\sigma^{-1}(H)$.
  - If τ contract a horizontal ~~component~~ divisor then it must be an irreducible component of $\sigma^{-1}(H)$ otherwise $E \cap$ general member S_c is a disjoint union of (-1)-curves. But $\sigma(E) \subset \mathbb{P}^3$ is ample and intersects then a general member along (-1)-curves only: absurd!

$\rightsquigarrow$ τ preserves $\mathbb{A}^3 = \tilde{\mathbb{P}}^3 \setminus \sigma^{-1}(H)$

- Criterion:
  - If $H \cap S$ is irreducible then $\begin{matrix}\tilde{\mathbb{P}}' \\ \downarrow \\ \mathbb{P}^1\end{matrix}$ dP_3-fibration

    So not a conic bundle!
  - If $H \cap S = 3$ lines then the output is necessarily MCB: can contract at most an irreducible horizontal component so relative Picard rk is 2.

- Suppose $H \cap S =$ Conic $\cup$ line. Then $\exists$ good resolution $\begin{matrix}\tilde{\mathbb{P}}^3 \\ \downarrow \\ \mathbb{P}^3\end{matrix}$ and a relative MMP $\tilde{\mathbb{P}}^3 \xrightarrow{\tau} \tilde{\mathbb{P}}'$ whose output is a dP_4-fibration: contract the Horizontal component corresponding to L.