<u>Motivation</u> (Golyshev '07, Coates-Corti-Golyshev-K '13)

"Given a classification problem in algebraic geometry, use mirror ~~symmetry~~ ~~to~~ duality to translate it into a problem in [combinatorics]; solve this problem and translate the result back into ~~a~~ geometry."

differential equation

~~Q~~ Can we classify Fano manifolds using this idea?

X Fano manifold, $\dim X = n$.

$\rightsquigarrow$ Regularised quantum period $\hat{G}_X(t) = \sum_{k \geq 0} k!\, a_k t^k$

Gromov-Witten curve count of deg k on X.

$f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm n}]$ Laurent poly

$\rightsquigarrow$ Classical period
$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^n \int_{|x_1| = \dots = |x_n| = 1} \frac{1}{1 - tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$
$$= \sum_{k \geq 0} b_k t^k ,$$

$\mathrm{coeff}_1(f^k)$

Say f is <u>mirror dual</u> to X if $\pi_f = \hat{G}_X$.

Hope is it might be easier to find & classify f than X.

<u>Example</u>.

$X = \mathbb{P}^2$

By Givental '98 we have

$$\hat{G}_X = \sum_{k\geq 0} \frac{(3k)!}{(k!)^3} t^{3k} = 1 + 6t^3 + 90t^6 + 1680t^9 + \dots$$

Take $f = x + y + \frac{1}{xy}$.

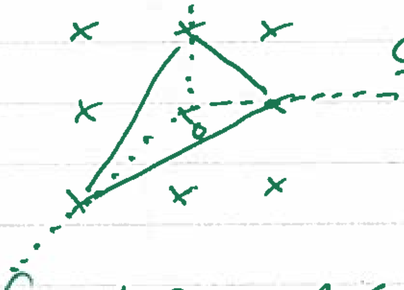
$$\cancel{\pi_f = \sum_{k\geq 0} \binom{3k}{k\ k\ k} t^{3k}}$$

$$\mathrm{coeff}_1(f^n) = \begin{cases} \binom{n}{\frac{n}{3}\ \frac{n}{3}\ \frac{n}{3}} & \text{if } 3 \mid n \\ 0 & \text{otherwise} \end{cases}$$

so $\pi_f = \sum_{k\geq 0} \binom{3k}{k\ k\ k} t^{3k} = \hat{G}_X$.

If we draw the Newton poly. for f:

$P := \mathrm{Newt}(f)$:



$\subseteq N_{\mathbb{Q}}$

Notice the spanning fan of P is $\mathbb{P}^2$ (as a toric var.).

Expectation

The expectation is that in general if f is mirror dual to X, then f gives an ~~...~~ n-dim toric var. X_P, $P = \mathrm{Newt}(f) \subseteq N_{\mathbb{Q}}$, and X_P is a ~~X_P is $\mathbb{Q}$-Gor~~ $\mathbb{Q}$-Gorenstein deformation of X.

Call X_P a toric degeneration of X.

Example

Hacking–Prokhorov '10 tell us the toric degenerations of $\mathbb{P}^2$.

Using Karpov–Nogin '98 we have a tree of deformations

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{P}^2 \arrow[d, dash] & \\
 & \mathbb{P}(1,1,4) \arrow[d, dash] & \\
 & \mathbb{P}(1,4,25) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{P}(4,25,29^2) & & \mathbb{P}(1,25,13^2) \\
\cdots & & \cdots
\end{tikzcd}
$$

where $\mathbb{P}(a^2,b^2,c^2)$ is a sol^n to Markov eq^n: $a^2+b^2+c^2=3abc$.

At the level of f we have what we call a mutation (Galkin–Usnich '10, Akhtar–Coates–Galkin–K '12):

Write $x = XY,\ y = Y$ (c.o.b.)

so $f = x + y + \dfrac{1}{xy} = (1+X)Y + \dfrac{1}{XY^2}$

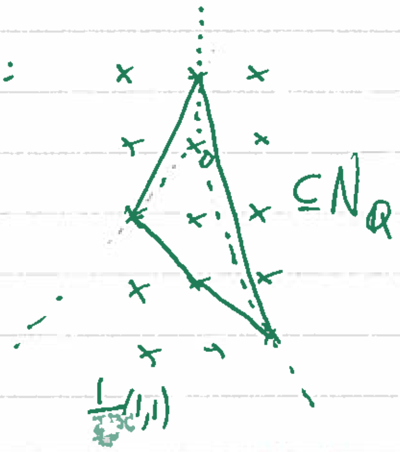
Define $\varphi : (X,Y) \longmapsto \left(X, \dfrac{Y}{1+X}\right)$

$$g := \varphi^*(f) = Y + \frac{(1+X)^2}{XY^2} = y + \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3}$$

Easy to check that $\pi_g = 1 + 6t^3 + 90t^6 + 1680t^9 + \ldots = \pi_f$

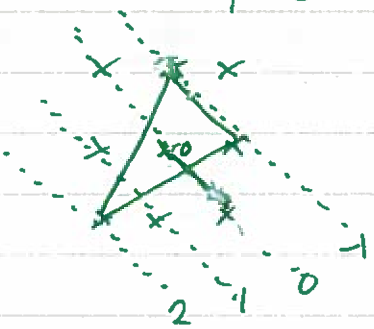
so g is mirror dual to $\mathbb{P}^2$, and:

$Q = \mathrm{Newt}(g)$:



with $X_Q = \mathbb{P}(1,1,4)$.

Translating this purely into combinatorics we have a mutation of P:



Pick $w \in M$, w primitive.

$\rightsquigarrow$ imposes a grading on N

$w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

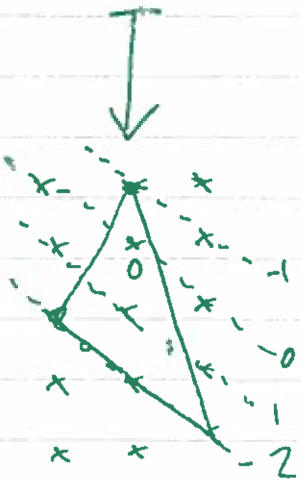
Pick $F \subseteq w^\perp$ a lattice polytope satisfying some condition...

$F = \mathrm{conv}\{(0,0), (1,-1)\}$

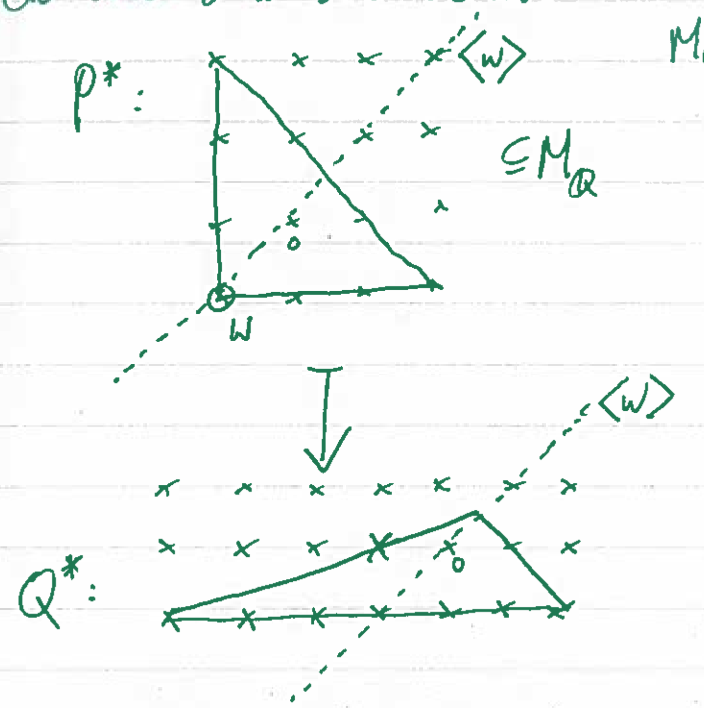
~~At the height~~ Slice P by height w.r.t. w.

At $-$'ve height $-h$ ~~you~~ we "Minkowski subtract" hF

$+$'ve h Minkowski add hF!



On the dual side we have:



Mutation is:

$$\varphi: u \longmapsto u - w \min\{w(v) \mid v \in F\}.$$

Piecewise-$GL(M)$ transformation.

$$\Rightarrow |P^* \cap M| = |Q^* \cap M|$$

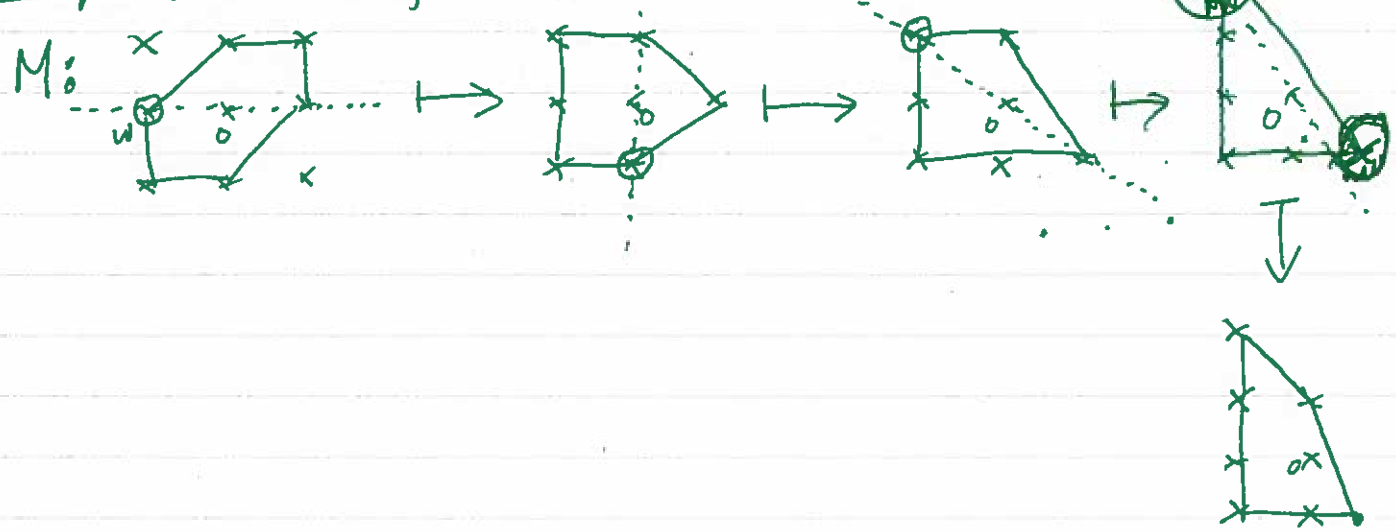
$$\mathrm{Vol}(P^*) = \mathrm{Vol}(Q^*)$$

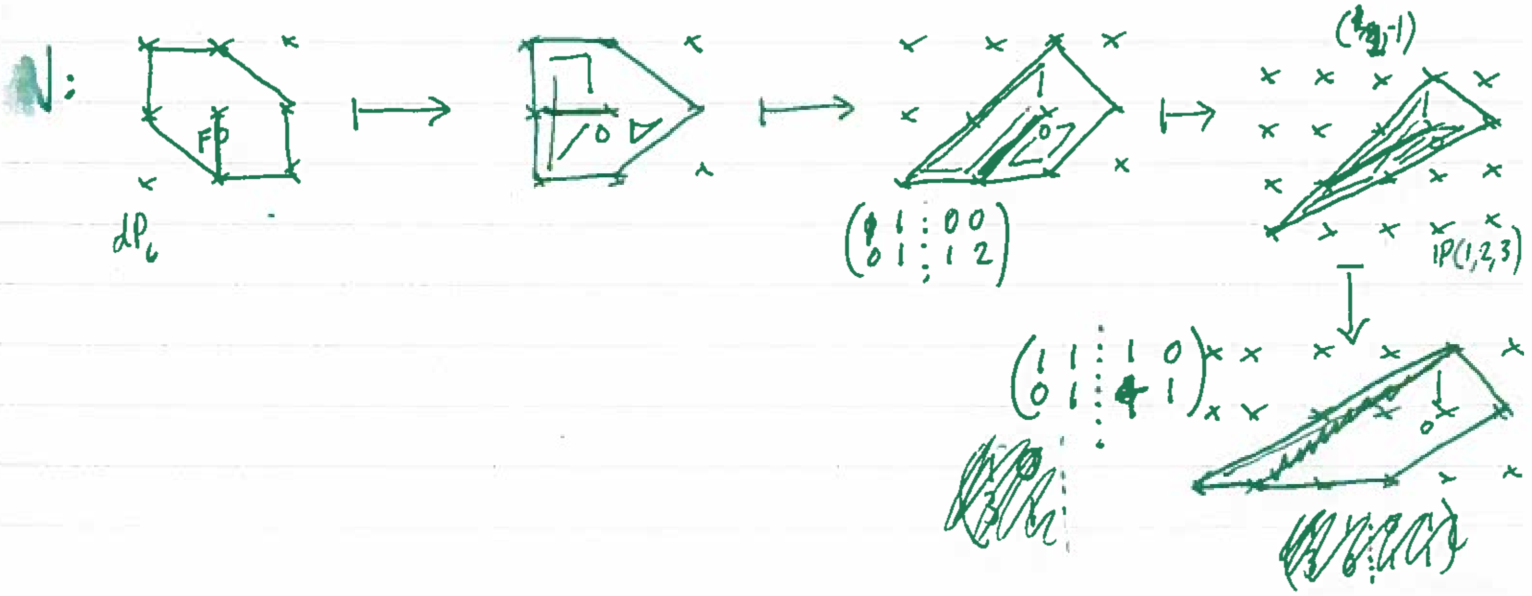
Hence ~~$(-K_{X_P})^n = (-K$~~

$$\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}(X_Q, -K_{X_Q})$$

$$(-K_{X_P})^n = (-K_{X_Q})^n.$$

Example (dP_6) (c.f. Elisa)





~~The~~ ~~[illegible]~~ . Singularity content (Akhtar–K. '14)

Notice all the sings are T-singularities: ~~[illegible]~~ .

$\rightsquigarrow$ Kollár – Shepherd-Barron '88

a toric surface sing. is $\mathbb{Q}$-Gorenstein smoothable iff it's a T-sing: ~~[illegible]~~ $\frac{1}{dr^2}(1, dra-1)$, $\gcd\{r,a\}=1$.

T-sings are particularly well behaved under mutation — you can mutate a cone in the fan iff it is a T-sing.

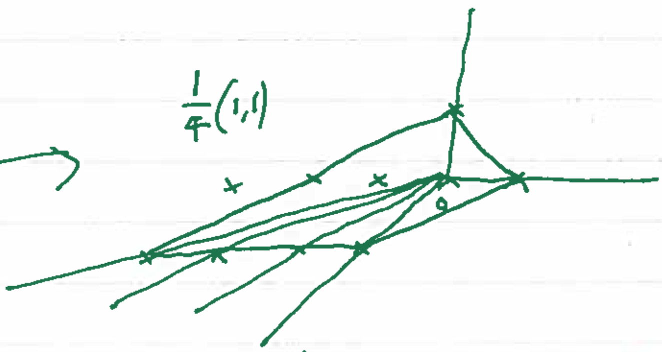
$\longrightarrow$ the atomic pieces of mutation in 2-dim are the primitive T-sings $\frac{1}{r^2}(1, ra-1)$ ~~[illegible]~~ .

$\Longrightarrow$ If you/subdivide your fan into primitive T-sings crepant then that # is preserved under mutation.

Example (dP_6).

N:

$\longleftrightarrow$



6 primitive T-sings — 6 primitive T-sings

No local-to-global obstruction to this smoothing.

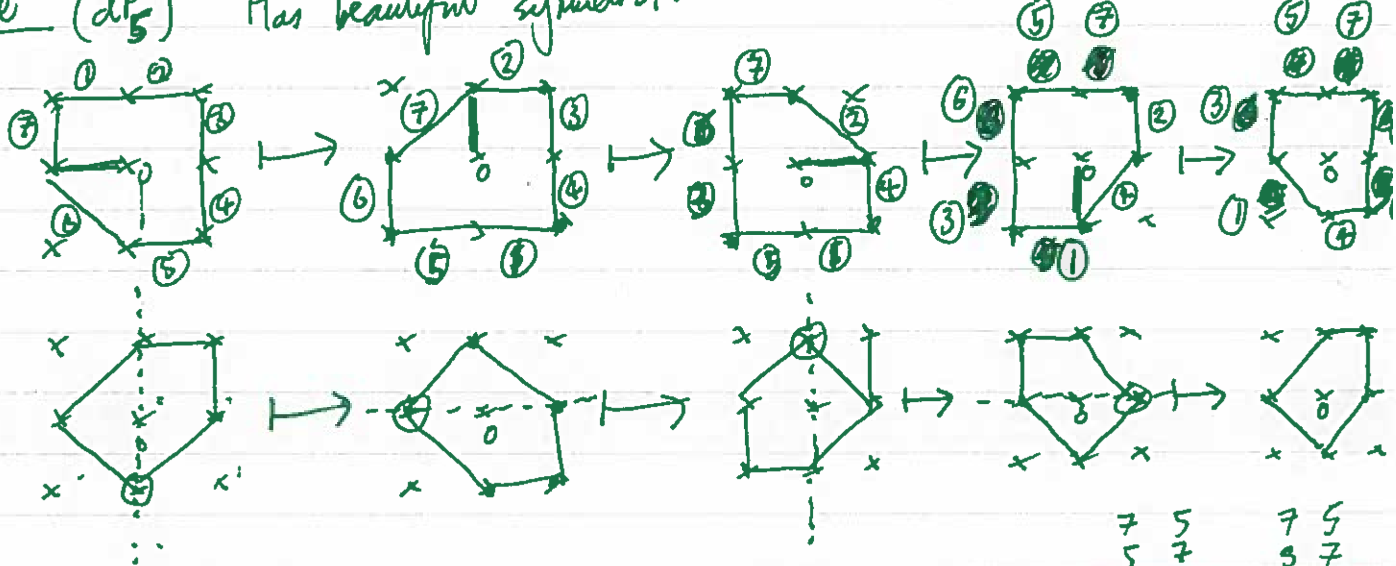
Thm (K-Nill-Prince '17)

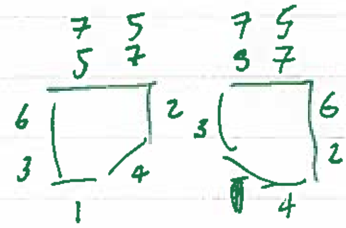
$$\left\{ \text{Fano polygons with only T-sings} \right\} \Big/ \text{mutation} \xleftrightarrow{\;1:1\;} \text{10 smooth del Pezzo surfaces.}$$

Example (dP_5) Has beautiful symmetries:

N:

M:

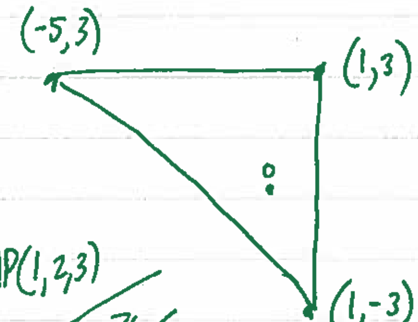




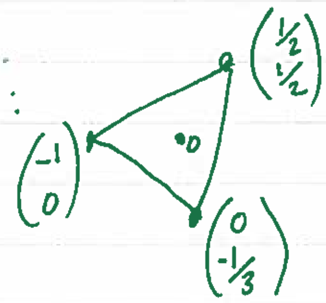
1 2 3 4 5 6 7
7 5 2 6 4 1 3

Example (dP_4)

N:



M:



$\mathbb{P}(1,2,3) / \mathbb{Z}/6$

$\frac{1}{6}(0,1,5)$

$$X_6 \subseteq \mathbb{P}(1\,1\,2\,3)$$

What about higher dim?

There is no known analogue of T-sing, but we use the fact that T-sings are exactly the mutable cones in 2-dims.

~~Thm (K–Tveiten '17) In 2-dims, a mutation equivalence class~~.

Let P be a Fano polytope (arb. dim) and let f be a Laurent, $\mathrm{Newt}(f) = P$, such that for every sequence of mutations starting at P $\exists$ sequence of mutation starting at f.

We call f max. mut.

Thm (K–, Tveiten '17)

In dim $= 2$, f is unique (up to some normalising cond's) iff P has only T-sing's.

iff X_P is Q-Gorenstein smoothable

Example (dP_1)

$$f = \frac{y^3\left(1+x+\frac{x}{y}\right)^6}{x^5} - 60 \cdots$$

$$\pi_f = 1 + 10260t^2 + \ldots \quad \text{which is} = \hat{G}_{dP_1}(t)$$

Conjecture

In dim ≥ 3, if X_P is smoothable, then f is unique.

Thm (K–Tveiten '17)

Mirror duals f are known for all 105 smooth Fano 3-folds
(Coates–Corti–Galkin–K '16)

~~They have been~~ and are known for all known smooth Fano 4-folds
(Coates–Galkin–K–Strangeway '14).

In every case f satisfies the conjecture.