Differentiation of Genus 3 Hyperelliptic Functions

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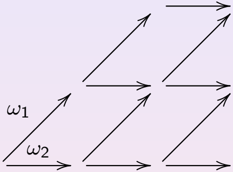
$$\mathcal{V}_\lambda = \{(x, y) \in \mathbb{C}^2 \colon y^2 = x^3 + \lambda_4 x + \lambda_6\}$$

$$\updownarrow$$

$$\mathbb{C}/\Gamma \qquad\qquad -4\lambda_4^3 \neq 27\lambda_6^2$$



$$\frac{\wp(u)'^2}{4} = \wp(u)^3 + \lambda_4 \wp(u) + \lambda_6$$

$$\frac{\partial}{\partial u}\wp(u) = \wp'(u), \quad \frac{\partial}{\partial u}\wp'(u) = 6\wp(u)^2 + 2\lambda_4.$$

$$\wp(u) = \frac{1}{u^2} + \sum_{(k,m)\neq(0,0)} \left(\frac{1}{(u - 2k\omega_1 - 2m\omega_2)^2} - \frac{1}{(2k\omega_1 + 2m\omega_2)^2} \right),$$

$$\wp(u) = \frac{1}{u^2} - \frac{1}{80}\lambda_4 u^2 - \frac{1}{168}\lambda_6 u^4 + \frac{1}{19200}\lambda_4^2 u^6 + \frac{1}{49280}\lambda_4 \lambda_6 u^8 + \ldots$$

F. G. Frobenius, L. Stickelberger, *Über die Differentiation der elliptischen Functionen nach den Perioden und Invarianten*, J. Reine Angew. Math., 92 (1882), 311–337.

$$\mathcal{L}_0 = L_0 - u\partial_u,$$
$$\mathcal{L}_1 = \partial_u,$$
$$\mathcal{L}_2 = L_2 - \zeta(u;\lambda_4,\lambda_6)\partial_u.$$

$$\mathcal{U} \xrightarrow{\mathbb{C}/\Gamma} \mathcal{B}$$

Here $\wp(u;\lambda_4,\lambda_6)$ and $\zeta(u;\lambda_4,\lambda_6)$ are Weierstrass functions, and the fields L_k on $\mathcal{B}$ are

$$L_0 = 4\lambda_4\partial_{\lambda_4} + 6\lambda_6\partial_{\lambda_6}, \qquad L_2 = 6\lambda_6\partial_{\lambda_4} - \frac{4}{3}\lambda_4^2\partial_{\lambda_6}.$$

Lie algebra

$$[\mathcal{L}_0,\mathcal{L}_1] = \mathcal{L}_1, \qquad [\mathcal{L}_0,\mathcal{L}_2] = 2\mathcal{L}_2, \qquad [\mathcal{L}_1,\mathcal{L}_2] = \wp(u;\lambda_4,\lambda_6)\mathcal{L}_1.$$

Hyperelliptic functions of genus g

An *Abelian function* is a meromorphic function on $\mathbb{C}^g$ with a lattice of periods $\Gamma \subset \mathbb{C}^g$ of rank $2g$. $T^g = \mathbb{C}^g/\Gamma$.

$$\mathcal{V}_\lambda = \{(X,Y) \in \mathbb{C}^2 \colon Y^2 = X^{2g+1} + \lambda_4 X^{2g-1} + \ldots + \lambda_{4g} X + \lambda_{4g+2}\}$$

where $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}$. $\mathcal{B} = \mathbb{C}^{2g} \backslash \Sigma$

A hyperelliptic function of genus g
is a smooth function defined on
an open dense subset of $\mathbb{C}^g \times \mathcal{B}$,
such that for each $\lambda \in \mathcal{B}$ it's restriction to $\mathbb{C}^g \times \lambda$
is Abelian with T^g the Jacobian $\mathcal{J}_\lambda$ of $\mathcal{V}_\lambda$.

$$\mathcal{U} \xrightarrow{\mathcal{J}_\lambda = \mathbb{C}^g/\Gamma} \mathcal{B}$$

Problem of Differentiation of Hyperelliptic Functions

Denote the field of hyperelliptic functions by $\mathcal{F}$.

- Find the generators of the $\mathcal{F}$-module $\mathsf{Der}\,\mathcal{F}$ of derivations of the field $\mathcal{F}$ and their action on $\mathcal{F}$.
- Describe the structure of Lie algebra $\mathsf{Der}\,\mathcal{F}$ (i.e. find the commutation relations).

V. M. Buchstaber, D. V. Leikin, *Solution of the Problem of Differentiation of Abelian Functions over Parameters for Families of* (n, s)*-Curves*, Funct. Anal. Appl., 42:4, (2008).

V. M. Buchstaber,
Polynomial dynamical systems and Korteweg–de Vries equation,
Proc. Steklov Inst. Math., 294 (2016), 176–200.

Polynomial map

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$$
\begin{array}{ccc}
\mathcal{U} & \overset{\varphi}{\dashrightarrow} & \mathbb{C}^{3g} \\
\downarrow \pi & & \downarrow p \\
\mathcal{B} & \hookrightarrow & \mathbb{C}^{2g}
\end{array}
\qquad
\begin{array}{cccc}
\mathcal{L}_1 & \mathcal{L}_3 & \dots & \mathcal{L}_{2g-1} \\
\mathcal{L}_0 & \mathcal{L}_2 & \dots & \mathcal{L}_{4g-2}
\end{array}
$$

The map φ will be given by a set of generators of $\mathcal{F}$
and p will be a polynomial map.

We will now describe φ and p.

Hyperelliptic Kleinian functions

Let $\sigma(u;\lambda)$ be the hyperelliptic sigma function
(or Weierstrass elliptic sigma function in genus $g = 1$ case).
We use the notation

$$\zeta_k = \frac{\partial}{\partial_{u_k}} \ln \sigma(u;\lambda), \qquad \wp_{i;k_1,\ldots,k_n} = -\frac{\partial^{i+n}}{\partial^i_{u_1}\partial_{u_{k_1}}\cdots\partial_{u_{k_n}}} \ln \sigma(u;\lambda),$$

where $u = (u_1, u_3, \ldots, u_{2g-1})$, $n \geqslant 0$, $i + n \geqslant 2$.
In the case $n = 0$ we will skip the semicolon.

$$\mathcal{U} \overset{\varphi}{\dashrightarrow} \mathbb{C}^{3g} : \qquad (u,\lambda) \mapsto \begin{pmatrix} \wp_{1;1} & \wp_{1;3} & \cdots & \wp_{1;2g-1} \\ \wp_{2;1} & \wp_{2;3} & \cdots & \wp_{2;2g-1} \\ \wp_{3;1} & \wp_{3;3} & \cdots & \wp_{3;2g-1} \end{pmatrix}.$$

V. M. Buchstaber, V. Z. Enolskii, D. V. Leikin,
Kleinian functions, hyperelliptic Jacobians and applications,
Reviews in Mathematics and Math. Physics, 10:2, (1997).

For $i, k \in \{1, 3, \ldots, 2g-1\}$ we have the relations

$$\begin{aligned}
\wp_{3;i} =& 6\wp_2\wp_{1;i} + 6\wp_{1;i+2} - 2\wp_{0,3;i} + 2\lambda_4\delta_{i,1}, \\
\wp_{2;i}\wp_{2;k} =& 4\left(\wp_2\wp_{1;i}\wp_{1;k} + \wp_{1;k}\wp_{1;i+2} + \wp_{1;i}\wp_{1;k+2} + \wp_{;k+2,i+2}\right) - \\
& - 2(\wp_{1;i}\wp_{;3,k} + \wp_{1;k}\wp_{;3,i} + \wp_{;k,i+4} + \wp_{;i,k+4}) + \\
& + 2\lambda_4(\delta_{i,1}\wp_{1;k} + \delta_{k,1}\wp_{1;i}) + 2\lambda_{i+k+4}(2\delta_{i,k} + \delta_{k,i-2} + \delta_{i,k-2}).
\end{aligned}$$

Corollary

Consider the map $\varphi : \mathcal{U} \dashrightarrow \mathbb{C}^{\frac{g(g+9)}{2}}$,

$$\varphi : (u, \lambda) \mapsto (x_{i,j}, w_{k,l}, \lambda_s) = (\wp_{i;j}, \wp_{0;k,l}, \lambda_s).$$

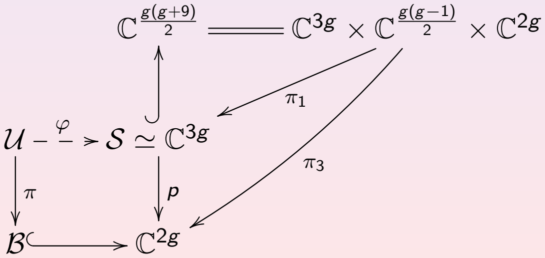
Then the image of φ *lies in* $\mathcal{S} \subset \mathbb{C}^{\frac{g(g+9)}{2}}$,
where $\mathcal{S}$ *is determined by the set of* $\frac{g(g+3)}{2}$ *equations.*

Theorem

The projection $\pi_1\colon \mathbb{C}^{\frac{g(g+9)}{2}} \to \mathbb{C}^{3g}$ *on the first* $3g$ *coordinates gives the isomorphism* $\mathcal{S} \simeq \mathbb{C}^{3g}$.

Corollary

The projection $\pi_3\colon \mathbb{C}^{\frac{g(g+9)}{2}} \to \mathbb{C}^{2g}$ *on the last* $2g$ *coordinates gives a polynomial map* $p\colon \mathbb{C}^{3g} \to \mathbb{C}^{2g}$.



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Denote the field of hyperelliptic functions by $\mathcal{F}$.

- Find the generators of the $\mathcal{F}$-module $\mathsf{Der}\,\mathcal{F}$ of derivations of the field $\mathcal{F}$ and their action on $\mathcal{F}$.
- Describe the structure of Lie algebra $\mathsf{Der}\,\mathcal{F}$.

$\mathsf{Der}\,\mathcal{F}$ is determined by it's action on the generators of $\mathcal{F}$.
We take the $3g$ functions $\wp_{i;j}(u, \lambda)$ as these generators.

$$\begin{array}{ccc} \mathcal{U} & \stackrel{\varphi}{\dashrightarrow} & \mathbb{C}^{3g} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \mathcal{B} & \hookrightarrow & \mathbb{C}^{2g} \end{array}$$

Problem of Differentiation of Hyperelliptic Functions

Denote the ring of polynomials in $\lambda \in \mathbb{C}^{2g}$ by $\mathcal{P}$.
A vector field $\mathcal{L}$ in $\mathbb{C}^{3g}$ will be called projectable for p if there exists a vector field L in $\mathbb{C}^{2g}$ such that

$$\mathcal{L}(p^* f) = p^* L(f) \quad \text{for any} \quad f \in \mathcal{P}.$$

The vector field L will be called the pushforward of $\mathcal{L}$.
For a projectable vector field $\mathcal{L}$ we have $\mathcal{L}(p^*\mathcal{P}) \subset p^*\mathcal{P}$.

- Find $3g$ polynomial vector fields in $\mathbb{C}^{3g}$ projectable for $p\colon \mathbb{C}^{3g} \to \mathbb{C}^{2g}$ and independent at any point in $p^{-1}(\mathcal{B})$.
- Construct their polynomial Lie algebra.

Genus 1

The projection p takes the form

$$\lambda_4 = \frac{1}{2}x_4 - 3x_2^2,$$
$$\lambda_6 = \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 + 2x_2^3.$$

The vector fields are

$$\begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 3x_3 & 4x_4 \\ x_3 & x_4 & 12x_2x_3 \\ \frac{2}{3}x_4 - 2x_2^2 & 3x_2x_3 & 2x_2x_4 + 3x_3^2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_4} \end{pmatrix}$$

The polynomial Lie algebra is

$$[\mathcal{L}_0, \mathcal{L}_1] = \mathcal{L}_1, \quad [\mathcal{L}_0, \mathcal{L}_2] = 2\mathcal{L}_2, \quad [\mathcal{L}_1, \mathcal{L}_2] = x_2\mathcal{L}_1.$$

Genus 2 polynomial map

In the formulas below we write y_4 instead of $x_{1,3}$, y_5 instead of $x_{2,3}$ and y_6 instead of $x_{3,3}$ to shorten down the formulas.

The projection p takes the form

$$\begin{aligned}
\lambda_4 &= -3x_2^2 + \frac{1}{2}x_4 - 2y_4, \\
\lambda_8 &= -\frac{1}{2}(x_4 y_4 - x_3 y_5 + x_2 y_6) + \left(4x_2^2 + y_4\right) y_4, \\
\lambda_6 &= 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2 x_4 - 2x_2 y_4 + \frac{1}{2}y_6, \\
\lambda_{10} &= 2x_2 y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4 y_6.
\end{aligned}$$

Genus 2 Lie algebra

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k, \quad k = 1, 2, 3, 4, 6, \qquad [\mathcal{L}_1, \mathcal{L}_3] = 0,$$

$$[\mathcal{L}_1, \mathcal{L}_2] = \wp_2\mathcal{L}_1 - \mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_4] = \wp_{1;3}\mathcal{L}_1 + \wp_2\mathcal{L}_3, \quad [\mathcal{L}_1, \mathcal{L}_6] = \wp_{1;3}\mathcal{L}_3,$$

$$[\mathcal{L}_3, \mathcal{L}_2] = \left(\wp_{1;3} + \frac{4}{5}\lambda_4\right)\mathcal{L}_1, \qquad [\mathcal{L}_3, \mathcal{L}_6] = \frac{3}{5}\lambda_8\mathcal{L}_1 + \wp_{0;3,3}\mathcal{L}_3,$$

$$[\mathcal{L}_3, \mathcal{L}_4] = \left(\wp_{0;3,3} + \frac{6}{5}\lambda_6\right)\mathcal{L}_1 + (\wp_{1;3} - \lambda_4)\,\mathcal{L}_3,$$

$$[\mathcal{L}_2, \mathcal{L}_4] = \frac{8}{5}\lambda_6\mathcal{L}_0 - \frac{8}{5}\lambda_4\mathcal{L}_2 + 2\mathcal{L}_6 - \frac{1}{2}\wp_{2;3}\mathcal{L}_1 + \frac{1}{2}\wp_3\mathcal{L}_3,$$

$$[\mathcal{L}_2, \mathcal{L}_6] = \frac{4}{5}\lambda_8\mathcal{L}_0 - \frac{4}{5}\lambda_4\mathcal{L}_4 - \frac{1}{2}\wp_{1;3,3}\mathcal{L}_1 + \frac{1}{2}\wp_{2;3}\mathcal{L}_3,$$

$$[\mathcal{L}_4, \mathcal{L}_6] = -2\lambda_{10}\mathcal{L}_0 + \frac{6}{5}\lambda_8\mathcal{L}_2 - \frac{6}{5}\lambda_6\mathcal{L}_4 + 2\lambda_4\mathcal{L}_6 - \frac{1}{2}\wp_{0;3,3,3}\mathcal{L}_1 + \frac{1}{2}\wp_{1;3,3}\mathcal{L}_3.$$

Genus 2 generators

$$\mathcal{U} \xrightarrow{\ \pi\ } \mathcal{B}$$

$$\mathcal{L}_1 = \partial_{u_1},$$
$$\mathcal{L}_3 = \partial_{u_3},$$

$$\mathcal{L}_0 = L_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3},$$
$$\mathcal{L}_2 = L_2 + \left(-\zeta_1 + \frac{4}{5}\lambda_4 u_3\right)\partial_{u_1} - u_1\partial_{u_3},$$
$$\mathcal{L}_4 = L_4 + \left(-\zeta_3 + \frac{6}{5}\lambda_6 u_3\right)\partial_{u_1} - \left(\zeta_1 + \lambda_4 u_3\right)\partial_{u_3},$$
$$\mathcal{L}_6 = L_6 + \frac{3}{5}\lambda_8 u_3\partial_{u_1} - \zeta_3\partial_{u_3},$$

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- Find $3g$ polynomial vector fields in $\mathbb{C}^{3g}$ projectable for $p\colon \mathbb{C}^{3g} \to \mathbb{C}^{2g}$ and independent at any point in $p^{-1}(\mathcal{B})$.
- Construct their polynomial Lie algebra.

Polynomial vector fields in $\mathcal{B}$

We consider $\mathbb{C}^{2g}$ with coordinates $(\lambda_4, \lambda_6, \dots, \lambda_{4g}, \lambda_{4g+2})$. Set $\lambda_s = 0$ for every $s \notin \{4, 6, \dots, 4g, 4g+2\}$.
For $k, m \in \{1, 2, \dots, 2g\}$, $k \leqslant m$ set

$$T_{2k,2m} = 2(k+m)\lambda_{2k+2m} + \sum_{s=2}^{k-1} 2(k+m-2s)\lambda_{2s}\lambda_{2k+2m-2s} - \\ - \frac{2k(2g-m+1)}{2g+1}\lambda_{2k}\lambda_{2m},$$

and for $k > m$ set $T_{2k,2m} = T_{2m,2k}$. For $k = 0, 1, 2, \dots, 2g-1$ we have the vector fields

$$L_{2k} = \sum_{s=2}^{2g+1} T_{2k+2,2s-2}\frac{\partial}{\partial \lambda_{2s}}.$$

Proposition

1. *The vector field L_0 is the Euler vector field on $\mathcal{B}$.*
   We have $L_0(\lambda_k) = k\lambda_k$, $[L_0, L_{2k}] = 2kL_{2k}$.
2. *We have $L_{2k}(\lambda_{2s+4}) = L_{2s}(\lambda_{2k+4})$.*
3. *The vector fields L_{2k} are independent at any point of $\mathcal{B}$.*
4. *The vector fields L_{2k} are tangent*
   to the discriminant curve Σ of $\mathcal{V}_\lambda$.

$$\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_0] \\ [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ \wp_2 & -1 & 0 \\ \wp_{1;3} & \wp_2 & -1 \\ \wp_{1;5} & \wp_{1;3} & \wp_2 \\ 0 & \wp_{1;5} & \wp_{1;3} \\ 0 & 0 & \wp_{1;5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix},$$

Genus 3 polynomial map

$$\lambda_4 = -3x_2^2 + \frac{1}{2}x_4 - 2y_4,$$

$$\lambda_6 = 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 - 2x_2y_4 + \frac{1}{2}y_6 - 2z_6,$$

$$\lambda_8 = 4x_2^2y_4 - \frac{1}{2}(x_4y_4 - x_3y_5 + x_2y_6) + y_4^2 - 2x_2z_6 + \frac{1}{2}z_8,$$

$$\lambda_{10} = 2x_2y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4y_6 - \frac{1}{2}(x_4z_6 - x_3z_7 + x_2z_8) + \left(4x_2^2 + 2y_4\right)z_6,$$

$$\lambda_{12} = 4x_2y_4z_6 - \frac{1}{2}(y_6z_6 - y_5z_7 + y_4z_8) + z_6^2,$$

$$\lambda_{14} = 2x_2z_6^2 + \frac{1}{4}z_7^2 - \frac{1}{2}z_6z_8.$$

Genus 3 generators

$$\mathcal{L}_1 = \partial_{u_1}, \qquad \mathcal{L}_3 = \partial_{u_3}, \qquad \mathcal{L}_5 = \partial_{u_5},$$

$$\mathcal{L}_0 = L_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3} - 5u_5\partial_{u_5},$$

$$\mathcal{L}_2 = L_2 - \left(\zeta_1 - \frac{8}{7}\lambda_4 u_3\right)\partial_{u_1} - \left(u_1 - \frac{4}{7}\lambda_4 u_5\right)\partial_{u_3} - 3u_3\partial_{u_5},$$

$$\mathcal{L}_4 = L_4 - \left(\zeta_3 - \frac{12}{7}\lambda_6 u_3\right)\partial_{u_1} - \left(\zeta_1 + \lambda_4 u_3 - \frac{6}{7}\lambda_6 u_5\right)\partial_{u_3} - (u_1 + 3\lambda_4 u_5)\partial_{u_5}$$

$$\mathcal{L}_6 = L_6 - \left(\zeta_5 - \frac{9}{7}\lambda_8 u_3\right)\partial_{u_1} - \left(\zeta_3 - \frac{8}{7}\lambda_8 u_5\right)\partial_{u_3} - (\zeta_1 + \lambda_4 u_3 + 2\lambda_6 u_5)\,\partial_{u_5},$$

$$\mathcal{L}_8 = L_8 + \left(\frac{6}{7}\lambda_{10} u_3 - \lambda_{12} u_5\right)\partial_{u_1} - \left(\zeta_5 - \frac{10}{7}\lambda_{10} u_5\right)\partial_{u_3} - (\zeta_3 + \lambda_8 u_5)\,\partial_{u_5},$$

$$\mathcal{L}_{10} = L_{10} + \left(\frac{3}{7}\lambda_{12} u_3 - 2\lambda_{14} u_5\right)\partial_{u_1} + \frac{5}{7}\lambda_{12} u_5\partial_{u_3} - \zeta_5\partial_{u_5}.$$

Genus 3 Lie algebra

$$[\mathcal{L}_0, \mathcal{L}_k] = k\mathcal{L}_k,$$

$$[\mathcal{L}_1, \mathcal{L}_3] = 0, \qquad [\mathcal{L}_1, \mathcal{L}_5] = 0, \qquad [\mathcal{L}_3, \mathcal{L}_5] = 0,$$

$$\begin{pmatrix} [\mathcal{L}_1, \mathcal{L}_2] \\ [\mathcal{L}_1, \mathcal{L}_4] \\ [\mathcal{L}_1, \mathcal{L}_6] \\ [\mathcal{L}_1, \mathcal{L}_8] \\ [\mathcal{L}_1, \mathcal{L}_{10}] \end{pmatrix} = \begin{pmatrix} \wp_2 & -1 & 0 \\ \wp_{1;3} & \wp_2 & -1 \\ \wp_{1;5} & \wp_{1;3} & \wp_2 \\ 0 & \wp_{1;5} & \wp_{1;3} \\ 0 & 0 & \wp_{1;5} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix},$$

$$\begin{pmatrix}[\mathcal{L}_3,\mathcal{L}_2]\\ [\mathcal{L}_3,\mathcal{L}_4]\\ [\mathcal{L}_3,\mathcal{L}_6]\\ [\mathcal{L}_3,\mathcal{L}_8]\\ [\mathcal{L}_3,\mathcal{L}_{10}]\end{pmatrix} = \begin{pmatrix}\wp_{1;3}-\lambda_4 & 0 & -3\\ \wp_{0;3,3} & \wp_{1;3}-\lambda_4 & 0\\ \wp_{0;3,5} & \wp_{0;3,3} & \wp_{1;3}-\lambda_4\\ 0 & \wp_{0;3,5} & \wp_{0;3,3}\\ 0 & 0 & \wp_{0;3,5}\end{pmatrix}\begin{pmatrix}\mathcal{L}_1\\ \mathcal{L}_3\\ \mathcal{L}_5\end{pmatrix} + \frac{3}{7}\begin{pmatrix}5\lambda_4\\ 4\lambda_6\\ 3\lambda_8\\ 2\lambda_{10}\\ \lambda_{12}\end{pmatrix}\mathcal{L}_1,$$

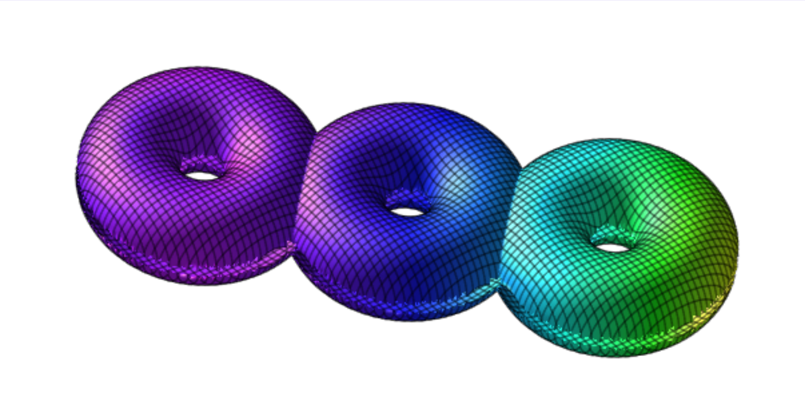
$$\begin{pmatrix}[\mathcal{L}_5,\mathcal{L}_2]\\ [\mathcal{L}_5,\mathcal{L}_4]\\ [\mathcal{L}_5,\mathcal{L}_6]\\ [\mathcal{L}_5,\mathcal{L}_8]\\ [\mathcal{L}_5,\mathcal{L}_{10}]\end{pmatrix} = \begin{pmatrix}\wp_{1;5} & 0 & 0\\ \wp_{0;3,5} & \wp_{1;5} & 0\\ \wp_{0;5,5} & \wp_{0;3,5} & \wp_{1;5}\\ -\lambda_{12} & \wp_{0;5,5} & \wp_{0;3,5}\\ -2\lambda_{14} & -\lambda_{12} & \wp_{0;5,5}\end{pmatrix}\begin{pmatrix}\mathcal{L}_1\\ \mathcal{L}_3\\ \mathcal{L}_5\end{pmatrix} + \frac{2}{7}\begin{pmatrix}2\lambda_4\\ 3\lambda_6\\ 4\lambda_8\\ 5\lambda_{10}\\ 6\lambda_{12}\end{pmatrix}\mathcal{L}_3 - \begin{pmatrix}0\\ 3\lambda_4\\ 2\lambda_6\\ \lambda_8\\ 0\end{pmatrix}\mathcal{L}_5,$$

$$\begin{pmatrix} [\mathcal{L}_2, \mathcal{L}_4] \\ [\mathcal{L}_2, \mathcal{L}_6] \\ [\mathcal{L}_2, \mathcal{L}_8] \\ [\mathcal{L}_2, \mathcal{L}_{10}] \\ [\mathcal{L}_4, \mathcal{L}_6] \\ [\mathcal{L}_4, \mathcal{L}_8] \\ [\mathcal{L}_4, \mathcal{L}_{10}] \\ [\mathcal{L}_6, \mathcal{L}_8] \\ [\mathcal{L}_6, \mathcal{L}_{10}] \\ [\mathcal{L}_8, \mathcal{L}_{10}] \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathcal{L}_0 \\ \mathcal{L}_2 \\ \mathcal{L}_4 \\ \mathcal{L}_6 \\ \mathcal{L}_8 \\ \mathcal{L}_{10} \end{pmatrix} + \mathcal{K} \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_3 \\ \mathcal{L}_5 \end{pmatrix},$$

$$\mathcal{K} = \frac{1}{2}\begin{pmatrix} -\wp_{2;3} & \wp_3 & 0 \\ -\wp_{1;3,3} - \wp_{2;5} & \wp_{2;3} & \wp_3 \\ -2\wp_{1;3,5} & \wp_{2;5} & \wp_{2;3} \\ -\wp_{1;5,5} & 0 & \wp_{2;5} \\ -\wp_{0;3,3,3} & \wp_{1;3,3} - 2\wp_{2;5} & 2\wp_{2;3} \\ -2\wp_{0;3,3,5} & 0 & 2\wp_{1;3,3} \\ -\wp_{0;3,5,5} & -\wp_{1;5,5} & 2\wp_{1;3,5} \\ -2\wp_{0;3,5,5} & 2\wp_{1;5,5} - \wp_{0;3,3,5} & \wp_{0;3,3,3} \\ -\wp_{0;5,5,5} & -\wp_{0;3,5,5} & \wp_{0;3,3,5} + \wp_{1;5,5} \\ 0 & -\wp_{0;5,5,5} & \wp_{0;3,5,5} \end{pmatrix},$$

$$\mathcal{M} = \frac{2}{7}\begin{pmatrix} 8\lambda_6 & -8\lambda_4 & 0 & 7 & 0 & 0 \\ 6\lambda_8 & 0 & -6\lambda_4 & 0 & 14 & 0 \\ 4\lambda_{10} & 0 & 0 & -4\lambda_4 & 0 & 21 \\ 2\lambda_{12} & 0 & 0 & 0 & -2\lambda_4 & 0 \\ -7\lambda_{10} & 9\lambda_8 & -9\lambda_6 & 7\lambda_4 & 0 & 7 \\ -14\lambda_{12} & 6\lambda_{10} & 0 & -6\lambda_6 & 14\lambda_4 & 0 \\ -21\lambda_{14} & 3\lambda_{12} & 0 & 0 & -3\lambda_6 & 21\lambda_4 \\ -7\lambda_{14} & -7\lambda_{12} & 8\lambda_{10} & -8\lambda_8 & 7\lambda_6 & 7\lambda_4 \\ 0 & -14\lambda_{14} & 4\lambda_{12} & 0 & -4\lambda_8 & 14\lambda_6 \\ 0 & 0 & -7\lambda_{14} & 5\lambda_{12} & -5\lambda_{10} & 7\lambda_8 \end{pmatrix},$$

$$\begin{pmatrix} [L_2, L_4] \\ [L_2, L_6] \\ [L_2, L_8] \\ [L_2, L_{10}] \\ [L_4, L_6] \\ [L_4, L_8] \\ [L_4, L_{10}] \\ [L_6, L_8] \\ [L_6, L_{10}] \\ [L_8, L_{10}] \end{pmatrix} = \mathcal{M} \begin{pmatrix} L_0 \\ L_2 \\ L_4 \\ L_6 \\ L_8 \\ L_{10} \end{pmatrix}.$$



Thank you!