THE DISCRIMINANT OF A CUBIC SURFACE

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ABSTRACT

The discriminant of a quaternary cubic form, of degree 32 in the coefficients, was obtained as a polynomial in four invariants of the form by Salmon (1861 *Phil. Trans. R. Soc.* **150**, 229-239). There appears to be an incorrect numerical multiplier and the error has persisted through the several editions of his treatise. This paper offers a correction.

1. The equation of an algebraic surface of order n in projective space of three dimensions involves a homogeneous quaternary polynomial F of degree n; we take the liberty of using F also to denote this surface $F = 0$. The discriminant Δ is that invariant of F whose vanishing is the necessary and sufficient condition for F to have a node; it is the eliminant of the four first polars or partial derivatives of F. As each of these polars has degree 1 in the coefficients of F and order $n-1$ in the variables the eliminant is [5, pp 70-71] of degree $4(n-1)^3$ in the coefficients.

When, as here, $n = 3$ Δ has degree 32. Salmon [4, p.235] offered an expression as a polynomial in four invariants of F that he and Clebsch [2] independently and, to all intents and purposes, simultaneously discovered, but it seems that his form for Δ is not correct. Whereas he prints $\Delta = 0$ as

$$(A^2 - 64B)^2 = 16384(D + 2AC),$$

the capital letters being the invariants in question and the multiplier on the right 2^{14}, the proposed emendation is that $2AC$ ought to be $\frac{1}{8}AC$. Did Salmon carry through all the calculations accurately and then unfortunately misread some superscript of a power of 2? Has the correct form of Δ ever yet been printed?

This mistake has persisted. In the respective editions it appears on pages 398 [6a], 435 [6b], 478 [6c], 509 [6d], 198 [6e], 427 [7], 75 [8], and perhaps librarians and readers in general could make the correction in their copies. The same erroneous form is printed in at least one standard work of reference [1, p.803].

2. Salmon sets the stage with that luminous clarity with which his readers are so often favoured. The equation of F referred to its Sylvester pentahedron is

$$F \equiv ax^3 + by^3 + cz^3 + dt^3 + eu^3 = 0 \tag{2.1}$$

and one can take the linear identity between the five planes to be

$$x + y + z + t + u \equiv 0. \tag{2.2}$$

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In polarising (2.1) one must handle F as a quaternary cubic form, the fifth of the five supernumerary coordinates being, by (2.2), the negative of the sum of the other four.

The simultaneous vanishing of four polars is therefore equivalent to the conditions

$$ax^2 = by^2 = cz^2 = dt^2 = eu^2$$

so that, again by (2.2), when all these conditions hold,

$$a^{-\frac{1}{2}} + b^{-\frac{1}{2}} + c^{-\frac{1}{2}} + d^{-\frac{1}{2}} + e^{-\frac{1}{2}} = 0.$$

This implies that $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$, $d^{\frac{1}{2}}$, $e^{\frac{1}{2}}$, appropriately signed, are, the sum of their reciprocals being zero, roots of an equation

$$x^5 + \alpha x^4 + \beta x^3 + \gamma x^2 + \varepsilon = 0$$

with the linear term absent. The five oppositely signed square roots therefore satisfy

$$x^5 - \alpha x^4 + \beta x^3 - \gamma x^2 - \varepsilon = 0$$

so that all ten square roots satisfy

$$(x^5 + \beta x^3)^2 - (\alpha x^4 + \gamma x^2 + \varepsilon)^2 = 0. \tag{2.3}$$

But Salmon takes the equation satisfied by a, b, c, d, e themselves to be

$$x^5 - px^4 + qx^3 - rx^2 + sx - t = 0$$

so that (2.3) is the same equation as

$$x^{10} - px^8 + qx^6 - rx^4 + sx^2 - t = 0$$

and

$$p = \alpha^2 - 2\beta, q = \beta^2 - 2\gamma\alpha, r = \gamma^2 + 2\alpha\varepsilon, s = -2\gamma\varepsilon, t = \varepsilon^2. \tag{2.4}$$

Elimination of $\alpha, \beta, \gamma, \varepsilon$ between these five equations gives a relation between p, q, r, s, t and so between Salmon's invariants which, as he defines them, are

$$A = s^2 - 4rt, B = t^3p, C = t^4s, D = t^6q, E = t^8. \tag{2.5}$$

But these equations are, by (2.4)

$$A = -8\alpha\varepsilon^3, B = \varepsilon^6(\alpha^2 - 2\beta), C = -2\gamma\varepsilon^9, D = \varepsilon^{12}(\beta^2 - 2\gamma\alpha), E = \varepsilon^{16}$$

so that

$$(A^2 - 64B)^2 = 2^{14}\varepsilon^{12}\beta^2 = 2^{11}(8D + AC). \tag{2.6}$$

This, it is submitted, gives the correct form of Δ.

3. Clebsch, in his paper on quaternary cubics, obtains five invariants; but his mistaken impression [2, p.119] that the order of every invariant is divisible by 8 may be the reason why he never found Salmon's skew invariant of degree 100. Clebsch of course appreciated that the condition for the cubic surface to have a node must be expressible in terms of invariants and he even goes so far [2, p.124] as to show that the discriminant will be a linear combination of terms which, in Salmon's notation, are

$$D,\ CA,\ B^2,\ BA^2,\ A^4.$$

But there is no evidence of any attempt on Clebsch's part to determine the "reine Zahlen" that occur as multipliers.

4. The suspicion that Salmon had made a slip was aroused by applying his expression for Δ to the nodal surface

$$t(x^2 + y^2 + z^2) + 2\lambda xyz = 0$$

having a canonical form [3, p.36]

$$\{t+\lambda(x+y+z)\}^3+\{t+\lambda(x-y-z)\}^3+\{t+\lambda(y-z-x)\}^3+\{t+\lambda(z-x-y)\}^3+\frac{(-4t)^3}{16}=0.$$

Here $a = b = c = d = 1, e = \dfrac{1}{16}$, roots of

$$x^5 - \frac{65}{16}x^4 + \frac{25}{4}x^3 - \frac{35}{8}x^2 + \frac{5}{4}x - \frac{1}{16} = 0$$

and Salmon's Δ *is not zero*. Indeed, by (2.5),

$$A = 2^{-5}.3.5,\quad B = 2^{-16}.5.13,\quad A^2 - 64B = 2^{-5}.5,\quad AC = 2^{-23}.3.5^2,\quad D = 2^{-26}.5^2,$$

so that, if $(A^2 - 64B)^2$ is a linear combination of D and AC,

$$2^{-10}.5^2 = \mu.2^{-26}.5^2 + \nu.2^{-23}.3.5^2.$$

If $\mu = 2^{14}m$ and $\nu = 2^{14}n$ this is

$$\tfrac{1}{4}m + 6n = 1;$$

Salmon's $m = 1, n = 2$ do not accord with this requirement. The correct values are, as has been shown, $m = 1, n = \frac{1}{8}$.

5. In conclusion there is, albeit not directly concerned with Δ, a further matter to be mentioned.

Salmon and Clebsch are not the only independent discoverers of invariants of a cubic surface; these were rediscovered many years later [9, p.362] by Young as an incidental by-product of his powerful techniques in quantitative substitutional analysis, and indeed Young hoped to present subsequently the complete system for a single cubic in any number of variables. He was not aware of Salmon's achievement when he found, by a completely different and completely ungeometrical procedure, *the same six invariants;* these include the skew invariant of degree 100 that eluded Clebsch. When, after publication, Young learnt that these had all been found in 1860, he at once consulted the original authority and, in a letter of 5 March 1934, wrote "I have read Salmon's paper on the quaternary cubic in the *Phil. Trans.* 1861. It is a beautiful piece of work." And so say all of us. And, as Young adds in the same letter, the fact that the invariants were known before does not destroy their value as an illustration of his method.

Acknowledgement

An earlier draft of this note used a far longer and more laborious method to obtain (2.6). The much shorter route by using (2.3) was pointed out and is a great improvement.

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