An orthogonal group of order $2^{13}\cdot 3^5\cdot 5^2\cdot 7$

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Summary. - *The group of automorphisms of the non-singular quadric consisting of 135 points in a finite projective space* [7] *is investigated by using the geometry of the figure.*

CONTENTS

Introduction

1. The non-singular quadratic forms in any even number of variables over $GF(2)$, or $\boldsymbol{F}$ as this field will be called, are of two types-distinct in that no form of either type can be changed into a form of the other by any linear transformation whose coefficients all belong to $\boldsymbol{F}$. When equated to zero a quadratic form in $2n$ variables gives a quadric in a projective space $[2n-1]$, and the root of the above distinction lies in the fact that such a quadric may, or may not, contain spaces $[n-1]$ lying wholly on it; it may be ruled or non-ruled. The orders of the groups of automorphisms of the ruled quadrics were calculated by JORDAN (**10**, 244) and by DICKSON (**3**, 206), and DICKSON calculates the order for the non-ruled quadrics too; but the geometry that underlies these groups and accounts for their structure has been ignored, even for low values of n, until quite recently.

It is, then, proposed to investigate, describe and exploit the figure in [7] set up by a ruled quadric $\mathfrak{S}$; the initial $\mathfrak{S}$ of STUDY is used because the quadric, consisting of 135 points and containing 270 solids that fall into two systems of 135, affords an instance, indeed the simplest instance, of his principle of triality (**12**, 477). Its group A of automorphisms has a subgroup A^+ of index 2 - the subgroup called $FH\,(8,2)$ by DICKSON (**3**, 310); automorphisms in the coset of A^+ transpose the two systems of solids on $\mathfrak{S}$ whereas those in A^+ itself do not.

There are simplexes Σ such that, when any one of them is used as simplex of reference, the quadratic form on the left-hand side of the equation of $\mathfrak{S}$ is the sum of the 28 products of pairs of the 8 homogeneous coordinates. The matrices μ of the automorphisms of this quadratic form can be identified, precisely as for the analogous groups in 6 or 7 variables

(**6** and **7**): the 8 columns of μ are linearly independent, each is the coordinate vector of some point m on $\mathfrak{S}$, and no two of these 8 m can be conjugate in regard to $\mathfrak{S}$. In other words: the columns of μ are the coordinate vectors of the vertices of some Σ. It is helpful, when endeavouring to construct those μ satisfying supplementary conditions (e. g. μ for which the points of some subspace are all invariant) to remember that, in consequence of no column being conjugate to any other, each column *is* conjugate to the sum of any even number of columns exclusive of itself. These defining attributes of μ are relied upon constantly; for instance, if one seeks the general form of μ when certain points are prescribed to be invariant, their coordinate vectors being latent column vectors of μ (§§ 22, 23, 30, 34), or if one requires the general matrix form of the normaliser of some operation of A (§§ 41, 43, 46, 48). But it is, upon occasion, advisable to use some alternative canonical form for the equation of $\mathfrak{S}$ and then μ will be defined by quite different properties. In investigating the structure of a SYLOW 2-group of A^+ one takes the canonical form to be not (6.1) but (10.4), and bases the discussion on the 2^{12} matrices (50.1).

2. In geometry based on $\boldsymbol{F}$ there is, when a simplex is given, a single point of the space that does not lie in any bounding prime of the simplex. When the 8 vertices of any Σ are so supplemented the ennead $\mathcal{E}$ constituted by the 9 points is such that, *whichever* of its points is omitted, the other 8 are vertices of a Σ. This explains the occurrence of symmetric subgroups $\mathfrak{S}_9$ in A, whose order can be deduced from the geometry, without any allusion to JORDAN'S and DICKSON'S general formulae, in different ways (§§ 7, 11). Rich though the geometry is in detail nothing is irrelevant; it is summarised in Table I, some of which is mere transcription of facts ascertained in earlier work while those parts which are not are expeditiously compiled in §§ 12–14.

There follow, in §§ 15 and 16, certain properties of the enneads, and then some paragraphs point contrasts and underline affinities between the geometry here and that in **9**, the paper wherein MISS HAMILL, using a different representation, succeeded in decomposing A into its 67 conjugate classes. Her numbering of these classes will be followed here, and the title of this paper is chosen to echo hers.

3. There are, in a symmetric group $\mathfrak{S}_9$, 30 conjugate classes associated one with each partition $\{\lambda\}$ of 9; when $\mathfrak{S}_9$ is a subgroup of A these furnish members of 30 of the conjugate classes of A. Each operation μ of A leaves certain points of [7] unmoved; these fill the *latent space* σ of μ whose dimension is always (§ 19) 2 less than the number of parts in $\{\lambda\}$. But it is not merely the dimension of σ that one finds; one determines, in § 20 and § 21, its precise relation to $\mathfrak{S}$. The results are shown in Table II on

p. 26. All these are accumulated by setting out from μ in $\mathfrak{S}_9$ and determining σ; but one can argue oppositely by taking some space σ and finding those operations of A for which it is latent. This investigation progresses by gradual descent: when seeking those operations for which σ is latent one presumes known all those whose latent spaces include σ.

Those operations of A for which every point of σ is invariant form a group A_σ, with a subgroup A_σ^+ of index 2. A_σ includes all those operations whose latent space contains σ and these, if the space properly contains σ, can be discarded if already known. These discarded operations embrace the whole of A_σ^+ or of its coset according as σ has odd or even dimension, so that the excess of the number of discards in the exhausted half of A_σ over the number in the other half is the number of operations of A for which σ is latent. In the course of the discussion explicit matrix forms for several A_σ are found–they depend, of course, on how σ is chosen in relation to the simplex of reference of the coordinates; one may mention, as based on the symmetric canonical form (6.1), the 128 matrices (23.3) and, subject to (34.4), the 192 matrices (34.2), as well as sets of 64 matrices, based on the canonical form (10.4), occurring in §§ 35, 36.

The details tend inevitably towards greater elaboration as the work progresses through decreasing dimensions of σ, and there is abundant interest in the underlying geometry. Of those conjugate classes of A that do not intersect any of its subgroups $\mathfrak{S}_9$ 29 are encountered, and representatives of many of these are given. This part of the paper closes when σ, its dimension progressively lessening, has shrunk to a point. This point is labelled m or p according as it is on or off $\mathfrak{C}$, and A_p happens to be the direct product of a cyclic group of order 2 and a group on which much has been written already. But A_m, of index 135 in A, is not familiar and Table III on p. 68 gives the numbers of its $2^{13}\cdot 3^2\cdot 5\cdot 7$ members in those 48 of the 67 conjugate classes of A among which they are distributed, as well as the mode in which these members of A permute the 135 m of the geometry The analogous information for those A_σ with σ a line - to mention only those A_σ of relatively high order - could easily be compiled. The orders of the various groups A_σ are assembled in Table IV on p. 69.

4. The only classes in MISS HAMILL'S list of 67 (9, 76–7) whose operations are devoid of invariant points and which are therefore not encountered here are those numbered

LVI, LVII, LIX, LX, LXIV, LXV, LXVI, LXVII

whose members have, respectively, periods

3, 6, 12, 5, 10, 15, 6, 12.

All of these have some power of their operations in either class LVI or class LX, and class LXV is so subordinated to both these classes. These eight outstanding classes can be brought within the ambit of the geometry once this is enlarged by invoking STUDY's triality and imposing his semilinear transformations (12, 477): he appropriates the adjective semilinear in full awareness of the fact that three successive such transformations $\mathfrak{T}$ may be necessary to generate a collineation. $\mathfrak{T}$ imposes an outer automorphism of period 3 on A^+, transforming A into two other groups which also have A^+ for a subgroup of index 2. For example: the last class LXVII falls, in A^+, into two conjugate classes of 7257600 operations; class XXVII has also this number of operations, and of the same period 12, and there is an outer automorphism of A^+ which permutes these 3 classes cyclically. And this same automorphism transforms the members of classes

IX, XI, IV, III

which are powers of members of class XXVII, into the members of classes

LXVI, LXII, LVI, LXI

which are the corresponding powers of members of class LXVII. But these matters of triality and the geometry pertaining to them must be postponed to another day.

When the occasion comes to consider them any facts ascertained in this present paper will be available.

5. A SYLOW subgroup S_3 of A, and so of A^+, is the direct product of a cyclic group of order 3 and a SYLOW subgroup, of order 81, of a cubic surface group, so that nothing further need be said about S_3 here. But the SYLOW subgroups S_2^+ of A^+ are associated with the flags on $\mathfrak{S}$ and this fact points one way towards an exploration of their properties. The last part of the paper (§§ 50–69) is given over to these matters. The 2^{12} matrices (50.1) furnish an S_2^+ and their law (50.3) of composition affords a ready access to its upper central series and to many other features of this group whose structure, whether in its broad outline or fine detail, lies open to scrutiny by examining the figure in [7].

Table I shows the numbers of subordinate spaces, of different kinds, in any subspace σ of [7]: σ, with the number of such subspaces in the geometry, heads the column, and the symbol for the subordinate space flanks the row, containing the entry. Reciprocating in $\mathfrak{S}$ gives the number of spaces, of different kinds, that contain the polar space σ' of σ; these latter numbers being so deducible instantaneously need not be printed and so the table is left triangular. It is used almost incessantly once it is on record. An entry written as a sum indicates that σ meets $\mathfrak{S}$ in a singular quadric and the different constituents o the sum imply subordinate spaces differently related to the vertex of the section.

I.

The geometry of the ruled quadric $\mathfrak{S}$; the 960 inscribed enneads and the symmetric subgroups, of degree 9, of the orthogonal group.

6. When $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are homogeneous coordinates in a projective space [7] the equation

$$\sum_{i<j} x_i x_j = 0, \tag{6.1}$$

with a sum of 28 products on left, represents a quadric $\mathfrak{S}$. The base field $\boldsymbol{F}$ is the GALOIS FIELD $GF(2)$ consisting of only two marks 0, 1 with $1+1=0$; every x_i is a mark of $\boldsymbol{F}$ so that, with two choices for each of the 8 coordinates which, as coordinates of a point, are debarred from being all 0 simultaneously, there are $2^8 - 1 = 255$ points in [7]. If a point m is on $\mathfrak{S}$ the number of products $x_i x_j$ that are 1 is even; the number of non-zero coordinates of m is then one of 1, 4, 5, 8 so that $\mathfrak{S}$ consists of $8+70+56+1=135$ points m. The $255-135=120$ points off $\mathfrak{S}$ will be labelled p. The quadripartite partitioning of 135 is consequent upon the introduction of the coordinate system: the first 8 m are the vertices of the simplex of reference Σ_0, the last single m is the unit point U. The partitioning is indeed significant, but it does not confer privilege on any m. The 135 m all have the same geometrical attributes; they are permuted transitively by by the group A of automorphisms of $\mathfrak{S}$.

Since $\Sigma x_i x_j$ is symmetric the symmetric group $\mathfrak{S}_8$ is a subgroup of A; subgroups $\mathfrak{S}_8$ are associated one with each simplex Σ in regard to which $\mathfrak{S}$ has the same equation (6.1) as it has when referred to Σ_0. Label that vertex of Σ_0, all of whose coordinates save x_i are zero, X_i. These X_i all lie on $\mathfrak{S}$ and, moreover, no two of them are conjugate: their join is a chord c, not a generator, of $\mathfrak{S}$; the remaining point on c has two non-zero coordinates and so is off $\mathfrak{S}$. Any simplex Σ having these properties gives, when taken as simplex of reference, (6.1) as the equation of $\mathfrak{S}$: the square of each coordinate is absent because each vertex is on $\mathfrak{S}$, while the product of each pair of coordinates is present because no two vertices are conjugate. Describe such a simplex Σ as *eligible*. Now not only is it that no two vertices of Σ_0 are conjugate to one another: none of them is conjugate to the unit point U whereat the tangent prime of $\mathfrak{S}$ is the unit prime

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0. \tag{6.2}$$

Thus U completes, with the X_i, an ennead $\mathcal{E}_0$ – a set of 9 m, each the sole point of [7] not in any bounding prime of the simplex whose vertices are the others, all of them on $\mathfrak{S}$ and no two conjugate. All 9! permutations of points of $\mathcal{E}_0$ can be imposed by projectivities that belong to A because all transpositions of pairs in $\mathcal{E}_0$ can be so imposed: those of pairs of X_i are imposed by permutation matrices, and if one requires to transpose, say, X_0 with U and yet leave every other X_i unmoved one takes

$$
(6.3)\qquad
\begin{bmatrix}
1 & . & . & . & . & . & . & . \\
1 & 1 & . & . & . & . & . & . \\
1 & . & 1 & . & . & . & . & . \\
1 & . & . & 1 & . & . & . & . \\
1 & . & . & . & 1 & . & . & . \\
1 & . & . & . & . & 1 & . & . \\
1 & . & . & . & . & . & 1 & . \\
1 & . & . & . & . & . & . & 1
\end{bmatrix}
$$

as premultiplier of the coordinate column vector of any point. This matrix has the two properties (see **6** for forms in 6, **7** for forms in 7 variables) that characterise the matrices of A: each column is the coordinate vector of an m, while no two m whose coordinate vectors so occur in the same matrix of A can be conjugate: the columns, indeed, are coordinate vectors of the vertices of an eligible simplex Σ.

A has therefore symmetric subgroups $\mathcal{S}_9$. Every eligible Σ provides, by the addition to its 8 vertices of the unique point of [7] that does not lie in any of its bounding primes, an ennead $\mathcal{E}$ whose members undergo all 9! permutations under one of these subgroups. A is transitive on these $\mathcal{E}$; there is a projectivity in A that transforms Σ_0 into Σ and so U into the unique point which, adjoined to Σ, completes $\mathcal{E}$. The order of A is $9!N$ where N is the number of $\mathcal{E}$.

7. In order to calculate N one notes, first, that the prime P spanned by any 7 members of an ennead has for its pole the point p on the join of the remaining two: for example, $x_7=0$ is spanned by all X_i save X_7 and is the polar of $(1, 1, 1, 1, 1, 1, 1\ \cdot)$ on UX_7. Now it was remarked in **7** that there are, in $x_7=0$, 288 simplexes σ in regard to which the section of $\mathfrak{S}$ has the equation $\Sigma x_i x_j = 0$ $(0 \leq i < j \leq 6)$. The $21p$ which lie one on each edge of σ are all conjugate both to U and to X_7; for if neither of two points, here vertices of σ, is conjugate to a given point O the remaining point of their join has to be, since it is the intersection of the join with the polar of O. Hence the $21p$ lie in the polar [5] with respect to $\mathfrak{S}$ of UX_7.

So, in order to construct $\mathcal{E}$, one can take any of the 288 σ in the polar of any of the 120 p, $\mathcal{E}$ being then completed unambiguously by the two points analogous to U and X_7 above. Thus, since any of the 36 joins of $\mathcal{E}$ can fulfil this function of completion,

$$N = 288 \times 120/36 = 960.$$

The order of A is therefore

$$960\cdot 9! = 2^{13}\cdot 3^5\cdot 5^2\cdot 7.$$

8. The projectivity imposed by (6.3) has period 2, and a glance at the matrix tells that all points in $x_0 = 0$, and only these, are invariant. The join of every two points that are paired with one another passes through the third point Y_0 on UX_0; one may therefore, and this is in accord with earlier nomenclature (**1**, **9**), describe the projectivity as a projection $\mathfrak{J}$ and call Y_0 its *centre*.

The prime of invariant points is the polar of Y_0 and includes every point of $\mathcal{E}_0$ other than U and X_0. There are 120 projections in A, each centred at one of the 120 p, every point in the polar of the centre being invariant. The projection centred on the join of two members of an ennead $\mathcal{E}$ transposes them while leaving every other member of $\mathcal{E}$ invariant. The 36 projections centred on the 36 joins of members of $\mathcal{E}$ answer to the 36 transpositions of a subgroup $\mathfrak{S}_9$ of A.

9. The 8 p on the joins of one member of an ennead to the remaining members form a simplex in regard to which also $\mathfrak{S}$ has a symmetric equation. Take, for example the points Y_i on the joins to U of the remaining points X_i of $\mathcal{E}_0$. They are linearly independent, the matrix of their column vectors (the single zero of each being placed on the diagonal) having **I** for its square. None is on $\mathfrak{S}$, so that the square of every y_i is present in the equation of $\mathfrak{S}$; nor are any two conjugate, as is seen by testing Y_0 and Y_1, so that every product y_iy_j is present too and the equation of $\mathfrak{S}$ referred to the simplex whose vertices are Y_i is

$$\sum_{i=0}^{7} y_i^2 + \sum_{i<j} y_iy_j = 0.$$

This quadratic form therefore has its group of automorphisms isomorphic to A.

10. There are, over $\boldsymbol{F}$, two projectively distinct types of non-singular quadric in [7]; one is ruled, having solids ω on it, the other not. $\mathfrak{S}$ is ruled.

Solids on $\mathfrak{S}$ are detected by taking any plane d on $\mathfrak{S}$ - a plane is on $\mathfrak{S}$ if it is spanned by three non-collinear m that are mutually conjugate; d lies in its own polar [4] D which, should $\mathfrak{S}$ be ruled, meets $\mathfrak{S}$ in two solids. Take, then, to span d, U and a pair of conjugate m in (6.2) that are not collinear with U, say

$$(1, 1, 1, 1, 0, 0, 0, 0) \text{ and } (0, 0, 1, 1, 1, 1, 0, 0). \tag{10.1}$$

The equations for D are

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = x_6 + x_7. \tag{10.2}$$

If, in (6.1), one writes

$$x_0 + \xi,\ x_2 + \xi,\ x_4 + \xi,\ x_6 + \xi$$

for x_1, x_3, x_5, x_7 respectively the outcome is

$$(x_0 + x_2 + x_4 + x_6)\ (x_0 + x_2 + x_4 + x_6 + \xi) = 0,$$

and either factor determines, with (10.2), a solid on $\mathfrak{S}$. Equations for the solid determined by the vanishing of the second factor are, for example,

$$x_0 + x_3 + x_5 + x_7 = x_2 + x_5 + x_7 + x_1 = x_4 + x_7 + x_1 + x_3 =$$
$$= x_6 + x_1 + x_3 + x_5 = 0,$$

and (6.1) admits the corresponding form

$$(x_0 + x_3 + x_5 + x_7)\ (x_2 + x_4 + x_7) + (x_2 + x_5 + x_7 + x_1)\ (x_6 + x_7 + x_3) +$$
$$(x_4 + x_7 + x_1 + x_3)\ (x_6 + x_2 + x_1) + (x_6 + x_1 + x_3 + x_5)\ (x_0 + x_1 + x_7) = 0 \tag{10.3}$$

which, the 8 linear forms occurring being linearly independent, is the standard equation

$$Y_1Z_1 + Y_2Z_2 + Y_3Z_3 + Y_4Z_4 = 0 \tag{10.4}$$

of a ruled quadric in [7].

Since (6.1) is symmetric any equation derived from (10.3) by permuting suffixes is also equivalent to (6.1) and allows equations for 16 solids ω to be written down. Had one, however, to give but one instance, set out by spanning d by X_7 and the two points (10.1) one would have found that D met $\mathfrak{S}$ in the two solids

$$x_6+x_1+x_3+x_5=x_0+x_3+x_5=x_2+x_5+x_1=x_4+x_1+x_3=0,$$

$$x_6+x_0+x_2+x_4=x_1+x_2+x_4=x_3+x_4+x_0=x_5+x_0+x_2=0.$$

The solids on $\mathfrak{S}$ fall, as is well known (**11**, 43), into two systems. It is important to note that these systems are transposed by each of the 120 projections $\mathfrak{J}$. For the prime P of invariant points of $\mathfrak{J}$, polar of the centre of $\mathfrak{J}$, meets $\mathfrak{S}$ in a non-singular quadric which, belonging to a [6], does not contain any solids; thus P meets every solid ω on $\mathfrak{S}$ in a plane d and transforms ω into another solid ω' on $\mathfrak{S}$ through d. But ω, ω' cannot meet in a plane if they belong to the same system; they belong therefore to opposite systems.

Since the projections transpose the systems A has a subgroup A^+, of index 2, of operations that leave both systems invariant. The products of even numbers of $\mathfrak{J}$ belong to A^+, which has 960 alternating subgroups $\mathfrak{A}_9$.

11. One now sets out to describe the figure in detail, obtaining the numbers of the various subspaces and their relations to each other; the geometry will afford much information about the structure of A. The designations, already used on earlier occasions (**6**, **7**), for the subspaces of [7] in their relation to $\mathfrak{S}$ are retained here: for example, a line is labelled s, t, c, g according as 0, 1, 2, 3 of its points are on $\mathfrak{S}$.

Prime sections of $\mathfrak{S}$ are of two kinds: non-singular sections by polars P of points p and singular sections by tangent primes M at points m. The former section, a non-singular quadric in [6], has its kernel (**6**, § 9) at p; the 63 lines in P that pass through p are all t while the 64 lines outside P that pass through p are either c or s, those that meet $\mathfrak{S}$ (once, and therefore twice) being c, the others s. Since there are $135-63=72$ m outside P there are $36c$, and so $28s$, through p. Hence the total number of

$$t \text{ is } 120\times 63/2=3780,$$

$$c \text{ is } 120\times 36 \quad =4320,$$

$$s \text{ is } 120\times 28/3=1120.$$

Both the c and the s fall into *quartets*; the four lines of a quartet span the whole [7] and each of them is the polar of the [5] spanned by the remaining three. The order of A is, again, quickly found as soon as the number of quartets of either kind is known. That quartets of c occur is shown by (10.4), the vertices of the simplex of reference being intersections of $\mathfrak{S}$ with such a quartet. Conversely: when $\mathfrak{S}$ is referred to a simplex whose vertices are the intersections of $\mathfrak{S}$ with a quartet of chords its equation has the form (10.4); no square can occur because every vertex of the simplex is on $\mathfrak{S}$, while all but 4 of the 28 product terms are absent because all but 4 of the 28 pairs of vertices of the simplex are conjugate pairs of points. A is transitive on the quartets of chords. Since, once such a quartet is chosen, its members can undergo 4! permutations, and since the two m on any member can be transposed, the order of A is $2^4\cdot 4!N_c$ where N_c is the number of quartets of c. In order to calculate N_c note that the polar C of any of the 4320 c meets $\mathfrak{S}$ in a Klein quadric $\mathfrak{K}$ having planes on it; $\mathfrak{K}$ has (see the last column of Table I in 7) 280 chords each of which has, with respect to $\mathfrak{K}$, a polar solid $\varkappa$ containing 18 c, these being paired as polars of one another with respect to the section of $\mathfrak{S}$ and $\mathfrak{K}$ by $\varkappa$. Hence, as the quartet could have been assembled in any order,

$$4!\ N_c = 4320\cdot 280\cdot 18,$$

and the order of A is

$$2^4\cdot 4320\cdot 280\cdot 18 = 2^{13}\cdot 3^5\cdot 5^2\cdot 7\cdot$$

The discussion of quartets of s is similar. The polar S of any of the 1120 s meets $\mathfrak{S}$ in a quadric $\mathfrak{L}$ without any planes on it; $\mathfrak{L}$ has (see the last column but one of Table I in 7, or the line of numbers along the top of Table II in **6**) 120 lines in S skew to it, each of which has, with respect to $\mathfrak{L}$, a polar solid $\varkappa$ containing two s, polars of each other with respect to the section of $\mathfrak{S}$ and $\mathfrak{L}$ by $\varkappa$. Hence, as the quartet could have been assembled in any order,

$$4!\ N_s = 1120\cdot 120\cdot 2$$

where N_s is the number of quartets of s. If the vertices of the simplex of reference consist of two points on each of the four s in a quartet the equation of $\mathfrak{S}$ is

$$\xi_1^2 + \xi_1\eta_1 + \eta_1^2 + \xi_2^2 + \xi_2\eta_2 + \eta_2^2 + \xi_3^2 + \xi_3\eta_3 + \eta_3^2 + \xi_4^2 + \xi_4\eta_4 + \eta_4^2 = 0.$$

But as the members of a quartet can undergo 4! permutations and as there are, having regard to order, 6 choices for the two vertices on each s, the order of A is

$$6^4\cdot 4!\ N_s = 6^4\cdot 1120\cdot 120.2 = 2^{13}\cdot 3^5\cdot 5^2\cdot 7.$$

12. Planes d and solids ω lie wholly on $\mathfrak{S}$. It will be recalled (6, § 10) that the letters used to indicate the sections of $\mathfrak{S}$ by other planes and solids are as follows.

	Planes		Solids
e:	a single line.	γ:	a single plane.
f:	two lines.	φ:	two planes.
h:	a single point.	χ:	a single line.
j:	3 non-collinear points.	ψ:	3 concurrent non-coplanar lines.
		$\varkappa$:	a non-singular ruled quadric.
		λ:	a non-singular non-ruled quadric.

The tangent prime M to $\mathfrak{S}$ at m cuts $\mathfrak{S}$ in a cone, vertex m, whose section by a [5] in M and not passing through m is a KLEIN quadric $\mathcal{K}$. The numbers of subspaces, and their relation to $\mathcal{K}$, in the [5] are given in the last column of Table I in 7; these are

$$28p,\ 35m;\quad 105g,\ 210t,\ 280c,\ 56s;\quad 30d,\ 105e,\ 630f,\ 70h,\ 560j.$$

It follows that there pass through m

$$28t,\ 35g;\quad 105d,\ 210e,\ 280f,\ 56h;\quad 30\omega,\ 105\gamma.\ 630\varphi,\ 70\chi,\ 560\psi.$$

which, at the same time, lie in M. One can deduce immediately the numbers of subspaces in [7] but, on noting the numbers of m in the divisors below, one must remark that in f and ψ the 'contact' is unique while in φ it is one of 3 points. Thus the total number of

$$\begin{aligned}
t &\text{ is } 135 \times 28 = 3780\\
g &\text{ is } 135 \times 35/3 = 1575\\
d &\text{ is } 135 \times 105/7 = 2025\\
e &\text{ is } 135 \times 210/3 = 9450\\
f &\text{ is } 135 \times 280 = 37800\\
h &\text{ is } 135 \times 56 = 7560\\
\omega &\text{ is } 135 \times 30/15 = 270\\
\gamma &\text{ is } 135 \times 105/7 = 2025
\end{aligned}$$

$$\varphi \quad \text{is } 135\times 630/3 = 28350$$

$$\chi \quad \text{is } 135\times 70/3 = 3150$$

$$\psi \quad \text{is } 135\times 560 = 75600$$

13. The sections of $\mathfrak{S}$ by the spaces just listed are singular; it remains to find how many planes j meet $\mathfrak{S}$ in non-singular conics and how many solids $\varkappa$, λ meet $\mathfrak{S}$ in non-singular quadric surfaces - ruled and non-ruled respectively. Once this has been done the numbers of all types of subspace will be known, since a space of dimension exceeding 3 is the polar of one of dimension less than 3.

Any three m, no two of them conjugate, span a j. Choose any of the 135 m and call it m_1. There are 70 others, two on each of 35 g through m_1, in the tangent prime M_1, and so 64 outside M_1; it is these 64 that are not conjugate to m_1. Choose any of these 64, and call it m_2. The tangent primes M_1, M_2 meet in the polar C of m_1m_2; C contains 35 m composing the KLEIN section of $\mathfrak{S}$ therein, so that there are 36 m outside C but in M_1, another 36 in M_2. Hence there are, outside both M_1 and M_2,

$$135-35-36-36=28$$

m; it is these 28 that are not conjugate either to m_1 or to m_2. Choose any of these 28 and call it m_3. Since m_1, m_2, m_3 could have been selected in any sequence the number of j is

$$135\cdot64\cdot28/6=40320.$$

There are, as explained in § 11, 11200 quartets of s; any two members of such a quartet span a $\varkappa$, so that each quartet affords 6 $\varkappa$. Moreover there is one, and only one, way of spanning a given $\varkappa$ by a pair of s, polars of each other with respect to the section of $\mathfrak{S}$: namely by what have been called (**6**, 631) the DANDELIN lines of $\varkappa$. It follows that the number of $\varkappa$ is 67200. This number is also obtainable by using the N_c quartets of c for here, too, any two members of a quartet span a $\varkappa$ and are polars of each other with respect to the section of $\mathfrak{S}$. But $\varkappa$ can be spanned by any of 9 polar pairs of c, so that any one $\varkappa$ arises from 81 quartets, there being also 9 choices in the polar of $\varkappa$ for the two members of the quartet outside $\varkappa$. It follows that the number of $\varkappa$ is $6N_c/81$.

A solid λ includes 5 m no two of which are conjugate; conversely (take, for example, the solid $X_0X_1X_2X_3$) any 4 m, no two being conjugate, span a solid λ. Thus each of the 960 $\mathcal{S}$ provides 126 λ and so, since each λ, including 5 sets of 4 mutually non-conjugate m, arises from 5 different $\mathcal{S}$, the number of λ is $960.126/5 = 24192$. They form 12096 pairs of polar solids: for instance the polar of $X_0X_1X_2X_3$ is

$$x_0 = x_1 = x_2 = x_3 = x_4 + x_5 + x_6 + x_7;$$

herein are 5 m, no two conjugate, namely those whose coordinates are the columns

.	1	1	1	1
.	1	1	1	1
.	1	1	1	1
.	1	1	1	1
1	1	.	.	.
1	.	1	.	.
1	.	.	1	.
1	.	.	.	1

14. One can now build Table I, each column showing how many subspaces of different categories lie in the space whose symbol heads the column. Save for those columns headed by ω, D, G, M the table is largely a transcript of earlier ones: compare Table I of 7 with its first column, and five columns headed by letters with zero suffixes, deleted, its top row amalgamated with its second and each row led by a letter with zero suffix amalgamated with that led by the same letter without the suffix. The column headed by P consists of the numbers down the extreme left of Table I of 7 (having regard to the amalgamations just prescribed); the column headed by J consists of the numbers, correspondingly amalgamated, along the top of Table I of 6; that headed by S of the numbers along the top of Table II of 6. Capitals now denote spaces that are polars, with respect to $\mathfrak{S}$, of those denoted by corresponding small letters.

If an entry appears as a sum it is in a column corresponding to a space whose section of $\mathfrak{S}$ is singular, the different components of the sum indicating different relations of the subspaces to the vertex of the section: recall the note to Table II of 6; for instance, of the 75 g in T 15 pass through the contact of T with $\mathfrak{S}$ whereas the other 60 do not.

The numbers of spaces through a given subspace are deducible instantly by polarising in $\mathfrak{S}$; there are, for instance, 75 G through t of which 15 lie in the tangent M to $\mathfrak{S}$ at its contact with t.

Column D is readily compiled, since D meets $\mathfrak{S}$ in two solids ω, ω'

through its polar plane d. The other solid through d in D is γ. There are 28 solids in D which do not contain d and which therefore meet $\mathfrak{S}$ in two planes, one in ω and one in ω', intersecting in a line in d. The g in D consist of 7 in d, 28 outside d in ω, 28 outside d in ω'; the c join the 8 points of ω outside d to the 8 points of ω' outside d; the p are the 8 points of γ outside d; a plane f in D joins one of the 28 lines g in ω outside d to one of those 4 lines in ω' that pass through the intersection of g with d and yet do not lie in d; and so on.

The entries in column G follow from the observation that the line cone, with vertex the polar g of G, in which G cuts $\mathfrak{S}$ projects from g a non-singular ruled quadric, or hyperboloid, in [3]. Let, for definiteness, g join the points (10.1), so that G is

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 . \tag{14.1}$$

If, in (6,1), one writes

$$x_0 + \xi, \quad x_2 + \xi, \quad x_4 + \xi$$

for x_1, x_3, x_5 respectivety the outcome is

$$(x_0 + x_2 + x_4)\ (x_0 + x_2 + x_4 + \xi) + (x_6 + \xi)\ (x_7 + \xi) = 0,$$

the canonical form $xy + zt = 0$ for a hyperboloid. All solids in G that are skew to g are $\varkappa$. There is no λ in G; every solid in G other than these $\varkappa$ meets $\mathfrak{S}$ in a singular quadric.

The number of solids in G skew to g is quickly found by a standard procedure. Such a solid is spanned by taking

A_1, any of the 63–3 points in G not on g;

A_2, any of the 63–7 points in G not in the plane gA_1;

A_3, any of the 63–15 points in G not in the solid gA_1A_2:

A_4, any of the 63–31 points in G not in the [4] $gA_1A_2A_3$.

But this same solid would have been spanned by taking

B_1, any of its 15 points;

B_2, any of its 14 points other than B_1;

B_3, any of its 12 points not on the line B_1B_2;

B_4, any of its 8 points not on the plane $B_1B_2B_3$.

Hence there are

$$60\cdot 56\cdot 48\cdot 32/15\cdot 14\cdot 12\cdot 8 = 4^4 = 256$$

solids in G skew to g. The discussion has also shown that there are, skew to a given line therein, 64 planes in [4] and 16 lines in [3].

Since the subspaces in $\varkappa$ consist of

$$9m,\ 6p;\ \ 6g,\ 18c,\ 9t,\ 2s;\ \ 9f,\ 6j;$$

it follows that there are, containing g and lying in G,

$9d$,	each including	$4m$	off g;	
$6e$,	,, ,,	$4p$	off g;	
6ω,	,, ,,	$16g$	skew to g;	
18φ,	,, ,,	$16c$	skew to g;	
9γ,	,, ,,	$16t$	skew to g;	
2χ,	,, ,,	$16s$	skew to g;	
$9D$,	,, ,,	$64f$	skew to g;	
$6E$,	,, ,,	$64j$	skew to g;	

All these numbers provide entries in column G; for instance $9.64 = 576f$ and $6.64 = 384j$. The only entries now outstanding are those for spaces which neither contain g nor are skew to g, and so meet it at one of its 3 m. They are quickly found. As instances: the 96 g skew to g lie 4 in each of 24 d joining them to any m on g, so that 72 such d occur; the 576 f skew to g lie 8 in each of 72 φ joining them to any m on g, so that 216 such Φ occur; and so on.

Since the tangent prime M to $\mathfrak{S}$ at m meets $\mathfrak{S}$ in a point cone projecting a KLEIN quadric every [5] in M that does not contain m is a C; the last column in the table can thus be disposed of summarily. For, C having the subspaces

$$28p,\ 35m;\ \ 105g,\ 210t,\ 280c,\ 56s;\ \ 30d,\ 105e,\ 630f,\ 70h,\ 560j;$$
$$105\varphi,\ 210\psi,\ 280\varkappa,\ 56\lambda;\ \ 28J,\ 35F$$

M contains

$$28t,\ 35g;\ \ 105d,\ 210e,\ 280f,\ 56h;\ \ 30\omega,\ 105\gamma,\ 630\varphi,\ 70\chi,\ 560\psi;$$
$$105D,\ 210E,\ 280F,\ 56H;\ \ 28T,\ 35G$$

each of which includes (the m being in addition to the contact while the g, t, c do not contain this contact)

$$2p,\ 2m;\ \ 4g,\ 4t,\ 4c,\ 4s;\ \ 8d,\ 8e,\ 8f,\ 8h,\ 8j;\ \ 16\varphi,\ 16\psi,\ 16\varkappa,\ 16\lambda;\ \ 32J,\ 32F$$

respectively.

TABLE I

	135	120	1575	37 80	4320	1120	2025	9450	7560	37800	40320	270	2025	28350	3150	75600	67200	24192	40320	37800	7560	9450	2025	1120	4320	3780	1575	120	135
	m	p	g	t	c	s	d	e	h	f	j	ω	γ	φ	χ	ψ	$\varkappa$	λ	J	F	H	E	D	S	C	T	G	P	M
m			3	1	2	·	7	3	1	1+4	3	15	7	3+8	3	1+6	9	5	15	1+18	1+10	3+12	7+16	27	35	1+30	3+36	63	1+70
p			·	2	1	3	·	4	6	2	4	·	8	4	12	8	6	10	16	12	20	16	8	36	28	32	24	64	56
g							7	1	·	2	·	35	7	1+12	1	3	6	·	15	9+24	5	.1+18	7+56	45	105	15+60	1+54+96	315	35+420
t							·	6	3	1	3	·	28	6	18	4+12	9	15	60	6+36	10+60	24+48	28	270	210	16+240	36+144	1008	28+840
c							·	·	·	4	3	·	·	16	·	12	18	10	60	72	40	48	64	216	280	240	288	1008	1120
s							·	·	4	·	1	·	·	·	16	4	2	10	20	8	40	16	·	120	56	80	32	336	224
d												15	1	2	·	·	·	·	·	6	·	3	1+28	·	30	15	9+72	135	105+240
e												·	14	1	3	3	·	·	15	9	15	4+36	14	135	105	60+120	6+108	1260	210+840
h												·	·	·	12	1	·	5	15	2	10+40	12	·	270	70	20+120	24	1260	56+560
f												·	·	12	·	3	9	·	45	18+72	10	36	112	270	630	60+360	216+576	3780	280+5040
j												·	·	·	·	8	6	10	80	48	80	64	·	720	560	640	384	5376	4480
ω																			·	·	·	·	2	·	·	·	6	·	30
γ																			·	·	·	3	1	·	·	15	9	135	105
φ																			·	9	·	3	28	·	105	45	18+216	945	630+1680
χ																			·	·	5	1	·	45	·	15	2	315	70
ψ																			15	6	10	24	·	270	210	80+240	144	5040	560+3360
$\varkappa$																			10	16	·	·	·	120	280	160	256	3360	4480
λ																			6	·	16	·	·	216	56	96	·	2016	896
J																								36	28	32	·	1344	896
F																								·	35	10	48	630	280+1120
H																								27	·	6	·	378	56
E																								·	·	15	6	315	210
D																								·	·	·	9	·	105
S																												28	·
C																												36	64
T																												63	28
G																												·	35

15. Before passing on to further matters a word or two about the enneads will be in place. Each pair of vertices of $\mathcal{E}$ has for its join a c, each set of 3 vertices spans a plane j; hence the number of $\mathcal{E}$ having

a given m for vertex is	$960\cdot 9/135 \quad = 64,$
a given c for edge is	$960\cdot 36/4320 = 8.$
a given j as plane face is	$960\cdot 84/40320 = 2.$

The solid spanned by 4 vertices of $\mathcal{E}$ is a λ, and so contains a single m that is not a vertex of $\mathcal{E}$. All $9+126$ m are accounted for by the vertices of $\mathcal{E}$ and by these supplementary m that lie one in each of the 126λ determined by sets of 4 of the 9 vertices. After a vertex of $\mathcal{E}$ is isolated from the other 8 there are 35 ways of partitioning these as 4^2; each such bisection affords a pair of λ, and the two supplementary m, one in each λ, are collinear with the isolated vertex. Thus all 35 g through this vertex are accounted for. The collinearity is verified at once by isolating U in $\mathcal{E}_0$.

Each of the 28 plane faces that contain a given vertex m of, say, $\mathcal{E}_1$ provides, as sharing this plane face, one of the 64 $\mathcal{E}$ having m for vertex. These 28 $\mathcal{E}$ account, with $\mathcal{E}_1$ itself, for all 8 enneads sharing any edge of $\mathcal{E}_1$ through m: if enneads share 2 vertices they share 3. The remaining 35 that have m for vertex share no other vertex with $\mathcal{E}_1$; each of them is linked with one of the 35 separations into tedrads τ, τ' of the 8 vertices of $\mathcal{E}_1$ other than m. Supplementary to τ is μ, completing the section of $\mathfrak{S}$ by the λ which τ spans; $m\mu$ is completed by the supplement μ' to τ' and that tedrad which is in perspective with τ from μ' belongs to the new ennead, which is completed by the tetrad in perspective with τ' from μ. Take, in illustration, m to be U and $\mathcal{E}_1$ to be $\mathcal{E}_0$. The other ennead having vertices U, X_0, X_1 is given, apart from U, by the columns

$$
\begin{matrix}
1 & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . \\
. & . & . & 1 & 1 & 1 & 1 & 1 \\
. & . & 1 & . & 1 & 1 & 1 & 1 \\
. & . & 1 & 1 & . & 1 & 1 & 1 \\
. & . & 1 & 1 & 1 & . & 1 & 1 \\
. & . & 1 & 1 & 1 & 1 & . & 1 \\
. & . & 1 & 1 & 1 & 1 & 1 & .
\end{matrix}
\tag{15.1}
$$

The ennead which shares no vertex other than U with $\mathcal{E}_0$ and which is linked with the separation $X_0X_1X_2X_3/X_4X_5X_6X_7$ is given, apart from U, by the columns

$$(15.2)\qquad \begin{array}{cccccccc}
1 & . & . & . & 1 & 1 & 1 & 1\\
. & 1 & . & . & 1 & 1 & 1 & 1\\
. & . & 1 & . & 1 & 1 & 1 & 1\\
. & . & . & 1 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 1 & 1 & . & . & .\\
1 & 1 & 1 & 1 & . & 1 & . & .\\
1 & 1 & 1 & 1 & . & . & 1 & .\\
1 & 1 & 1 & 1 & . & . & . & 1
\end{array}$$

Here μ is $(1\ 1\ 1\ 1\ .\ .\ .\ .)$ and μ' is $(.\ .\ .\ .\ 1\ 1\ 1\ 1)$.

Arrays in clear analogy with (15.1) give all 28 $\mathcal{E}$ that share with $\mathcal{E}_0$ U and two other vertices; arrays in clear analogy with (15.2) give all 35 $\mathcal{E}$ that share only U with $\mathcal{E}_0$.

16. Any 3 vertices of an $\mathcal{E}$ with their centroid compose a conic with its kernel. If the 9 vertices are partitioned as 3^3 the kernels of the 3 triads are collinear. If, for example, $\mathcal{E}$ is $\mathcal{E}_0$ and the partition is

$$(16.1)\qquad UX_0X_1:\ X_2X_3X_4:\ X_5X_6X_7$$

the kernels are

$$(16.2)\qquad (.\ .\ 1\ 1\ 1\ 1\ 1\ 1),\quad (.\ .\ 1\ 1\ 1\ .\ .\ .),\quad (.\ .\ .\ .\ .\ 1\ 1\ 1).$$

The line s of the kernels may be called an *axis* of $\mathcal{E}$; each $\mathcal{E}$ has ${}^9C_3\cdot{}^6C_3/3! = 280$ axes, and each s is an axis of $960\times 280/1120 = 240\,\mathcal{E}$. This number can be found otherwise. If s_0 is an axis of $\mathcal{E}$ there are planes j_1, j_2, j_3 that meet s_0 at p_1, p_2, p_3 and together account for all 9 vertices of $\mathcal{E}$; in j_i lies s_i which, being the only line in j_i skew to $\mathfrak{S}$, is in the polar of every vertex of $\mathcal{E}$ outside j_i. Thus s_1, s_2, s_3 complete a quartet with s_0. Conversely: let s_1, s_2, s_3 complete a quartet with s_0 and let j_1, j_2, j_3 be one of the 6 sets of planes joining them to the p on s_0. Each plane includes $3m$; no two in the same j_i are conjugate; nor are any two in different j_i since, were such a pair of m conjugate, it would follow that two of the p on s_0 were. The $9m$ therefore give an $\mathcal{E}$ of which s_0 is an axis. Since there are 40 quartets having s_0 as a member there are $240\,\mathcal{E}$ having s_0 as an axis.

Each triad of vertices of an ennead $\mathcal{E}_1$ is shared by another $\mathcal{E}$ so that the partitioning (of vertices of $\mathcal{E}_1$) leads to $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$. One of the partitionings as 3^2 of those 6 vertices of $\mathcal{E}^i$ that it does not share with $\mathcal{E}_1$ has both its two kernels on the axis s of $\mathcal{E}_1$ linked to the original partitioning.

Thus each of $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$ leads, *via* s, to three $\mathcal{E}$ one of which is $\mathcal{E}_1$; it appears that these are always the same $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$. So one obtains a closed set of 6 $\mathcal{E}$ falling into complementary triads; each member of either triad shares 3 vertices with each member of the other and s serves as an axis for all 6$\mathcal{E}$. It is enough, by way of proving these facts, to give the instance that derives from (16.1). The enneads are as follows, with their triads of common vertices bracketed. The coordinate vector of the kernel of a triad is the sum of those of its three members; all 6 $\mathcal{E}$ here displayed have (16.2) for an axis. Each triad is labelled by the script capital $\mathcal{A}$ $\mathcal{B}$, or $\mathcal{C}$; triads with the same kernel have the same script label and are distinguished one from an another ther by its suffix.

1 1 .	. . .	. . .	1 1 .	. . .	. . .	1 1 .	. . .	. . .
1 . 1	. . .	. . .	1 . 1	. . .	. . .	1 . 1	. . .	. . .
1 . .	1 . .	. . .	1 . .	. 1 1	1 1 1	. 1 1	1 . .	1 1 1
1 . .	. 1 .	. . .	1 . .	1 . 1	1 1 1	. 1 1	. 1 .	1 1 1
1 . .	. . 1	. . .	1 . .	1 1 .	1 1 1	. 1 1	. . 1	1 1 1
1 . .	. . .	1 . .	. 1 1	1 1 1	1 . .	1 . .	1 1 1	. 1 1
1 . .	. . .	. 1 .	. 1 1	1 1 1	. 1 .	1 . .	1 1 1	1 . 1
1 . .	. . .	. . 1	. 1 1	1 1 1	. . 1	1 . .	1 1 1	1 1 .
$\mathcal{A}_1$	$\mathcal{B}_1$	$\mathcal{C}_1$	$\mathcal{B}_2$	$\mathcal{C}_2$	$\mathcal{A}_2$	$\mathcal{C}_3$	$\mathcal{A}_3$	$\mathcal{B}_3$

1 1 .	. . .	. . .	1 1 .	. . .	. . .	1 1 .	. . .	. . .
1 . 1	. . .	. . .	1 . 1	. . .	. . .	1 . 1	. . .	. . .
1 . .	. 1 1	1 1 1	1 . .	1 . .	. . .	. 1 1	1 1 1	1 . .
1 . .	1 . 1	1 1 1	1 . .	. 1 .	. . .	. 1 1	1 1 1	. 1 .
1 . .	1 1 .	1 1 1	1 . .	. . 1	. . .	. 1 1	1 1 1	. . 1
1 . .	1 1 1	. 1 1	. 1 1	1 1 1	1 . .	1 . .	1 . .	. . .
1 . .	1 1 1	1 . 1	. 1 1	1 1 1	. 1 .	1 . .	. 1 .	. . .
1 . .	1 1 1	1 1 .	. 1 1	1 1 1	. . 1	1 . .	. . 1	. . .
$\mathcal{A}_1$	$\mathcal{C}_2$	$\mathcal{B}_3$	$\mathcal{B}_2$	$\mathcal{A}_3$	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{A}_2$	$\mathcal{B}_1$

17. It is inevitable that Table I should have many features in common with Table II of **9**, for both tables relate to representations of the same group as a group of projectivities in [7], although over different fields. The dictionary given in § 18 of **6** extends to

$$\begin{array}{cccccccccccccc} p_0 & e_1 & \gamma_1 & \gamma_1\times p_0 & c_2 & \gamma_1\times\gamma_1 & \gamma_3 & \beta_3 & \gamma_4 & \beta_4 & \beta_5 & m_5 & m_6 \\ p & t & s & j & h & \varkappa & \lambda & \chi & J & H & T & S & P. \end{array}$$

The relations (**9**: 57,58) between those spaces labelled β and γ (the suffix denoting the number of dimensions, and c_2 being used for β_2, γ_2 as these planes happen to be identical) accord with analogous relations here. That β_n includes $(n+1)\beta_{n-1}$ points to there being

$$4h \text{ in } \chi, \quad 5\chi \text{ in } H, \quad 6H \text{ in } T$$

and that β_n includes $2^n\gamma_{n-1}$ points to there being

$$4s \text{ in } h, \quad 8h \text{ in } \chi, \quad 16\lambda \text{ in } H, \quad 32J \text{ in } T;$$

all this is as it should be, allowing the $12h$ in χ to be partitioned. And there are, too,

$$4s \text{ in } h, \quad 5h \text{ in } \lambda, \quad 6\lambda \text{ in } J$$

in agreement with there being $(n+2)\gamma_{n-1}$ in γ_n.

All points in MISS HAMILL's geometry are analogues of p, so that the top row of her table corresponds to the second of ours. The few apparent discrepancies are easily explained away. That J contains 16 p and γ_4 only 15 p_0 is because the 16 include the kernel of the section of $\mathfrak{S}$ by J; the same explanation reconciles P containing 64 p with m_6 containing only 63 p_0. The section of $\mathfrak{S}$ by T is a point cone projecting a non-singular quadric in [4] and so having a kernel-generator; if the two p hereon are omitted there remain 30 p which are analogues of the 30 p in β_5.

Take now the 28 p in C, and recall, from § 9 of **7**, their distribution among 8 heptagons; omit one heptagon, and consider the remaining 21 p. If one takes for C, with the coordinates and the heptagons as in § 9 of **7**, the unit [5] and omits $\mathfrak{h}_0$ the remaining p are all those, and only those, having 2 of their 7 coordinates non-zero. This fits the duad notation (**9**, 57) for the vertices of γ_5 perfectly, and vertices of γ_5 that are polar to each other correspond to p that are conjugate with respect to $\mathfrak{S}$. Thus C includes 8 analogues of γ_5, one for each omitted heptagon, and that there are 4320 C accords with there being 34560 γ_5. Take next, from the 21 p in C that are not vertices of $\mathfrak{h}_0$, those 15 for which $x_0 = 0$; they are analogues of vertices of a γ_4. But they belong to two sets of 21 p that are analogues of vertices of a γ_5, for one can supplement them by 6 vertices either of $\mathfrak{b}_0$ or of $\mathfrak{b}_1$ (the common vertex of $\mathfrak{b}_0$ and $\mathfrak{b}_1$ not permitted). And so, just as there are 7 γ_4 in γ_5, there are $7.8/2 = 28\,J$ in C.

As no analogues of m were available in **9** neither could there be any of g, c; d, f; ω, φ; D; for the p in such spaces, if existent at all, are inadequate to span them. The presence of m in the finite space simplifies the description of the analogues of γ_n: that of γ_n consists of those $\frac{1}{2}(n+1)(n+2)\,p$ that lie one on each join of a pair of $n+2\ m$ no two of which are conjugate, of $n+2\ m$, that is, that belong to the same ennead. The eligible simplexes Σ thus afford the analogues of $8640\gamma_6$. the enneads $\mathcal{E}$ those of $960\gamma_7$. It was remarked, at the first mention of γ_n in **9**, that it could "be visualized as the general prime section of a simplex in $n+1$, though it does not arise in that way here": but this, in the finite space, is precisely the way in which it does arise. MISS HAMILL states (**9**, 60) that there are 135 β_7 and 960 γ_7, and the clear implication that A has permutation representations of degrees 135 and 960 must be credited to her, as also must, to go no higher, representations of degrees 120 and 1120. That there are as many β_7 as there are m is because the $56p$ in any M are analogues of the vertices of a β_7. For let M be the unit prime. The p therein are those points either 2 or 6 of whose 8 coordinates are non-zero. If those having $x_i = x_j = 1$. with the remaining coordinates 0, and $x_i = x_j = 0$, with the remaining coordinates 1, are labelled (ij) and (ji) one has the non-commuting duads (**9**, 58) that serve to describe β_7 and the differing relations among its vertices.

Those of the $56p$ for which some definite x_i is equal to one, and only one, other coordinate are the 42 vertices of a β_6, and so the 8 β_6 in this β_7 are identified.

18. Whether it be the geometrical definition, or the mere number, of any space of the long list displayed along the top of Table II in **9** that one considers it has its analogue in the finite geometry. Consider some of the [5] 's; the concordance of β_5, m_5, γ_5 with T, S. C has already been verified, but these are only 3 of 11 types of [5] occurring in **9**. Let $a \oplus b$ denote the direct sum of two subspaces a and b, meaning thereby the space spanned by skew spaces a, b every point of either of which is conjugate to every point of the other. What, then, is $h \oplus h$, and in how many ways can it be thus spanned? It is, being spanned by skew planes, a [5]; since the section of $\mathfrak{S}$ by h is singular, with a point vertex, that by $h \oplus h$ is singular, with a line vertex; this implies G. Conversely, given G, take h_1, any of the $24\,h$ therein; the polar g of G is joined to h_1 by a solid χ_1 whose polar χ_2 is the intersection of G and the polar H_1 of h_1; the second plane h_2 required to span G must lie in χ_2 but must not meet g in the same m

as h_1 does. Thus there are 8 choices for h_2, there being $4h$ in χ_2 through either of the two points other than m on g.

So G can be spanned as $h\oplus h$ in $\frac{1}{2}(24\times 8)=96$ ways. Even so are there $151200=96\times 1575$ spaces $c_2\times c_2$ in **9**. Or take $\lambda\oplus t$, a [5] meeting $\mathfrak{S}$ in a point cone and so a T. Given T there are 96λ therein; once one of these is chosen t has to be the intersection of T with its polar λ', and indeed there are $362880=96\times 3780$ spaces $\gamma_3\times e_1$ in **9**. Then $s\oplus s\oplus t$, which is also a T because it meets $\mathfrak{S}$ in a point-cone, is the same as $\varkappa\oplus t$. Given T there are $160\varkappa$ therein; once one of these is chosen t has to be the intersection of T with its polar $\varkappa'$, and there are $604800=160\times 3780$ spaces $\gamma_1\times\gamma_1\times e_1$ in **9**. The section of $\mathfrak{S}$ by a [5] $\lambda\oplus s$ is non-singular; it is a C wherein λ and s are polars, and each C can be so spanned in 56 ways. Just so are there $56\times 4320=241920$ spaces $\gamma_3\times\gamma_1$ in **9**. And so forth.

19. When $\mathfrak{S}_9$ is a subgroup of A each of its conjugate classes is part of a conjugate class of A. If the ennead of which $\mathfrak{S}_9$ is the stabiliser is $\mathcal{E}_0$ one can at once write down a matrix μ belonging to any of these 30 classes of A; its columns are the coordinate vectors of those 8 members of $\mathcal{E}_0$ into which the vertices of Σ_0, ordered so that their coordinate vectors form the unit matrix, are transformed.

A projectivity that permutes the members of $\mathcal{E}_0$ among themselves is the section by [7] of a projectivity in [8] that permutes the 9 vertices of a simplex therein; one takes the [8] to be spanned by the vertices of Σ_0 with a point X_8 outside [7]. Any 9-rowed permutation matrix permutes the 9 X_i while leaving the unit point V in [8] unmoved; it imposes a projectivity Π on the star of lines through V and so induces, in $x_8=0$, one π on the points of [7]. This permutes the members of $\mathcal{E}_0$, consisting as they do of the vertices of Σ_0 together with the intersection U of VX_8 with [7]; all 9! permutations of members of $\mathcal{E}_0$ are so obtainable.

The dimension of the latent space Λ of Π is (cf. **7**, 595) less by 1 than the number ν of parts in $\{\lambda\}$, the partition of 9 that corresponds to the cyclic decomposition of the permutation imposed by Π; the $[\nu-2]$ in which Λ meets $x_8=0$ is latent for π. Now it is, *a priori*, possible for a point a to be latent for π but not for Π; it may happen that Π transposes a with the third point on Va.

But scrutiny discloses that this occurs when, and only when, every part of $\{\lambda\}$ is even; since this is impossible for any partition of an odd number the contingency may here be disregarded. For if $\{\lambda\}=\{\lambda_1\lambda_2\ldots\}$ consider the effect of Π^{λ_1} on a point a in $x_8=0$ that is presumed invariant for π but not for Π. Since Π is effected by a 9-rowed permutation matrix Π^{λ_1}

leaves a certain λ_1 of the 9 coordinates of a unchanged. Yet Π itself adds 1, and so Π^{λ_1} adds λ_1, to every coordinate of a; hence λ_1 cannot be odd, nor can any of $\lambda_2, \ldots$. Thus the latent space of any projectivity in any subgroup $\mathcal{S}_9$ of A has dimension $\nu-2$ where ν is the number of cycles in the permutation of members of the appropriate ennead.

20. Once the matrix μ is written down the equations of the latent space follow: its points are those whose coordinate vectors are latent for μ. Thus not only the dimension but also the category of the space in its relation to $\mathfrak{S}$ can be determined. Of the 30 instances a few suffice by way of illustration; let them be concerned with the partitions

$$1^2 34,\quad 3^3,\quad 1^4 5,\quad 1^4 23,\quad 2^2 5,\quad 9.$$

$1^2 34$
$$\mu = 1 \oplus \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix}$$

whose latent vectors fill the space

$$x_1 = x_2 = x_3, \quad x_4 = x_5 = x_6 = x_7.$$

This is a plane f; the only p therein are

$$(.\ 1\ 1\ 1\ .\ .\ .\ .) \text{ and } (.\ 1\ 1\ 1\ 1\ 1\ 1\ 1).$$

3^3
$$\mu = \begin{bmatrix} . & 1 & . & . & . & . & . & 1 \\ . & . & 1 & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & 1 \end{bmatrix}$$

whose latent vectors are those having

$$x_0 = x_1 = x_2, \quad x_3 = x_4 = x_5, \quad x_6 = x_7 = 0.$$

This is a line s.

$1^4 5$ $$\mu = 1 \oplus 1 \oplus 1 \oplus \begin{bmatrix} . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \\ 1 & . & . & . & . \end{bmatrix}$$

for which the invariant points satisfy $x_3 = x_4 = x_5 = x_6 = x_7$. This is a solid λ, indeed $X_0X_1X_2U$.

$1^4 23$ $$\mu = 1 \oplus 1 \oplus 1 \oplus \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix}$$

for which the invariant points are given by $x_3 = x_4$, $x_5 = x_6 = x_7$. This is a [4] J, containing $X_0X_1X_2U$.

$2^2 5$ $$\mu = \begin{bmatrix} . & 1 & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & 1 \\ . & . & . & 1 & . & . & . & 1 \\ . & . & 1 & . & . & . & . & 1 \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & . & 1 & . & 1 \\ . & . & . & . & . & . & 1 & 1 \end{bmatrix}$$

having for invariant points those which satisfy

$$x_0 = x_1, \quad x_2 = x_3, \quad x_4 = x_5 = x_6 = x_7 = 0,$$

a line t.

9 Here
$$|\mu + \lambda I| = \begin{vmatrix} \lambda & . & . & . & . & . & . & 1 \\ 1 & \lambda & . & . & . & . & . & 1 \\ . & 1 & \lambda & . & . & . & . & 1 \\ . & . & 1 & \lambda & . & . & . & 1 \\ . & . & . & 1 & \lambda & . & . & 1 \\ . & . & . & . & 1 & \lambda & . & 1 \\ . & . & . & . & . & 1 & \lambda & 1 \\ . & . & . & . & . & . & 1 & 1+\lambda \end{vmatrix} = 1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6 + \lambda^7 + \lambda^8$$

as is seen by subtracting each row, multiplied by λ, from the one immediately above it, beginning the operations from the bottom. Since $|\mu+\lambda I|=0$ has no root in $\boldsymbol{F}$ there is no invariant point.

21. Once equations determining the latent space are known the numbers of m and p therein are known too. These mere numbers do not always identify the space; they fail to distinguish e from j, γ from ψ, E from J. But these distinctions can be realised on scrutinising the distribution of the points; in e, for example, the m are collinear whereas in j they are not. And another criterion can be appealed to: the latent space for any partition $\{\lambda\}$ necessarily includes spaces latent for any partitions to which $\{\lambda\}$ is subordinate. $1^4 23$ gave J, not E, because it is subordinate to $1^4 5$ whose latent space is λ, and E does not include any λ.

Likewise $1^3 24$ and $1^2 2^2 3$ are both subordinate to $1^2 34$ whose latent plane is f; hence their latent solids cannot be γ and so, as including 8 p, have to be ψ. And $12^2 4$, subordinate to 14^2 with invariant g, has latent e, not j; while $1^3 6$ and 123^2, subordinate to $1^2 7$ and 3^3 respectively, have latent j.

Proceeding on these lines one compiles Table II. This displays in its different columns, reading from the left,

the partition determining the cycle type of the permutation of 9 members of an ennead.

the dimension of the latent space,

the number of invariant m,

the number of invariant p,

the category of the latent space,

the conjugate class of A to which the group operation belongs in MISS HAMILL'S enumeration.

The number of invariant p has to accord with that in MISS HAMILL'S v-column (9, Table III), and here there is no discrepancy to explain away. The period, and the category of the latent space, usually fix the class numeral; in the few instances where they do not one can, to decide which numeral to attach, note those already attached to classes whose members are powers of those of the class in question. For example: 123^2 and $1^3 6$ both give classes whose operations have period 6, and points of a j invariant. But the former class consists of operations whose squares are in X($1^3 3^2$) and cubes in II($1^7 2$), and so is XVII, whereas the latter, since its operations have their squares in X and their cubes in V($1^3 2^3$), is XXII.

A sign + is affixed to partitions having an odd number of parts to indicate that the corresponding classes of operations are in A^+. The dimension of the latent space of any operation in Table II is odd or even with the number of parts in the partition, and so according as the operation is in A^+ or in its coset. This is true of all operations in A, whether members of the 30 classes listed in Table II or not: any operation whose latent space has odd dimension is in A^+, any whose latent space has even dimension is in the coset.

TABLE II

		m	p	σ	
1^9+	7	135	120		I
1^72	6	63	64	P	II
1^63+	5	27	36	S	IV
1^54	4	11	20	H	VII
1^52^2+	5	31	32	T	III
1^423	4	15	16	J	VI
1^45+	3	5	10	λ	XII
1^36	2	3	4	j	XXII
1^324+	3	7	8	ψ	XI
1^32^3	4	15	16	E	V
1^33^2+	3	9	6	$\varkappa$	X
1^225	2	3	4	j	XX
1^22^23+	3	7	8	ψ	IX
1^234	2	5	2	f	XIX
1^27+	1	2	1	c	XXXIV
18	0	1	—	m	XLVIII
14^2+	1	3	—	g	XXVIII
12^24	2	3	4	e	XVIII
$126+$	1	1	2	t	XXXIII
123^2	2	3	4	j	XVII
$135+$	1	2	1	c	XXX
12^4+	3	7	8	γ	VIII
45	0	1	—	m	XLIV
3^3+	1	—	3	s	XXXVIII
36	0	—	1	p	L
$234+$	1	1	2	t	XXVII
27	0	—	1	p	XLV
2^25+	1	1	2	t	XXIX
2^33	2	3	4	e	XVI
$9+$	—	—	—	—	LVIII

II

Operations of the orthogonal group not belonging to any symmetric subgroup. The groups A_σ.

22. Certain spaces do not occur in the penultimate column of Table II; the absence of some is inevitable. In order to see whether M can figure as a latent space take the unit prime (6.2). Since every point having 2 non-zero coordinates is invariant

$$\mu = \begin{bmatrix} a+1 & a & a & a & a & a & a & a \\ b & b+1 & b & b & b & b & b & b \\ c & c & c+1 & c & c & c & c & c \\ d & d & d & d+1 & d & d & d & d \\ e & e & e & e & e+1 & e & e & e \\ f & f & f & f & f & f+1 & f & f \\ g & g & g & g & g & g & g+1 & g \\ h & h & h & h & h & h & h & h+1 \end{bmatrix}$$

where, to forestall linear dependence of the rows,

$$\Sigma\, a \equiv a+b+c+d+e+f+g+h=0.$$

In order that each column be the coordinate vector of an m

$$\Sigma\, ab+\Sigma a=a=b=c=d=e=f=g=h,$$

so that, there being an even number of products in the sum, $\Sigma ab=0$. Thus μ cannot belong to A unless $\mu=I$, and then M does not exhaust the invariant points. Nor can a [5] C ever be latent for an operation of A; its polar c involves a single p which would be invariant yet is not in C.

Certain spaces, then, never occur as latent spaces and are perforce absent from Table II. But when a space is present in Table II this in no way suggests that there are not other operations of A, outside its subgroups $\mathfrak{S}_9$, for which it is latent too. Other conjugate classes in A than those 30 which have members in these $\mathfrak{S}_9$ are encountered by pursuing this matter.

When a projectivity has every point of $x_0=x_1$ invariant its matrix must have the form

$$\mu = \begin{bmatrix} a+1 & a & . & . & . & . & . & . \\ b & b+1 & . & . & . & . & . & . \\ c & c & 1 & . & . & . & . & . \\ d & d & . & 1 & . & . & . & . \\ e & e & . & . & 1 & . & . & . \\ f & f & . & . & . & 1 & . & . \\ g & g & . & . & . & . & 1 & . \\ h & h & . & . & . & . & . & 1 \end{bmatrix}$$

wherein, to forestall singularity, $a = b$. Then, as above,

(22.1) $$\Sigma ab + \Sigma a = a = b.$$

Now, as μ belongs to A, the sum of any two of its last 6 columns is, as coordinate vector of a point, conjugate to both the first and second columns; hence

$$c = d = e = f = g = h, \quad = \alpha \text{ say},$$

so that (22.1) reduces to $ab + \alpha^2 = a = b$, and so $\alpha = 0$.

There are, then, two operations of A leaving every point in $x_0 = x_1$ invariant; they are I $(a = b = 0)$ and the matrix $(a = b = 1)$ of the projection centred at the pole of $x_0 = x_1$.

The only operation of A for which a [6] P is latent is the projection centred at it pole.

23. Consider, now, operations of A for which all points of some D are invariant. Take, as in (10.2), D to be

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = x_6 + x_7;$$

it includes the solid γ whose equations are

(23.1) $$x_0 = x_1, \quad x_2 = x_3, \quad x_4 = x_5, \quad x_6 = x_7,$$

and those projectivities for which every point of γ is invariant have matrices

$$(23.2)\qquad \begin{bmatrix} a_1+1 & a_1 & a_2 & a_2 & a_3 & a_3 & a_4 & a_4 \\ b_1 & b_1+1 & b_2 & b_2 & b_3 & b_3 & b_4 & b_4 \\ c_1 & c_1 & c_2+1 & c_2 & c_3 & c_3 & c_4 & c_4 \\ d_1 & d_1 & d_2 & d_2+1 & d_3 & d_3 & d_4 & d_4 \\ e_1 & e_1 & e_2 & e_2 & e_3+1 & e_3 & e_4 & e_4 \\ f_1 & f_1 & f_2 & f_2 & f_3 & f_3+1 & f_4 & f_4 \\ g_1 & g_1 & g_2 & g_2 & g_3 & g_3 & g_4+1 & g_4 \\ h_1 & h_1 & h_2 & h_2 & h_3 & h_3 & h_4 & h_4+1 \end{bmatrix}$$

Here, to forestall singularity,

$$a_1=b_1,\quad c_2=d_2,\quad e_3=f_3,\quad g_4=h_4.$$

Next, as the sum of any two columns has to be conjugate to each of the remaining six, we find, adding the first and second, then the third and fourth, and so on, that

$$a_2=b_2,\ a_3=b_3,\ a_4=b_4;\ c_1=d_1,\ c_3=d_3,\ c_4=d_4;$$

$$e_1=f_1,\ e_2=f_2,\ e_4=f_4;\ g_1=h_1,\ g_2=h_2,\ g_3=h_3.$$

One has also to prevent any two columns from being conjugate to each other; the second and third would be unless $a_2=c_1$, and other pairings show that (23.2) must be symmetric. Lastly, as the point whose coordinates fill any one column is to lie on $\mathfrak{S}$,

$$g_1=c_1+e_1,\quad g_2=a_2+e_2,\quad g_3=a_3+c_3.$$

The upshot is that any projectivity in A for which every point of (23.1) is invariant has a matrix of the form

$$(23.3)\qquad \begin{bmatrix} a+1 & a & h & h & g & g & g+h & g+h \\ a & a+1 & h & h & g & g & g+h & g+h \\ h & h & b+1 & b & f & f & h+f & h+f \\ h & h & b & b+1 & f & f & h+f & h+f \\ g & g & f & f & c+1 & c & f+g & f+g \\ g & g & f & f & c & c+1 & f+g & f+g \\ g+h & g+h & h+f & h+f & f+g & f+g & d+1 & d \\ g+h & g+h & h+f & h+f & f+g & f+g & d & d+1 \end{bmatrix}$$

The group property, expressible as

$$(23.4)\quad \begin{pmatrix} a & b & c & d \\ & f & g & h \end{pmatrix}\begin{pmatrix} a' & b' & c' & d' \\ & f' & g' & h' \end{pmatrix}=\begin{pmatrix} a+a' & b+b' & c'+c' & d+d' \\ & f+f' & g+g' & h+h' \end{pmatrix}$$

of such matrices is clear on multiplying them.

But it is, as yet, only the points of γ whose invariance has been insisted on. In order to ensure that every point of D is invariant it is necessary and sufficient to ensure the invariance of any one point of D outside γ, say of $(1\cdot 1\cdot 1\cdot 1\cdot)$. This demands $a=b=c=d=0$, and (23.3) becomes

$$(23.5)\quad \begin{bmatrix} 1 & . & h & h & g & g & g+h & g+h \\ . & 1 & h & h & g & g & g+h & g+h \\ h & h & 1 & . & f & f & h+f & h+f \\ h & h & . & 1 & f & f & h+f & h+f \\ g & g & f & f & 1 & . & f+g & f+g \\ g & g & f & f & . & 1 & f+g & f+g \\ g+h & g+h & h+f & h+f & f+g & f+g & 1 & . \\ g+h & g+h & h+f & h+f & f+g & f+g & . & 1 \end{bmatrix}$$

Thus there are 8 projectivities in A for which every point of D is invariant. They include $(f=g=h=0)$ the identity; the latent spaces of the others are the $7G$ through D, and no D can itself be latent for any operation of A. The $7G$ through (23.1), and the matrices (23.5) associated with them, are as follows.

$$\begin{array}{rl} x_0+x_1=x_2+x_3=x_4+x_5: & f=g=h=1. \\ x_2+x_3=x_4+x_5=x_6+x_7: & g=h=0,\ f=1. \\ x_0+x_1=x_4+x_5=x_6+x_7: & h=f=0,\ g=1. \\ x_0+x_1=x_2+x_3=x_6+x_7: & f=g=0,\ h=1. \\ x_0+x_1=x_2+x_3,\quad x_4+x_5=x_6+x_7: & f=g=1,\ h=0. \\ x_0+x_1=x_4+x_5,\quad x_2+x_3=x_6+x_7: & h=f=1,\ g=0. \\ x_0+x_1=x_6+x_7,\quad x_2+x_3=x_4+x_5: & g=h=1,\ f=0. \end{array}$$

24. This discussion has shown not only that no operation of A has D as its latent space but also that there is one, and only one, operation whose latent space is a given G. The 1575 such operations constitute class XIII; note the entry 24, on this level, under v in Table III of **9**; there are, indeed, 24 p in G. Such operations (over any field) were used by TITS: he names them *glissements* (**12**, 39; v, d there are g, G here), so that we call them *glides*.

The action of a glide is easily described. Every point in G is unmoved. Any point outside G lies in one, and only one, of the tangent primes to $\mathfrak{S}$ at the three m on the polar of G; such a point is transposed with the remaining point on the line joning it to m. Verification, using (23.5) with, say, $f = g = h = 1$, is immediate. Their actions show two glides to be permutable whenever their axes g_1, g_2 intersect; should their plane be d permutability is a consequence of (23.4). Should it be f, and m the intersection of the axes, their polars G_1, G_2 lie in the polar M of m, and the other [5] in M through their intersection F is the polar T of the other line through m in f. The action of both glides on any point W that lies in T but not in F is to transpose it with the other point on mW; hence, when the two glides act successively W is invariant, and that whatever the order of succession. So, since both products leave every point of T invariant, they are both the same as the involution for which T is latent because, as will be seen immediately, there are only four operations in A for which every point of T is invariant; two of these are projections which, belonging to the coset of A^+, cannot either of them be a product of two glides.

25. Table II includes 4 types of involution, and the glides furnish a a fifth. The operations II are the 120 projections $\mathfrak{J}$, already encountered in § 8.

Operations III are, as the partition $1^6 2^2$ shows, products of pairs of commuting $\mathfrak{J}$. The discussion in § 25 of **6** proves that $\mathfrak{J}_1$, $\mathfrak{J}_2$ commute if, and only if, the join of their centres is a t, whose polar T is then the latent space of $\mathfrak{J}_1\mathfrak{J}_2 = \mathfrak{J}_2\mathfrak{J}_1$

Conversely: when T is given its polar contains two p, say p_1 and p_2; the product of those projections centred at p_1 and p_2 is an involution whose latent space is T. There are 3780 such involutions in A. Any operation for which T is latent leaves every point of any of the 15 γ in T invariant, and (23.3) serves to show that T is latent for one involution.

For, taking T to be $x_0 = x_1$, $x_2 = x_3$ so that it contains (23.1), the only matrices (23.3) for which all its points are invariant have

$$c = d = f = g = h = 0;$$

these answer to I, $\mathfrak{J}_1$, $\mathfrak{J}_2$, $\mathfrak{J}_1\mathfrak{J}_2$ (this latter having $a = b = 1$) where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are those projections centred on the polar t of T.

When every pair of a set of projections commute the join of every pair of their centres is a t. Any 3 such centres span an e so that, as each e includes 4 sets of 3 non-collinear p, there are $4 \times 9450 = 37800$ involutions in class V, $1^3 2^3$ being the corresponding partition. The product of 4 projections whose centres are all in the same e is, in fact, a glide. Take, for example, the projections

$$x_0 \longleftrightarrow x_1, \quad x_2 \longleftrightarrow x_3, \quad x_4 \longleftrightarrow x_5; \tag{25.1}$$

their centres

$$(1\ 1\ .\ .\ .\ .\ .\ .),\quad (.\ .\ 1\ 1\ .\ .\ .\ .),\quad (.\ .\ .\ .\ 1\ 1\ .\ .)$$

span

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 = x_6 = x_7 = 0 \tag{25.2}$$

wherein the fourth p is $(1\ 1\ 1\ 1\ 1\ 1\ .\ .)$. The matrix of the projection centred here is

$$\begin{bmatrix} . & 1 & 1 & 1 & 1 & 1 & . & . \\ 1 & . & 1 & 1 & 1 & 1 & . & . \\ 1 & 1 & . & 1 & 1 & 1 & . & . \\ 1 & 1 & 1 & . & 1 & 1 & . & . \\ 1 & 1 & 1 & 1 & . & 1 & . & . \\ 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix}; \tag{25.3}$$

when it is multiplied by the matrices for (25,1) so that its first and second, third and fourth, fifth and sixth rows are transposed the product is (23.5) with

$$f = g = h = 1,$$

the matrix for the glide associated with

$$x_1 + x_1 = x_2 + x_3 = x_4 + x_5.$$

This is the polar of the g in (25.2), and each of the 6 e in any G must give, as the product of the 4 projections centred in any one of them, the same glide.

There are also sets of mutually commuting $\mathfrak{J}$ whose centres are not coplanar but vertices of a tetrahedron; since all 6 edges of such a tetrahedron are t its vertices span a γ. There are, in γ, $8.7.6.4/1.2.3.4 = 56$ tetrahedra whose vertices are all p; they fall into 28 complementary pairs, each pair exhausting, by its vertices, the 8 p in γ. The product of the 8 projections centred in γ (the order of factors, since they commute, being immaterial) is identity. Take, for example, γ to be (23.1) wherein the vertices of one tetrahedron are

$$(1\ 1\ .\ .\ .\ .\ .\ .),\quad (.\ .\ 1\ 1\ .\ .\ .\ .),\quad (.\ .\ .\ .\ 1\ 1\ .\ .),\quad (.\ .\ .\ .\ .\ .\ 1\ 1)$$

and of its complement

$$(.\ .\ 1\ 1\ 1\ 1\ 1\ 1),\quad (1\ 1\ .\ .\ 1\ 1\ 1\ 1),\quad (1\ 1\ 1\ 1\ .\ .\ 1\ 1),\quad (1\ 1\ 1\ 1\ 1\ 1\ .\ .).$$

The matrices of the projections centred at the former have product

$$\begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}. \tag{25.4}$$

The matrices of the projections centred at the latter are (25.3) and its analogues; in fact they are those instances of (23.3) with

$$a=g=h=0,\quad b=c=d=f=1;\quad b=h=f=0,\quad c=d=a=g=1;$$

$$c=f=g=0,\quad d=a=b=h=1;\quad d=0,\quad a=b=c=f=g=h=1;$$

so that (23.4) shows their product too to be (25.4). Thus 28 different products arise from tetrahedra of p in γ, and class VIII consists ef $28\times 2025=56700$ involutions. Notice, incidentally, that while each e in γ provides a glide the two glides so arising from e whose line g of intersection lies in the d in γ are the same, for both these e lie, with γ. in the polar G of this g.

The 8 projections centred in γ all commute with one another, and this is the maximum number that can do so. Since their product is identity they form an elementary abelian 2-group of order 2^7; when γ is (23.1) these 2^7 operations have matrices (23.3) which all, as (23.4) shows, have period 2 except for the unit matrix. They consist of

Identity;

8 projections, one for each p in γ;

28 involutions of class III, one for each t in γ;

56 involutions of class V, four for each e in γ;

28 involutions of class VIII, one for each pair of complementary tetrahedra with vertices among the p in γ;

7 glides, one for each pair of e through the same g in γ.

Since all these projectivities leave every point of γ invariant they form a normal subgroup of the stabiliser of γ in A. The order of the stabiliser is

$$2^{13}\cdot 3^5\cdot 5^2\cdot 7/2025 = 2^{13}\cdot 3\cdot 7\,;$$

its quotient group by the normal subgroup of order 2^7 has order

$$2^6\cdot 3\cdot 7 = 1344.$$

Its members, cosets of the elementary abelian group in the stabiliser, impose a group of 1344 permutations on the 8 p in γ (**4** § 19).

26. Were every point of each of two polar spaces S, s to be invariant under a projectivity every point of [7], since it completes a transversal of S and s, would be. Thus two differnt projectivities leaving every point of S invariant can never impose the same permutation on the 3 points of s. Now the projections $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{I}_3$ whose centres p_1, p_2, p_3 are on s all leave every point of S invariant and, simultaneously, impose the three possible transpositions on the p_i; they therefore, as any two of them generate the $\mathfrak{S}_3$ on these p_i, generate the whole of A_S – the subgroup of operations of A for which every point of S is invariant. The only spaces containing S are its joins P_i to the p_i; these are latent one for each $\mathfrak{I}_i$ and S is latent for the pair of inverse operations of period 3 that are products of pairs of these (non-commutative) $\mathfrak{I}_i$ but not for any other operations of A.

This corresponds to the sole appearance of S in Table II, and as there are 1120 S there are 224) operations in class IV.

27. Were every point of each of two polar space J, j to be invariant under a projectivity so would every point of their join P_0 be; for through any point of P_0 exterior to both J and j there passes a plane meeting them both in lines through their intersection p_0. This p_0 is the kernel of the sections of $\mathfrak{S}$ by both J and j, the latter section consisting of non-collinear points m_1. m_2, m_3. If different operations of A impose the same projectivity in J and the same projectivity in j it can only be because one is a product of the other and the projection $\mathfrak{I}_0$ centred at p_0.

A projectivity in j is determined by its action on any three non-collinear points in j. When a projectivity belongs to A it permutes m_1, m_2, m_3 and the transpositions of pairs of these are imposed by $\mathfrak{I}_1$, $\mathfrak{I}_2$, $\mathfrak{I}_3$ – the projections whose centres p_1, p_2, p_3 lie respectively on m_2m_3, m_3m_1, m_1m_2. These three projections all leave every point of J invariant, and any operation of A that does so must be generated by (any two of) them and $\mathfrak{I}_0$. Thus

A_J is found on adjoining $\mathfrak{J}_0$ to the $\mathfrak{S}_3$ that includes $\mathfrak{J}_1$, $\mathfrak{J}_2$. $\mathfrak{J}_3$; $\mathfrak{J}_0$ commutes with each $\mathfrak{J}_i$ and A_J is a dihedral group D_{12}. The spaces which contain J, namely

$$[7],\quad P_0,\quad P_1,\quad P_2.\quad P_3,\quad T_1,\quad T_2,\quad T_3$$

and S are respectively latent for

$$I,\ \mathfrak{J}_0\ \mathfrak{J}_1,\ \mathfrak{J}_2,\ \mathfrak{J}_3,\ \mathfrak{J}_0\mathfrak{J}_1,\ \mathfrak{J}_0\mathfrak{J}_1,\ \mathfrak{J}_0\mathfrak{J}_3$$

and a pair of inverse operations of period 3. There remain in D_{12} a pair of inverse operations of period 6; they have J for their latent space, and no other operation of A can do so. This accords with the sole appearance of J in Table II, and as there are 40320 J there are 80640 operations in class VI.

28. When one considers H with a view to identifying those operations of A for which it is latent it appears, since H spans an M with its polar h, that no two operations in A can impose the same projectivity in H if they impose the same projectivity in h. Now any operation in A permutes the four s in h, and any projectivity in h is determined when its action on this quadrilateral of s is known. The six possible transpositions of pairs of sides of this quadrilateral are imposed by the projections centred at its six vertices – projections all of which leave every point of H invariant.

Hence A_H is an $\mathfrak{S}_4$. The spaces containing H afford those latent for all operations in this $\mathfrak{S}_4$ save the six of period 4: these, and only these, have H for their latent space, and the $7560\,H$ account thereby for the 45360 operations of class VII.

29. D, as was seen in § 23, does not figure as a latent space, so that the only [4] 's still to be considered are E and F. E is quickly dismissed, but F will be seen to furnish, as did G, operations not represented in Table II.

The set A_E of operations for which every point of some E is invariant is an elementary abelian 2-group - common to the three subgroups A_γ, each elementary abelian and of order 2^7, that are associated one with each γ in E. Take E to be

$$x_0 = x_1,\quad x_2 = x_3,\quad x_4 = x_5$$

so that one γ therein is given by (23.1); (23.3) only leaves every point of E

invariant when $f=g=h$ and $d=0$. The matrices $\begin{pmatrix} a, & b, & c, & 0 \\ & f & f & f \end{pmatrix}$ form an elementary abelian group of order 2^4; the projections therein are centred at the 4 p in the polar e of E-projections, it will be recalled, that mutually commute and whose product is a glide. E is latent for the product of any three of them; indeed the sole appearence of E in Table II corresponds to the partition $1^3 2^3$. Since there are 9450 E there are 37800 members of class V.

30. Does A include operations whose latent space is a given $[4]\, F$? The subgroup A_F of operations of A for which all points of F are invariant includes

identity

two projections $\mathfrak{J}_1$, $\mathfrak{J}_2$ whose latent spaces are the two P through F,

two glides $\mathfrak{G}_1$, $\mathfrak{G}_2$ whose latent spaces are the two G through F, and

the involution $\mathfrak{J}_1\mathfrak{J}_2 = \mathfrak{J}_2\mathfrak{J}_1 = \mathfrak{G}_1\mathfrak{G}_2 = \mathfrak{G}_2\mathfrak{G}_1$ whose latent space is the T through F.

This set, of 6 operations none of which has period 3, cannot be a group; yet no other space through F is latent for any operation of A because neither M nor C can ever be. It is then certain that, despite its non-occurrence in Table II, operations having F latent do occur in A. Indeed the above 6 operations belong to a dihedral group $\mathfrak{D}_8$, and it is the pair of inverse operations of period 4 in $\mathfrak{D}_8$ that have F latent; they belong to class XXIII which, as there are 37800 F, has 75600 members.

Take, as in (14.1), G to be

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 .$$

The plane joining its polar g to p is an f provided that p is conjugate to one of the m on g but to neither of the other two; if, for example, p satisfies

$$x_0 + x_1 = x_2 + x_3 = x_4 + x_5 + 1.$$

So one may take $(1\ .\ 1\ .\ .\ .\ .)$ for p, when F is

$$x_0 = x_2,\quad x_1 = x_3,\quad x_0 + x_1 = x_4 + x_5 . \tag{30.1}$$

The projection centred at p is $x_0 \longleftrightarrow x_2$; when its matrix premultiplies that of the glide having G latent the product is (23.5) with $f=g=h=1$ and its first and third rows transposed. The latent column vectors of this matrix are indeed found to consist of those satisfying (30.1), and of no others. The other projection for which every point of (30.1) is invariant is $x_1 \longleftrightarrow x_3$; the G latent for the other glide is

$$x_0 + x_3 = x_1 + x_2 = x_4 + x_5 .$$

31. When polar spaces u, u' are skew the group of operations of A for which every point of u is invariant is the same as the group of projectivities induced by A in u'; the direct product of the groups so associated one with u and one with u' is a subgroup of A, as it is too when u, u' intersect at m and span M. The simplest instance occurs when, as in § 26, u, u' are S, s. Those operations for which every point of S is invariant compose an $\mathfrak{S}_3$ and impose all 3! permutations on the p on s. Those operations for which every point of s is invariant induce in S the 51840 operations of a "cubic surface" group $\mathfrak{G}$, and A has 1120 subgroups $\mathfrak{S}_3 \times \mathfrak{G}$. Its SYLOW 3-groups are each the direct product of $\mathfrak{S}_3$ and a SYLOW 3-group of $\mathfrak{G}$. One may note here a property of the operations in class IV, verified immediately by using the cyclic permutation $(x_0x_1x_2)$:

If a point lies neither in the latent S nor on its polar s the plane spanned by it and those two points which are its transforms under the operation and its square passes through s and meets S.

32. Take now a polar pair of solids $\varkappa$, $\varkappa'$; they meet $\mathfrak{S}$ in hyperboloids $\mathcal{H}$, $\mathcal{H}'$, each having a group of 72 automorphisms with the direct product $\mathfrak{S}_3 \times \mathfrak{S}_3$ as subgroup of index 2. Those automorphisms in the coset of $\mathfrak{S}_3 \times \mathfrak{S}_3$ transpose the reguli; those in $\mathfrak{S}_3 \times \mathfrak{S}_3$ itself do not, and each factor of this direct product imposes all 3! permutations on the 3 members of the appropriate regulus.

Since $\varkappa$ lies in

$$6P,\ 6G,\ 9T,\ 2S,\ 9F,\ 6J$$

the operations for which all its points are invariant include

Identity, 6 projections, 6 glides, 9 involutions of class III,

4 operations of class IV, 18 of class XXIII, 12 of class VI.

The projections are those centred on the two DANDELIN lines s, s' in $\varkappa'$ and transpose the reguli on $\mathcal{H}'$. For the P whose points are invariant under one of the projections $\mathfrak{J}$ meets $\varkappa'$ in a plane j through, say, s and meets $\mathcal{H}'$ in 3 points which are all invariant and which constitute a conic whose kernel k is the centre of $\mathfrak{J}$ and is the intersection of j with s'. The generators of $\mathcal{H}'$ through any one of these 3 points are transposed. For, were they not transposed, the two other points on either would be, whereas their join does not pass through k. All 36 members of the coset of $\mathfrak{S}_3 \times \mathfrak{S}_3$ are thus accounted for, namely by 6 projections, 18 operations of class XXIII and 12 of class VI.

The operations whose latent space is $\varkappa$ belong therefore to $\mathfrak{S}_3 \times \mathfrak{S}_3$, and there do remain, as yet unrecorded in this direct product,

4 operations of period 3 and 12 of period 6.

The former are those of class X recorded in Table II; there are 268800 such operations in A. The latter belong to a class of 806400 operations and impose permutations of periods 2 and 3 on the respective reguli of $\mathcal{K}'$. Their squares impose the identity on one regulus, a permutation of period 3 on the other. Now operations of class IV cannot do this, since they permute the 9 points of $\mathcal{K}'$ in 3 cycles of 3, the plane of each cycle containing either s or s'; hence the squares of the new operations are in class X. Their cubes impose identity on one regulus and a transposition on the other and so leave invariant, with every point of a g in this other regulus, the whole G joining g to $\varkappa$; they are glides. All these facts indicate class XXXIX.

33. The section of $\mathfrak{G}$ by any λ is an 'ellipsoid" - over $\boldsymbol{F}$ a set of 5 points, no 4 coplanar - whose group of automorphisms is an $\mathfrak{S}_5$. Since λ lies in

$$10P,\ 15T,\ 10S,\ 5H,\ 10J$$

the group of projectivities for which every point of λ is invariant, and which impose automorphisms on the ellipsoid in the polar λ' of λ, includes

identity, 10 projections, 15 involutions of class III and pairs of inverse operations,

10 of period 3, 15 of period 4, 10 of period 6.

These are the exact numbers and periods of operations in $\mathfrak{S}_5$, which is completed by 24 operations of period 5. These are in class XII, which thus has $24 \times 24192 = 580608$ members. And A has 12096 subgroups $\mathfrak{S}_5 \times \mathfrak{S}_5$.

34. Take now a solid χ. Since it lies in

$$12P,\ 1G,\ 18T,\ 16S,\ 3E,\ 12H$$

the operations of A for which every point of χ is invariant include

identity, 12 projections, 1 glide, 18 involutions of class III

and further operations: 32 in class IV, 12 in class V, 72 in class VII.

Of these $1+1+18+32$ are in A^+, the other $12+12+72$ outside A^+; hence there must be $96-52=44$ more operations of A^+ that leave every point of χ, but no further point, invariant. That there are precisely this number, that the subgroup A_χ so associated with χ has order 192, can also be shown by giving explicit matrix forms for them.

Take χ to be spanned by

(34.1) $(1\ .\ .\ .\ .\ .\ .\ .), \quad (.\ 1\ 1\ .\ .\ .\ .\ .), \quad (.\ .\ 1\ 1\ .\ .\ .\ .), (.\ 1\ 1\ 1\ 1\ .\ .\ .);$

a matrix for which these are all latent column vectors must have the form

$$\begin{bmatrix} 1 & a & a & a & a & A_1 & A_2 & A_3 \\ . & b+1 & b & b & b & B_1 & B_2 & B_3 \\ . & b & c+1 & c & c & C_1 & C_2 & C_3 \\ . & d & d & d+1 & d & D_1 & D_2 & D_3 \\ . & e & e & e & e+1 & E_1 & E_2 & E_3 \\ . & f & f & f & f & F_1 & F_2 & F_3 \\ . & g & g & g & g & G_1 & G_2 & G_3 \\ . & h & h & h & h & H_1 & H_2 & H_3 \end{bmatrix}$$

This has to belong to A. By requiring that the sum of any two of the second, third, fourth and fifth columns be conjugate to every other column one finds immediately

$$b=c=d=e, \quad B_i=C_i=D_i=E_i$$

for $i=1, 2, 3$ and then, prohibiting the first column from being conjugate to any other,

$$h=f+g, \quad H_i=F_i+G_i+1.$$

Next, since the columns are coordinate vectors of m,

$$a+b+f+g+fg=0, \quad A_i=F_iG_i,$$

and then, to prevent any of the last three columns being conjugate to the earlier ones,

$$b+B_i=(f+F_i)\ (g+G_i).$$

The matrix is therefore

$$(34.2)\qquad \begin{bmatrix} 1 & a & a & a & a & F_1G_1 & F_2G_2 & F_3G_3 \\ . & b+1 & b & b & b & B_1 & B_2 & B_3 \\ . & b & b+1 & b & b & B_1 & B_2 & B_3 \\ . & b & b & b+1 & b & B_1 & B_2 & B_3 \\ . & b & b & b & b+1 & B_1 & B_2 & B_3 \\ . & f & f & f & f & F_1 & F_2 & F_3 \\ . & g & g & g & g & G_1 & G_2 & G_3 \\ . & f+g & f+g & f+g & f+g & F_1+G_1+1 & F_2+G_2+1 & F_3+G_3+1 \end{bmatrix}$$

wherein

$$a = b + f + g + fg, \quad B_i = b + (f + F_i)\,(g + G_i).$$

The only further conditions necessary are those which prevent any two of the last three columns being conjugate:

$$(34.3)\qquad (F_2 + F_3 + 1)\,(G_2 + G_3 + 1) = (F_3 + F_1 + 1)\,(G_3 + G_1 + 1) =$$
$$= (F_1 + F_2 + 1)\,(G_1 + G_2 + 1) = 0$$

which imply, on addition,

$$(34.4)\qquad \begin{vmatrix} F_1 & F_2 & F_2 \\ G_1 & G_2 & G_3 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

Not only so; (34.4) implies, conversely, all of (34.3). For since the determinant is not zero the first two columns cannot be identical; hence at least one of

$$F_1 + F_2, \quad G_1 + G_2,$$

is 1 and the last equation of (34.3) holds; so, likewise, do the others.

There are 24 choices for F_i, G_i that conform to (34.4); if (F_1, F_2, F_3) and (G_1, G_2, G_3) are regarded as points in the 7-point plane the first can be any of the 6 other than the unit point, the second any of the 4 not on the join of the unit point to the first. And as each of b, f, g can be either 0 or 1 the number of matrices is $2^3\cdot 24 = 192$. This, then, is the order of A_χ.

The glide is solitary among the 192 operations. Since the g in χ joins the first and last of the points (34.1) its polar is

$$x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 = 0.$$

The glide is determined by stipulating that 2 points of this G whose join is skew to χ are invariant; these could be any 2 of those points whose only non-zero coordinates are two of x_5, x_6, x_7, points that are not invariant unless

$$f = g = F_2 = F_3 = G_3 = G_1 = 0, \quad F_1 = G_2 = 1.$$

The resulting matrices include, for $b = 0$, the unit matrix; the glide therefore has $b = 1$. It is found to commute with all 192 matrices, and so is in the centre of A_χ.

This group of order 192, with a centre of order 2, is the one encountered by other writers (2, 230; **13**, 149) and it was, more recently, found as a group of quaternary substitutions over $\boldsymbol{F}$ (**4** § 23). It includes 12 operations of period 4 all having the glide for their square. Any such operation must satisfy, among many other conditions,

$$f(F_1G_1 + F_3G_3) + g(F_2G_2 + F_3G_3) = 1,$$

so that

$$F_1G_1, \quad F_2G_2, \quad F_3G_3$$

cannot all be equal and f, g cannot both be zero. It is found that every requisite condition holds for each of the following sets of 4 matrices (b being free to be either 0 or 1 in each instance).

$$(i) \quad F_1 = G_2 = 0, \quad F_2 = G_1 = F_3 = G_3 = 1 = f + g$$

$$(ii) \quad F_2 = G_2 = F_3 = 0, \quad F_1 = G_1 = G_3 = 1 = f.$$

$$(iii) \quad F_1 = G_1 = G_3 = 0, \quad F_2 = G_2 = F_3 = 1 = g.$$

There is such a set of 12 operations associated with each of the 3150 χ, and the whole aggregate of 37800 constitutes class XV.

These are not the only members of A_χ to have χ for their latent space; the same holds for 32 operations of period 6, each of which has one of the 32 operations of period 3 for its square. Now the square of (34.2) has, for its bottom right-hand corner of elements. the square of the corresponding block in (34.2) for, on partitioning its 8 rows and 8 columns as $5+3$, the additional contribution to this block in the square of (34.2) is seen to be (over $\boldsymbol{F}$) the zero matrix. This fact is relevant because one S through χ is $x_5 = x_6 = x_7$, and those operations for which it is latent have matrices

$$1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{bmatrix} . & 1 & . \\ . & . & 1 \\ 1 & . & . \end{bmatrix}$$

(34.5) and

$$1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus \begin{bmatrix} . & . & 1 \\ 1 & . & . \\ . & 1 & . \end{bmatrix}$$

Any matrices (34.2) of which one of (34.5) is the square may have either

$$F_3 = G_1 = 1, \quad F_1 = F_2 = G_2 = G_3 = 0$$

or

$$F_2 = G_3 = 1, \quad F_1 = F_3 = G_1 = G_2 = 0$$

and it must also happen that $f = g = 0$. Were $b = 0$ too (34.2) would reduce to one of (34.5) which, of course, are squares of each other; but if $b = 1$ the resulting matrix has period 6 and χ for latent space. As each of the 32 operations of class IV has one such square root the whole group of order 192 is exhausted. These 32 operations of period 6 belong to class XIV, with a membership of 100800.

Through χ pass 3 E; these undergo permutations under A_χ and those operations of A_χ for which one of these E is stabilised constitute a SYLOW subgroup of order 64. When χ is spanned by (34.1) the E in question are

$$x_1 + x_2 + x_3 + x_4 = x_5 = 0, \quad x_6 = x_7;$$

$$x_1 + x_2 + x_3 + x_4 = x_6 = 0, \quad x_7 = x_5;$$

$$x_1 + x_2 + x_3 + x_4 = x_7 = 0, \quad x_5 = x_6.$$

The corresponding stabilisers occur for

(a) $F_1 + 1 = F_2 = F_3, \quad G_2 + G_3 = 1;$

(b) $G_1 = G_2 + 1 = G_3, \quad F_1 + F_3 = 1;$

(c) $F_1 = F_2 + 1, \quad G_1 = G_2 + 1, \quad F_3 + G_3 = F_1 + G_1 + 1.$

They intersect in the group, of order 32, given by

$$F_1 + 1 = F_2 = F_3, \quad G_1 = G_2 + 1 = G_3;$$

this includes those 12 operations of period 4 for which χ is latent, and it is a normal subgroup of A_χ.

35. In order to find operations whose latent space is a solid ω it is more expeditious to use (10.4) and find those automorphisms of

$$Y_1Z_1 + Y_2Z_2 + Y_3Z_3 + Y_4Z_4$$

for which every point of

$$Y_1 = Y_2 = Y_3 = Y_4 = 0$$

is invariant. They must, supposing the coordinates ranged in the order

$$Y_1, \ Y_2, \ Y_3, \ Y_4; \ Z_1, \ Z_2, \ Z_3, \ Z_4,$$

have matrices

$$\begin{bmatrix} A & \cdot \\ B & I \end{bmatrix}$$

where A, B, I are square with 4 rows; but the quadratic form is unaltered only when

$$A = I, \quad B = \begin{bmatrix} \cdot & c & b & a' \\ c & \cdot & a & b' \\ b & a & \cdot & c' \\ a' & b' & c' & \cdot \end{bmatrix}$$

Since

$$\begin{bmatrix} I & \cdot \\ B_1 & I \end{bmatrix} \begin{bmatrix} I & \cdot \\ B_2 & I \end{bmatrix} = \begin{bmatrix} I & \cdot \\ B_1 + B_2 & I \end{bmatrix}$$

every such operation that is not itself identity has period 2; A_ω is an elementary abelian 2-group whic , since each of a, b, c, a', b', c' can be 0 or 1, has order 64. It includes 35 glides, one for each G through ω; there remain 28 involutions for which ω is the latent space and so, with 28 for each of 270 ω, one finds the 7560 involutions of class LXI.

The rank of

$$\begin{bmatrix} I & . \\ B & I \end{bmatrix} + \begin{bmatrix} I & . \\ . & I \end{bmatrix}$$

is the rank of B, and so is 4 or 2 according as $aa' + bb' + cc'$ is 1 or 0; the invariant points fill a space of 5 or 3 dimensions to correspond, and the numbers, 35 and 28, are as they should be (**4**, 325).

36. The canonical form (10.4) is the more expeditious also in yielding information about operations having φ for their latent space. For

$$Y_1 = Y_2 = Y_3 = Z_3 = 0$$

is such a solid, and the general form of a matrix which leaves every point therein invariant can be written down; four of its columns are empty save for units on the main diagonal. But (10.4) is not automorphic under the resulting projectivity unless the matrix has one of the two following aspects, in both of which $e + f = ad + bc$:

$$\begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ a & b & . & . & . & . & 1 & . \\ . & . & . & 1 & . & . & . & . \\ ac & e & a & . & 1 & . & c & . \\ f & bd & b & . & . & 1 & d & . \\ c & d & 1 & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix} \equiv \mu_1, \qquad \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . \\ a & b & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ ac & e & c & . & 1 & . & a & . \\ f & bd & d & . & . & 1 & b & . \\ c & d & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{bmatrix} \equiv \mu_2.$$

Now the rank of $\mu_1 + I$ cannot, because of the restriction $e + f = ad + bc$, exceed 3; hence all 2^5 matrices μ_1 have their latent space of dimension at least 4; this space includes φ, but φ is not the whole of it. The rank of $\mu_2 + I$, on the other hand, is 4 unless $ad + bc = 0$. Hence if $ad + bc = 1$ the latent space of μ_2 has dimension 3 and so is φ itself. The number of solutions a, b, c, d of $ad + bc = 1$ is 6 (recall that there are 6 points not lying on a hyperboloid) and since there remains a choice of e and f subject to $e + f = 1$ there are 12 operations for which φ is latent.

The square of μ_2 depends only on $ad + bc$; if $ad + bc = 0$, $\mu_2^2 = I$; but if

$$ad + bc = 1,$$

μ_2 has period 4 and μ_2^2 imposes the transformation

$$Z_1 \longleftrightarrow Z_1 + Y_2, \quad Z_2 \longleftrightarrow Z_2 + Y_1,$$

all other coordinates being unchanged. Thus all 12 operations for which φ is latent have the same square, namely the glide associated with the polar, here $Y_1 = Y_2 = 0$, of the line of intersection of the two planes in which φ meets $\mathfrak{S}$. As there are 28350 φ there is a class of 340200 conjugate operations in A; it is numbered XXXVI in **9.**

37. The only type of solid yet to be considered is ψ; through it pass

$$8P;\ 3G,\ 16T,\ 4S;\ 3E,\ 1H,\ 3F,\ 8J$$

that are latent for operations of A. These account for

$$8 + 12 + 6 + 6 + 16 = 48$$

members of the coset of A^+, but only for

$$1 + 3 + 16 + 8 = 28$$

members of A^+ itself (identity included). Hence ψ is latent for 20 members of A^+ and A_ψ has order 96.

If ψ, ψ' are polar spaces $A_\psi \times A_{\psi'}$ is a subgroup of A, and there are 37800 such subgroups. ψ' is invariant as a whole under A_ψ, and each of the $3g$ therein is invariant for 32 operations of A_ψ; these are its SYLOW 2-groups and their intersection, for which all g in ψ' are invariant, is normal in A_ψ, of order 16, and will be seen to be elementary abelian.

In ψ' are three concurrent g:

$$g_1 \equiv mm_1m_1', \quad g_2 \equiv mm_2m_2', \quad g_3 \equiv mm_3m_3';$$

each pair spans a plane wherein a third line passes through m, these lines being

$$t_1 \equiv mp_1p_1', \quad t_2 \equiv mp_2p_2', \quad t_3 \equiv mp_3p_3'$$

where t_i is coplanar with the two g other than g_i. The seventh line through m in ψ' is $t \equiv mp_0p_0'$. Denote by $\mathfrak{J}_i$, $\mathfrak{J}_i'$ the projections centred at p_i, p_i'; any two $\mathfrak{J}$ with the same suffix commute, and both $\mathfrak{J}_0$ and $\mathfrak{J}_0'$ commute with all the others. Then every g_i is invariant under

$$(37.1)\qquad \left\{\begin{array}{cccc} \mathfrak{J}_0\mathfrak{J}'_0 & \mathfrak{J}_1\mathfrak{J}'_1 & \mathfrak{J}_2\mathfrak{J}'_2 & \mathfrak{J}_3\mathfrak{J}'_3 \\ I & \mathfrak{J}_0\mathfrak{J}'_0\mathfrak{J}_1\mathfrak{J}'_1 & \mathfrak{J}_0\mathfrak{J}'_0\mathfrak{J}_2\mathfrak{J}'_2 & \mathfrak{J}_0\mathfrak{J}'_0\mathfrak{J}_3\mathfrak{J}'_3 \end{array}\right.$$

as well as under

$$(37.2)\qquad \left\{\begin{array}{cccc} \mathfrak{J}_0 & \mathfrak{J}_0\mathfrak{J}_1\mathfrak{J}'_1 & \mathfrak{J}_0\mathfrak{J}_2\mathfrak{J}'_2 & \mathfrak{J}_0\mathfrak{J}_3\mathfrak{J}'_3 \\ \mathfrak{J}'_0 & \mathfrak{J}'_0\mathfrak{J}_1\mathfrak{J}'_1 & \mathfrak{J}'_0\mathfrak{J}_2\mathfrak{J}'_2 & \mathfrak{J}'_0\mathfrak{J}_3\mathfrak{J}'_3 \end{array}\right.$$

in all 16 operations and all, save I, of period 2. They exhaust those members of A_ψ for which g_1, g_2, g_3 are all invariant and form an elementary abelian 2-group, normal in A_ψ: (37.1) belong to A^+, (37.2) to the coset. Each S_2 of A_ψ consists of (37.1), (37.2) and 16 further operations of which 8 are in A^+; the aggregate of 32 operations composed of 8 in each S_2^+ supplementing (37.1) does not, since each of its constituents has a power of 2 for its period, include any of those 8, among the 28 operations listed above, of period 3. It then includes $32 - 20 = 12$ new operations, and these belong to class XI (see Table II) which therefore has 907200 members. There now remain $20 - 12 = 8$ further operations for which ψ is latent; they belong to class IX which has 604800 members.

The $\mathfrak{S}_4$ of operations for which every point of the H through ψ is invariant is, since H is unique, a normal subgroup of A_ψ. It intersects the elementary abelian normal subgroup in the 4 group, consisting of identity and three glides, that occupies the lower row in (37.1).

38. The detection, of operations with a given latent space σ, by cataloguing operations for which every point of σ is invariant and discarding those, all known from antecedent enquiries, whose latent spaces strictly include σ is now a matter of routine. The operations in question belong wholly to A^+ or to its coset according as the dimension of σ is odd or even – and that whether or not σ figures in the penultimate column of Table II. If σ has even dimension the discarded operations account for the whole intersection A^+_σ of A^+ with A_σ, and the order of A_σ is double the number of discards from A^+. The other discards are in the coset of A^+ and they fall short of half the order of A_σ by the number of operations for which σ is latent. Some, possibly all, of this deficiency may be made up by entries in Table II; when it is not all so made up the residue consists of operations in classes not encountered heretofore.

Take for σ a plane d. Operations of A^+ for which its points are all invariant have latent spaces of odd dimension that contain d; these are, in addition to the whole [7],

$$63G,\quad 28T;\ 2\omega,\ 28\varphi,\ 1\gamma$$

so that the number of operations of A^+ for which every point of d is invariant is

$$1+63+28+56+336+28=512$$

and A_d has order 1024. But the spaces of even dimension containing d are

$$8P;\ 14E,\ 112F$$

which yield only $8+56+224=288$ operations in the coset. Hence d is the latent space of $512-288=224$ operations. As there are 2025 d one thus accounts for 453600 operations; their period is a power of 2, they are in the coset, have not been previously encountered, and are devoid of invariant p; they constitute class LIII.

A_d permutes the 8 p in the γ through d. Those 128 operations (see § 23) for which every point of γ is invariant form a normal subgroup of A_d, the quotient group of order 8 being elementary abelian (**4**, 336).

39. The analogous facts for e are that it lies in

$$19G,\ 72T,\ 16S;\ 3\gamma,\ 3\varphi,\ 1\chi,\ 24\psi$$

so that all its points are invariant for

$$1+19+72+32+84+36+44+480=768$$

operations of A^+; whereas, lying as it does in

$$16P;\ 40E,\ 12H,\ 36F,\ 64J$$

there occur only

$$16+160+72+72+128=448$$

operations of the coset. Hence there are 320 operations, all of them in the coset, having e for their latent space. They include, according to Table II, members of classes XVI and XVIII.

Class XVIII consists, says the Table, of operations that permute the vertices of some ennead according to the partition $12^2 4$. It the ennead is $\mathscr{E}_0$

one such operation leaves U unmoved and imposes the permutation

$$(X_0X_1)\ (X_2X_3)\ (X_4X_5X_6X_7);$$

its latent e is

$$x_0 = x_1,\quad x_2 = x_3,\quad x_4 = x_5 = x_6 = x_7,$$

and this same e is latent for six permutations, any cyclic permutation of

$$X_4,\ X_5,\ X_6,\ X_7$$

being permissible. To span e take

the third point on X_0X_1,

the third point on X_2X_3,

the point (cf. § 15) supplementary to X_4, X_5, X_6, X_7.

Suppose then that a plane e is given, and let $m\mu\mu'$ be the g therein. Let $\mathcal{E}$ be any of the 64 enneads of which m is a vertex; its other vertices are automatically separated (see § 15) into tetrads τ, τ' supplemented respectively by μ, μ'. There is a second ennead $\mathcal{E}_1$, sharing m but no other vertex with $\mathcal{E}$, whose other vertices are likewise separated into tetrads τ_1, τ_1' supplemented by μ, μ'. If $\mathcal{E}$ is $\mathcal{E}_0$ and m is U then $\mathcal{E}_1$ could include (15.2).

Through μ pass transversals t, one to each pair of opposite edges of τ; since no 5 vertices of $\mathcal{E}$ can lie in a solid g cannot lie in the plane of any two of these t. Through μ' pass transversals t' one to each pair of opposite edges of τ'; as the solids spanned by τ and τ' are skew it is not possible for any t to intersect any t', Thus the six lines t, t' are joined to g by the six e which contain g. Here e is among the data; suppose it then to contain a transversal t. Transpose the pairs of vertices of τ on those of its edges that meet t, and impose any cyclic permutation on τ'; the consequent permutation of vertices of $\mathcal{E}$ is of type 12^24 and the resulting projectivity has e for its latent space. Since there are, on g, 3 choices for m and then 2 for μ, and since 6 cyclic permutations of τ' are available, there are 36 operations in A which leave $\mathcal{E}$ invariant and have e for their latent space; the number of operations in class XVIII for which e is latent is therefore 36ν where ν is the number of enneads, including $\mathcal{E}$ itself, into which $\mathcal{E}$ can be transformed by these operations.

One finds $\nu = 4$ by taking, again, $\mathcal{E}$ to be $\mathcal{E}_0$, m to be U; the other enneads into which $\mathcal{E}_0$ can be transformed are $\mathcal{E}^1$, defined by sharing U, X_0, X_1 with $\mathcal{E}_0$, $\mathcal{E}''$, defined by sharing U, X_2, X_3 with $\mathcal{E}_0$ and $\mathcal{E}_1$, sharing only U

with $\mathcal{E}_0$ and given by (15.2). Matrices of corresponding projectivities are

$$\begin{bmatrix} . & 1 & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ . & . & . & 1 & 1 & 1 & 1 & 1 \\ . & . & 1 & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 1 & 1 & . & 1 & 1 \\ . & . & 1 & 1 & 1 & 1 & . & 1 \\ . & . & 1 & 1 & 1 & 1 & 1 & . \\ . & . & 1 & 1 & . & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} . & 1 & . & . & 1 & 1 & 1 & 1 \\ 1 & . & . & . & 1 & 1 & 1 & 1 \\ . & . & . & 1 & 1 & 1 & 1 & 1 \\ . & . & 1 & . & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & . & . & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . & . & . \\ 1 & 1 & 1 & 1 & . & 1 & . & . \\ 1 & 1 & 1 & 1 & . & . & 1 & . \end{bmatrix}$$

Since each of the 9450 e is latent for 144 operations of class XVIII this class has 1360800 members.

Class XVI consists of operations permuting the vertices of some ennead according to the partition $2^3 3$; for example, all those projectivities belong to it which impose the permutation

$$(UX_0X_1)(X_2X_3)(X_4X_5)(X_6X_7)$$

on the vertices of $\mathcal{E}_0$, and the latent e is spanned by the p which lie one on each of

$$X_2X_3, \quad X_4X_5, \quad X_6X_7.$$

The number of $\mathcal{E}$ from which a given e so arises, transversal to 3 mutually skew edges, is

$$(960.36.21.10)/(9450.3!) = 128;$$

the remaining p in e is the kernel of the conic constituted by the trio of cyclically permuted vertices of $\mathcal{E}$. Now each such trio is shared by two of the 128 $\mathcal{E}$, which thus fall into 64 pairs; for instance, U, X_0, X_1 are shared by $\mathcal{E}_0$ with $\mathcal{E}'$, the ennead including (15.1); the p on

$$X_2X_3, \quad X_4X_5, \quad X_6X_7$$

also lie one on each of three joins of pairs of vertices of $\mathcal{E}'$, and the projectivity that imposes

$$(UX_0X_1)(X_2X_3)(X_4X_5)(X_6X_7)$$

imposes a permutation of this type too on the vertices of $\mathcal{E}'$. Since U, X_0, X_1

admit two cyclic permutations there are 128 members of class XVI having for their latent plane

$$x_0 = x_1 = 0, \quad x_2 = x_3, \quad x_4 = x_5, \quad x_6 = x_7$$

and so the class has 1209600 members.

As each e is latent for 128 operations of class XVI and 144 of class XVIII there remain, to complete A_e, $320-128-144=48$ new operations. They belong to class XXI.

Since there is a unique χ containing e the corresponding group A_χ, of order 192, is a normal subgroup of A_e, of order 1536. The glide in the centre of A_χ is also in the centre of A_e. It can be shown that all 48 operations of class XXI in A_e have this glide for their square.

40. Through a given f pass

$$33G,\ 42T,\ 8S;\ 9\varphi,\ 6\psi,\ 16\varkappa$$

so that all its points are invariant for

$$1+33+42+16+108+120+256=576$$

operations of A^+; there pass also through f

$$12P;\ 9E,\ 2H,\ 90F,\ 48J$$

yielding

$$12+36+12+180+96=336$$

operations of the coset. Hence f is the latent space for 240 operations, and these, according to Table II, include members of class XIX. Members of this class permute the vertices of some ennead according to the partition $1^2 34$; for instance, the projectivity determined by the permutation

$$(U)(X_0)(X_1X_2X_3)(X_4X_5X_6X_7) \tag{40.1}$$

of the vertices of $\mathcal{E}_0$ has for its latent plane

$$x_1 = x_2 = x_3, \quad x_4 = x_5 = x_6 = x_7; \tag{40.2}$$

and, there being two cyclic permutations of X_1, X_2, X_3 and six of X_4, X_5, X_6, X_7, the same latent f occurs for 12 operations in the stabiliser of $\mathcal{E}_0$.

The g in f join the supplement μ of X_4, X_5, X_6, X_7 to U and X_0 respectively; the kernel k of $X_1X_2X_3$ is that p in f which does not lie on UX_0. The 8 $\mathcal{E}$ that share both U and X_0 with $\mathcal{E}_0$ each yield the same f, with the same μ and k; note, as instances, those $\mathcal{E}$ whose other vertices than U have for their coordinate vectors the columns (so ordered intentionally) of

$$\begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . \\ . & . & . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & . & . & 1 & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & 1 & . \\ . & 1 & . & 1 & . & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & . & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & . & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & . & . & . & . & . & . & . \\ . & 1 & 1 & 1 & 1 & . & . & 1 \\ . & 1 & 1 & 1 & 1 & . & 1 & . \\ . & 1 & 1 & 1 & . & . & 1 & 1 \\ . & . & . & . & . & 1 & . & . \\ . & 1 & . & 1 & 1 & . & 1 & 1 \\ . & 1 & 1 & . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & 1 & . & 1 & 1 \end{bmatrix} \tag{40.3}$$

Under the projectivity that imposes (40.1) these other $\mathcal{E}$ move, with vertices of $\mathcal{E}_0$, in cycles of 3 and 4; none of them is stable. But the stabiliser of each furnishes, as does that of $\mathcal{E}_0$, 12 operations of class XIX for which (40.2) is latent; hence there are certainly 96 distinct members of this class having (40.2) for their latent f, and so 3628800 members in the whole class.

There are $36.35f$ obtainable from any $\mathcal{E}$ by taking two of its vertices and the kernel of the conic constituted by three others; the number of $\mathcal{E}$ from which a given f so arises is therefore

$$960.36.35/37800 = 32.$$

It was seen above that (40.2) so arises from those 8 $\mathcal{E}$ which include both U and X_0; each f contains 4 c, each of which is associated in this way with 8 $\mathcal{E}$, and so all 32 $\mathcal{E}$ are accounted for.

The matrix on the left in (40.3) is a representative of class XIX; it has period 12, its cube being a permutation matrix of period 4. But the matrix on the right, which also imposes a projectivity having (40.2) for its latent space, does not stabilise any $\mathcal{E}$; it has period 8 and so represents one of the $240 - 96 = 144$ operations still outstanding. These can only belong, in MISS HAMILL'S enumeration, to class LV. Because only the analogues of p occur as points in her geometry only the $2p$ on the t in f will be recorded in the penultimate column, and labelled e_1 in the final column, of Table III in **9**; since these operations are in the coset of A^+ LV is the only class eligible. The 144 operations so associated with each of the 37800 f exhaust the 5443200 members of the class. The fourth power of the matrix on the right in (40.3) imposes the glide whose latent G is

$$x_2 + x_3 = x_4 + x_7 = x_5 + x_6,$$

and indeed this same glide is the fourth power of all those 144 operations in class LV for which (40.2) is the latent plane.

41. Through a given j pass

$$15G,\quad 60T,\quad 20S;\quad 15\psi,\quad 10\varkappa,\quad 6\lambda$$

affording

$$1 + 15 + 60 + 40 + 300 + 160 + 144 = 720$$

operations of A^+ for which every point of j is invariant. There are, too,

$$16P;\quad 15E,\ 15H,\ 45F,\ 80J$$

providing

$$16 + 60 + 90 + 90 + 160 = 416$$

operations in the coset of A^+. A_j is of order 1440 and j is latent for $720 - 416 = 304$ operations. These, in fact, are all accounted for by the classes XVII, XX and XXII which Table II registers as having latent j.

The number of members in any of these classes can be obtained by finding the order of the normaliser. As this process has already been used elsewhere (**6** § 21; 7 §§ 15 — 17) it will be sufficient to apply it here to one of the classes, say to XX, and give the corresponding results for XVII and XXII without detailed substantiation. Members of class XX impose a permutation of type $1^2 2 5$ on the vertices of some $\mathcal{E}$, so one seeks the order of the normaliser in A of

$$(41.1)\qquad 1 \oplus \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix} \oplus \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

The general form of any matrix that commutes with (41.1) is

$$\begin{bmatrix} a & a' & a' & A & A & A & A & A \\ b & b' & c' & B & B & B & B & B \\ b & c' & b' & B & B & B & B & B \\ d & d' & d' & C & D & E & F & G \\ d & d' & d' & G & C & D & E & F \\ d & d' & d' & F & G & C & D & E \\ d & d' & d' & E & F & G & C & D \\ d & d' & d' & D & E & F & G & C \end{bmatrix}$$

and one now has to impose those conditions necessary for it to belong to A. First, to forestall singularity,

$$c' = b' + 1;$$

the second and third columns cannot then be coordinate vectors of points on $\mathfrak{S}$ unless

$$a' = d' = 0,$$

nor can the first column unless $b = ad$. The same condition, when imposed on any of the last five columns, gives

$$Ae_1 + B + e_2 = 0$$

where

$$e_1 = C + D + E + F + G,$$

$$e_2 = CD + CE + CF + CG + DE + DF + DG + EF + EG + FG.$$

The prohibition that no two of these last five columns be conjugate provides the restrictions

$$e_1 + CD + GC + FG + EF + DE = 1,$$

$$e_1 + CE + GD + FC + EG + DF = 1,$$

which combine to give $e_2 = 0$ — a condition also obtainable by noting that the sum of any four of the last five columns has to be the coordinate vector of an m. Further, as none of these last columns is permitted to be conjugate to either of the first two,

$$A + B + e_1 = 1 = ae_1 + dA.$$

The matrices of the normaliser therefore have the form

$$\begin{bmatrix} a & . & . & A & A & A & A & A \\ ad & b' & b'+1 & Ae_1 & Ae_1 & Ae_1 & Ae_1 & Ae_1 \\ ad & b'+1 & b' & Ae_1 & Ae_1 & Ae_1 & Ae_1 & Ae_1 \\ d & . & . & C & D & E & F & G \\ d & . & . & G & C & D & E & F \\ d & . & . & F & G & C & D & E \\ d & . & . & E & F & G & C & D \\ d & . & . & D & E & F & G & C \end{bmatrix}$$

with $e_2 = 0$. It is not possible for all of C, D, E, F, G to be 1 if the matrix is not singular; hence, as $e_2 = 0$, the number which are 1 is either 4 or 1 and the discussion bifurcates accordingly. In the former case $e_1 = 0$; hence $d = A = 1$ while a, b' stay free; there are 20 matrices. In the latter case $e_1 = 1$; hence $a = 1 + dA$ while A, b', d stay free; there are 40 matrices. The normaliser of (41.1) is of order 60, and the conjugate class in A to which (41.1) belongs has

$$960.9!/60 = 16.9! = 5806080$$

members. Each of the $40320j$ is latent for 144.

So much for class XX. There are also 40 members of class XVII and 120 of class XXII for which j is latent.

42. The spaces of odd dimension that pass through a plane h are, in addition to the whole [7],

$$5G,\ 70T,\ 40S;\ 5\chi,\ 10\psi,\ 16\lambda$$

so that every point of h is invariant for

$$1 + 5 + 70 + 80 + 220 + 200 + 384 = 960$$

operations in A^+. There are also, containing h,

$$20P;\ 15E,\ 50H,\ 10F,\ 80J$$

yielding

$$20 + 60 + 300 + 20 + 160 = 560$$

operations of the coset: A_h is of order 1920 and h is latent for 400 operations. These belong. since h never occurs in Table II, to classes not yet encountered.

There are, passing through the single m in an h, 5 g lying in the polar H; the join of each of these g to h is a χ. If μ has h latent and transposes the two points on g outside h μ^2 leaves every point of the corresponding χ invariant and, indeed, μ can be such that χ is the latent space of μ^2. Take, so that χ is spanned by (34.1), h to be

$$x_1 + x_2 + x_3 = x_4 = x_5 = x_6 = x_7 = 0 \tag{42.1}$$

and g to join

$$(.\ 1\ 1\ 1\ 1\ .\ .\ .) \text{ to } (1\ 1\ 1\ 1\ 1\ .\ .\ .).$$

The conditions that these latter two points be transposed, that every point satisfying (42.1) be invariant, and that μ belong to A impose restrictions on the first 5 columns of μ. These, coordinate veetors of 5 m in the same $\mathcal{E}$, are flanked by others, coordinate vectors of 3 of the 4 remaining members of $\mathcal{E}$; but these latter must be chosen and ordered so that the latent space of μ is h itself, not a space of larger dimension containing h. One instance is

$$\mu^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & . & 1 & . & 1 \\ . & 1 & . & . & . & . & 1 & . \\ . & . & 1 & . & . & . & 1 & . \\ . & . & . & 1 & . & . & 1 & . \\ . & 1 & 1 & 1 & . & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & . & 1 \\ . & . & . & . & . & . & 1 & 1 \\ . & . & . & . & . & 1 & 1 & . \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & . & . \\ . & 1 & . & . & . & . & . & 1 \\ . & . & 1 & . & . & . & . & 1 \\ . & . & . & 1 & . & . & . & 1 \\ . & . & . & . & 1 & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & . & . \\ . & . & . & . & . & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & 1 & . \end{bmatrix}$$

The matrix on the right is one of 12 found in § 34 to belong to class XV (it occurs under the second of three headings, with

$$F_2 = G_2 = F_3 = 0, \quad F_1 = G_1 = G_3 = 1 = f),$$

it has period 4 and its latent χ is (34.1). The matrix on the left, which is at once seen to have (42.1) for its latent plane, has therefore period 8. It permutes the remaining 4 of the 5 g in one cycle, and the m (apart from X_0) thereon in a cycle of 8.

Since each χ includes 12 h, and since an operation of period 8 has the same square as does its fifth power, each operation in class XV is the square of 24 of the new operations which therefore fill a class, XXV, of 1814400 members. Each of the 7560 h is latent for 240 of them.

The 5 χ which contain a given h join it to those 5 g that pass through the m in h and lie in its polar H. The 10 ψ in H are spanned each by 3 of these 5 concurrent g, and their polars are the 10 ψ containing h. An operation μ having h latent and transposing the two points outside h on a g in one of these latter has for its square an operation leaving every point of ψ invariant, and enquiry shows there to be μ such that ψ is the latent space of μ^2. When h is (42.1) one ψ containing it is

$$x_1 + x_2 + x_3 = x_4 + x_5 = x_6 = x_7 = 0. \tag{42.2}$$

One therefore demands that μ leaves every point satisfying (42.1) invariant, transposes through premultiplication the columns

$$(\,.\;.\;1\;1\;1\;1\;.\;.\,)' \text{ and } (\,1\;.\;1\;1\;1\;1\;.\;.\,)'$$

and does not leave invariant any point not satisfying (42.1). A general form for the first six columns of μ can be written down at once, and one then imposes the conditions for μ to belong to A. The details intervening may be suppressed; among the eligible matrices is

$$\mu = \begin{bmatrix} 1 & 1 & 1 & 1 & . & 1 & . & 1 \\ . & . & 1 & 1 & 1 & 1 & 1 & . \\ . & 1 & . & 1 & 1 & 1 & 1 & . \\ . & 1 & 1 & . & 1 & 1 & 1 & . \\ . & 1 & 1 & 1 & 1 & . & 1 & . \\ . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & 1 \end{bmatrix}$$

with (42.1) for its latent space. The latent space of μ^2 is (42.2), that of μ^3 is the F

$$x_1 + x_2 + x_3 = x_4 = x_5 = x_6 = x_7;$$

μ has period 12 and μ^2, of period 6, is (§ 37) one of the 604800 members of class IX. Since μ^2 is also the square of μ^7, and ψ includes only a single h, there are 1209600 members in this new class XXIV.

43. The spaces of even dimension that contain a given c include

$$28P;\ 105E,\ 70H,\ 630F,\ 560J;\ 28j,\ 35f$$

and operations of A for which such spaces are latent afford

$$28+420+420+1260+1120+8512+8400=20160$$

for which every point of c is invariant. On the other hand, in correspondence with the whole [7] and the

$$105G,\ 210T,\ 56S;\ 105\varphi,\ 210\psi,\ 280\varkappa,\ 56\lambda$$

that contain c, there are only

$$1+105+210+112+1260+4200+4480+1344=11712 \text{ members}$$

of A^+ that leave every point of c invariant. Thus c is latent for 8448 members of A^+.

The column of latent spaces in Table II enters c for those classes answering to 1^27 and 135; these classes are representable by 8-rowed permutation matrices whose normalisers, in the whole linear group, have the respective forms

$$(43.1)\qquad \begin{bmatrix} A & q & q & q & q & q & q & q \\ p & a & b & c & d & e & f & g \\ p & g & a & b & c & d & e & f \\ p & f & g & a & b & c & d & e \\ p & e & f & g & a & b & c & d \\ p & d & e & f & g & a & b & c \\ p & c & d & e & f & g & a & b \\ p & b & c & d & e & f & g & a \end{bmatrix}, \qquad \begin{bmatrix} A & B & C & q & q & q & q & q \\ C & A & B & q & q & q & q & q \\ B & C & A & q & q & q & q & q \\ p & p & p & a & b & c & d & e \\ p & p & p & e & a & b & c & d \\ p & p & p & d & e & a & b & c \\ p & p & p & c & d & e & a & b \\ p & p & p & b & c & d & e & a \end{bmatrix}.$$

When the conditions are imposed which ensure that these matrices belong to A it is found that, so far as the left-hand one is concerned, $A=1$, $q=0$ and only one of a, b, c, d, e, f, g, is non-zero. As there are 7 choices for this non-zero mark, and as p may be either 0 or 1, the normaliser in A is of order 14 and there is a conjugate class of

$$960.9!/14=24883200 \text{ members.}$$

This is class XXXIV, and each of the 4320 c is latent for 5760 of its operations. Then, taking the matrix on the right, it is found that $p=0$ while two of A, B, C are zero; and, further, that only one of a, b, c, d, e is non-zero. As there are 5 choices for this non-zero mark and 3 for that among A, B, C, and as q may be either 0 or 1, the normaliser in $\mathcal{A}$ is of order 30 and there is a conjugate class of

$$32\cdot 9! = 11612160 \text{ members.}$$

This is class XXX, and each of the 4320 c is latent for 2688 of its operations. All 8448 operations for which c is latent are thus accounted for.

The normalisers (43.1) anticipate representatives of two classes that have a single point latent. For while the permutation matrix chosen above as representative of class XXXIV leaves X_0 and U invariant its normaliser N includes, with $p=1$, operations of $\mathcal{A}$ that transpose X_0 and U. Indeed $N \cong \mathcal{C}_{14}$ and those 6 of its operations that have period 14, having themselves N for their normaliser, belong to a class XLV with the same number of members as XXXIV; it occurs in Table II. Again: the permutation matrix chosen above as representative of class XXX leaves U and $(\ldots 1\ 1\ 1\ 1\ 1)$ invariant, but its normaliser N' includes, with $q=1$, operations of $\mathcal{A}$ that transpose them. In fact $N' \cong \mathcal{C}_{30}$ and those 8 of its operations that have period 30, having themselves N' for their normaliser, belong to a class XLIII with the same number of members as XXX.

44. Those operations of $\mathcal{A}$ for which every point on s is invariant form a cubic surface group of order 51840; its subgroup of order 25920 consists of those operations whose latent spaces contain s and have odd dimension. Such spaces, apart from s itself, include the whole [7] together with

$$45G,\ 270T,\ 120S;\ 45\chi,\ 270\psi,\ 120\varkappa,\ 216\lambda;$$

they account for

$$1+45+270+240+1980+5400+1920+5184 = 15040$$

operations, leaving 10880 for which s is the latent space. These induce in the polar S of s operations of a cubic surface group that are devoid of invariant points; but one wishes to describe their effect on the whole [7].

Table II records that s is latent for permutations, according to the partition 3^3, of the vertices of any of those 240 $\mathcal{E}$ of which s is an axis (cf. § 16). Each triad of vertices of $\mathcal{E}$ whose kernel is on s can undergo two cyclic permutations so that there are, if η is the number of $\mathcal{E}$ which not only share s as an axis but whose vertices are permuted in 3 triads, with kernels on s, under some given operation of A. $240.8/\eta$ members of class XXXVIII for which s is latent. In order to find η take, as permuting the vertices of $\mathcal{E}_0$ in 3 cycles of 3,

$$\mu = \begin{bmatrix} 1 & 1 & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . \\ 1 & . & . & . & 1 & . & . & . \\ 1 & . & 1 & . & . & . & . & . \\ 1 & . & . & 1 & . & . & . & . \\ 1 & . & . & . & . & . & . & 1 \\ 1 & . & . & . & . & 1 & . & . \\ 1 & . & . & . & . & . & 1 & . \end{bmatrix}; \tag{44.1}$$

the kernels are on

$$x_0 = x_1 = 0, \quad x_2 = x_3 = x_4\,, \quad x_5 = x_6 = x_7 \tag{44.2}$$

and each point of [7] traces a cycle

$$\begin{array}{llllr} x_0 \to & x_0 + x_1 \to & x_1 & & . \\ x_1 & x_0 & x_0 + x_1 & \text{with centroid} & . \\ x_2 & x_0 + x_4 & x_1 + x_3 & & x_0 + x_1 + x_2 + x_3 + x_4 \\ x_3 & x_0 + x_2 & x_1 + x_4 & & x_0 + x_1 + x_2 + x_3 + x_4 \\ x_4 & x_0 + x_3 & x_1 + x_2 & & x_0 + x_1 + x_2 + x_3 + x_4 \\ x_5 & x_0 + x_7 & x_1 + x_6 & & x_0 + x_1 + x_5 + x_6 + x_7 \\ x_6 & x_0 + x_5 & x_1 + x_7 & & x_0 + x_1 + x_5 + x_6 + x_7 \\ x_7 & x_0 + x_6 & x_1 + x_5 & & x_0 + x_1 + x_5 + x_6 + x_7 \end{array}$$

The points of the cycle coincide if on s; they are collinear if

$$x_0 + x_1 = x_2 + x_3 + x_4 = x_5 + x_6 + x_7\,, \tag{44.3}$$

i. e. whenever they lie in S. Otherwise they span a plane in one of

$$x_2 + x_3 + x_4 = x_5 + x_6 + x_7, \quad x_0 + x_1 = x_5 + x_6 + x_7\,,$$
$$x_0 + x_1 = x_2 + x_3 + x_4\,.$$

These are the P which join S to the points of s, so that the plane of each cycle meets S in a line. Each of these P includes $63 - 27 = 36\; m$

outside S which, under μ, run in 12 cycles of 3; each cycle has its centroid on s and is the set of common vertices of two $\mathcal{E}$ which, as they cannot be transposed by an operation of period 3, are both invariant under μ. Since no point is invariant that is not on s each of these invariant $\mathcal{E}$ has its vertices permuted in 3 cycles and has s for an axis; thus $\eta = 24$, and there are 80 members of class XXXVIII for which s is latent. The whole class musters 89600, each of the 1120 s contributing 80.

A cubic surface group includes, with its conjugate class ϑ of 80 operations of period 3, certain other classes powers of whose members belong to ϑ; these are (5, 146) classes of 720, 4320, 5760 members of respective periods 6, 12, 9. One therefore expects, with these same periods, classes in A of

$$806400, \quad 4838400, \quad 6451200$$

members; and Table III of 9 lists such classes, namely XL, XLII, XLI. It will suffice to to give the matrix of one member of each of these classes, and to describe the action of the projectivity imposed thereby.

45. The above 24 $\mathcal{E}$ invariant under μ must consist of 4 such sets of 6 as were encountered in § 16. One set, there fully recorded, consists of

$$\begin{matrix} \mathcal{A}_1 & \mathcal{C}_2 & \mathcal{B}_3 \\ \mathcal{B}_1 & \mathcal{A}_2 & \mathcal{C}_3 \\ \mathcal{C}_1 & \mathcal{B}_2 & \mathcal{A}_3 \end{matrix} \tag{45.1}$$

The 6 $\mathcal{E}$ consist each of 3 triads labelled by the script capitals in a row or in a column; labels $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ indicate triads with the respective kernels

$$(. \, . \, 1\,1\,1\,1\,1\,1), \quad (. \, . \, 1\,1\,1 \, . \, . \, .), \quad (. \, . \, . \, . \, . \, 1\,1\,1). \tag{45.2}$$

The 80 members of class XXXVIII for which (44.2) is latent leave every one of the 9 triads invariant; yet one may allow, while keeping the latent space the same, operations of A to permute the triads. But any permissible permutation has to replace every 3 collinear triads in (45.1) by collinear triads (this is simply to insist that, under A, enneads remain enneads) and, moreover, it must replace every triad by one labelled by the same letter (this is to insist that the centroid of each triad, being on (44.2), is unmoved).

If $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are all invariant so is every triad; $\mathcal{B}_2$, for example, is the only one of $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ that is aligned in (45.1) both with $\mathcal{A}_2$ and $\mathcal{A}_3$. If $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are cyclically permuted the permutation that they undergo determines that undergone by the other triads; one has

$$(\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3)(\mathcal{B}_1\mathcal{B}_2\mathcal{B}_3)(\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3)$$

and its inverse. A matrix that imposes this must have two of the three coordinate vectors of $\mathcal{A}_2$ for its first two columns. followed by those of $\mathcal{B}_2$ for its next three and then by those of $\mathcal{C}_2$ for its last three columns; the columns must, within this prescription, be so ordered that no point off (44.2) is invariant, and any such ordering gives an admissible matrix. One example, whose cube is μ and which has therefore period 9, is

$$\begin{bmatrix} . & . & 1 & 1 & . & . & . & . \\ . & . & 1 & . & 1 & . & . & . \\ 1 & 1 & 1 & . & . & 1 & . & 1 \\ 1 & 1 & 1 & . & . & 1 & 1 & . \\ 1 & 1 & 1 & . & . & . & 1 & 1 \\ . & . & . & 1 & 1 & 1 & 1 & 1 \\ 1 & . & . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

One may also permit $\mathcal{A}_2$ and $\mathcal{A}_3$ to be transposed while $\mathcal{A}_1$ is invariant; this, by the restrictions above, implies the permutation

$$(\mathcal{A}_1)(\mathcal{A}_2\mathcal{A}_3)(\mathcal{B}_2)(\mathcal{B}_3\mathcal{B}_1)(\mathcal{C}_3)(\mathcal{C}_1\mathcal{C}_2).$$

A matrix that imposes it has for its first two columns the coordinate vectors of two members of $\mathcal{A}_1$; $\mathcal{B}_3$ provides the following three and $\mathcal{C}_2$ the last three columns; nor must any point be invariant unless it satisfies (44.2). One such matrix, whose square is μ and which has therefore period 6, is

$$\begin{bmatrix} . & 1 & . & . & . & . & . & . \\ 1 & 1 & . & . & . & . & . & . \\ . & 1 & 1 & 1 & 1 & 1 & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 & 1 & . \\ . & 1 & 1 & 1 & 1 & . & 1 & 1 \\ . & 1 & 1 & . & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & . & 1 & 1 & 1 \\ . & 1 & . & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{45.3}$$

Classes XL and XLI have been accounted for without substituting another scheme for (45.1); but operations in class XLII are only detected through permitting (45.1) to be replaced by another of the 4 schemes that are, as wholes, invariant under μ. There are, and they have been displayed in § 16, 27 m involved in the 9 triads of (45.1); in order to obtain another scheme one takes an m not included among these 27 yet which does not lie in (44.3). This m is one of a triad $\mathcal{A}'_1$ cyclically permuted by μ and having its centroid on (44.2); two $\mathcal{E}$, $\mathcal{A}'_1\mathcal{B}'_1\mathcal{C}'_1$ and $\mathcal{A}'_1\mathcal{B}'_3\mathcal{C}'_2$ include $\mathcal{A}'_1$ among their vertices and furnish the scheme. For example

$$
\begin{array}{ccc|ccc|ccc}
1\,.\,1 & 1\,1\,. & 1\,1\,. & 1\,1\,. & 1\,1\,. & 1\,.\,1 & .\,1\,1 & 1\,1\,. & 1\,.\,1 \\
.\,1\,1 & 1\,.\,1 & 1\,.\,1 & 1\,.\,1 & 1\,.\,1 & .\,1\,1 & 1\,1\,. & 1\,.\,1 & .\,1\,1 \\
.\,1\,. & 1\,1\,1 & 1\,.\,1 & .\,1\,. & .\,.\,1 & .\,.\,. & 1\,1\,1 & .\,1\,. & 1\,.\,1 \\
1\,1\,1 & .\,1\,. & 1\,1\,. & .\,.\,1 & 1\,1\,1 & .\,1\,1 & .\,.\,1 & .\,.\,1 & .\,.\,. \\
1\,.\,. & .\,.\,1 & .\,.\,. & 1\,1\,1 & .\,1\,. & 1\,.\,1 & 1\,.\,. & 1\,1\,1 & .\,1\,1 \\
.\,1\,. & 1\,.\,1 & 1\,1\,1 & 1\,1\,1 & 1\,1\,. & 1\,.\,. & .\,.\,1 & .\,.\,. & .\,1\,. \\
1\,1\,1 & 1\,1\,. & .\,1\,. & .\,1\,. & .\,.\,. & .\,1\,. & 1\,.\,. & 1\,.\,1 & 1\,1\,1 \\
\underbrace{1\,.\,.} & \underbrace{.\,.\,.} & \underbrace{.\,.\,1} & \underbrace{.\,.\,1} & \underbrace{1\,.\,1} & \underbrace{1\,1\,1} & \underbrace{1\,1\,1} & \underbrace{1\,1\,.} & \underbrace{1\,.\,.} \\
\mathcal{A}'_1 & \mathcal{B}'_1 & \mathcal{C}'_1 & \mathcal{A}'_2 & \mathcal{B}'_2 & \mathcal{C}'_2 & \mathcal{A}'_3 & \mathcal{B}'_3 & \mathcal{C}'_3
\end{array}
$$

Each triad, it will be noted, has its centroid or kernel at the appropriate point (45.2) according as its label is $\mathcal{A}$,$\mathcal{B}$ or $\mathcal{C}$. The two other schemes analogous to (45.1) involve those $\mathcal{E}$ derived from the above by cyclic permutation of the three bottom rows.

The matrices

$$
\begin{bmatrix}
1 & 1 & 1 & . & 1 & 1 & . & 1 \\
1 & . & 1 & 1 & . & 1 & 1 & . \\
. & . & k & k & 1 & k & k & . \\
1 & 1 & . & k+1 & k & 1 & k+1 & k \\
. & 1 & k+1 & 1 & k+1 & k+1 & . & k+1 \\
. & . & k & k & . & k & k & 1 \\
1 & 1 & 1 & k+1 & k & . & k+1 & k \\
. & 1 & k+1 & . & k+1 & k+1 & 1 & k+1
\end{bmatrix}
$$

impose on the triads the permutations

$$(\mathcal{A}_1\mathcal{A}'_1)(\mathcal{B}_1\mathcal{B}'_1)(\mathcal{C}_1\mathcal{C}'_1) \quad \text{for} \quad k=1,$$

$$(\mathcal{A}_1\mathcal{A}'_1)(\mathcal{B}_1\mathcal{B}'_3)(\mathcal{C}_1\mathcal{C}'_2) \quad \text{for} \quad k=0.$$

Each matrix is the product of the other by the glide that is the cube of (45.3), and both have (45.3) for their square. Each of the two other schemes eligible to replace (45.1) affords a pair of matrices of class XLII with each member of a pair the product of the other by the glide that is the cube of (45.3). Thus 6 matrices arise having (45.3) for their square; the other 4 are given by

$$\begin{bmatrix} 1 & 1 & . & 1 & 1 & . & 1 & 1 \\ 1 & . & 1 & . & 1 & 1 & . & 1 \\ 1 & 1 & k & k+1 & . & k & k+1 & 1 \\ . & 1 & 1 & k & k & . & k & k \\ . & . & k+1 & 1 & k+1 & k+1 & . & k+1 \\ 1 & 1 & k & k+1 & 1 & k & k+1 & . \\ . & 1 & . & k & k & 1 & k & k \\ . & . & k+1 & . & k+1 & k+1 & 1 & k+1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 & . & 1 & 1 & . \\ 1 & . & . & 1 & 1 & . & 1 & 1 \\ . & 1 & k & k & 1 & k & k & . \\ . & . & 1 & k+1 & k+1 & . & k+1 & k+1 \\ 1 & 1 & k+1 & . & k & k+1 & 1 & k \\ . & 1 & k & k & . & k & k & 1 \\ . & . & . & k+1 & k+1 & 1 & k+1 & k+1 \\ 1 & . & k+1 & 1 & k & k+1 & . & k \end{bmatrix}$$

46. The spaces of even dimension containing g consist of

$$24P;\ \ 114E,\ \ 24H,\ \ 792F,\ \ 384J;\ \ 48f,\ \ 6e,\ \ 9d;$$

hence the coset of A_g^+ consists of

$$24+456+144+1584+768+11520+1920+2016=18432$$

operations. But, apart from the whole [7] and g itself, the spaces of odd dimension that contain it include

$$151G,\ \ 180T,\ \ 32S;\ \ 6\omega,\ \ 9\gamma,\ \ 234\varphi,\ \ 2\chi,\ \ 144\psi,\ \ 256\varkappa;$$

hence there occur

$$1+151+180+64+168+252+2808+88+2880+4096=10688$$

operations of A_g^+, leaving 7744 for which g is latent.

Table II registers operations of class XXVIII as having g latent; they permute the vertices of some ennead according to the partition 14^2 and so are exemplified by those which permute the 8 vertices of Σ_0 in two cycles of 4. The size of this class can be deduced from the order of the normaliser of any one of its members. The permutation matrix

$$\begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix} \oplus \begin{bmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{bmatrix} \tag{46.1}$$

has, in the general linear group, a normaliser of the form

$$\begin{bmatrix} a & b & c & d & e & f & g & h \\ d & a & b & c & h & e & f & g \\ c & d & a & b & g & h & e & f \\ b & c & d & a & f & g & h & e \\ a' & b' & c' & d' & e' & f' & g' & h' \\ d' & a' & b' & c' & h' & e' & f' & g' \\ c' & d' & a' & b' & g' & h' & e' & f' \\ b' & c' & d' & a' & f' & g' & h' & e' \end{bmatrix} \tag{46.2}$$

This is to belong to A. Write

$$s=a+b+c+d,\quad t=e+f+g+h$$

$$s'=a'+b'+c'+d',\quad t'=e'+f'+g'+h'.$$

Since the first and third columns of (46.2) are not to be conjugate $s+s'=1$; one of s, s' is 0 and the other 1, and likewise with t, t'. But $st=s't'$ because the sum of the last 4 columns has to be conjugate to any of the other 4; hence

$$s=1+s'=1+t=t'. \tag{46.3}$$

Each column therefore has an odd number of units, and so either 1 or 5. Each of the first 4 columns has the same number, as does each of the last 4; but, indeed, every column has the same number of units. For were either the left or right hand half of (46.2) to consist of columns with a single unit and the other half to have 5 units in its columns one at least of these latter would, as coordinate vector of a point, be conjugate to one of the former. Now the first 2 columns of (46.2) are not to be conjugate; this in conjunction with (46.3), demands

$$(a+c)(b+d)=(a'+c')(b'+d')=0.$$

the zero on the right being necessary because 1 would contradict $s=1+s'$. Hence there are the mutually exclusive alternatives:

either $\quad a+c=b+d, \qquad a'+c'=b'+d'+1,$

or $\quad a+c=b+d+1, \quad a'+c'=b'+d'.$

We can confine attention to the former; the latter is then allowed for by reflecting in its horizontal bisector any matrix that occurs.
There are 4 possibilities:

$$\text{(i) } a=c=0,\quad b=d=0. \quad \text{(ii) } a=c=1,\quad b=d=1.$$

$$\text{(iii) } a=c=1,\quad b=d=0. \quad \text{(iv) } a=c=0,\quad b=d=1.$$

Each allows the first column, which has to include either 1 or 5 units, to be filled in 4 ways. It is then found that, in each instance, the last column, which has to include the same number of units as the first and is prohibited from being conjugate to it, can also be filled in 4 ways. Thus 64 matrices occur. They form, when taken with the reflections in the horizontal bisector, the group of order 128 that normalises (46.1). Thus (46 1) belongs to a coujugate class of

$$2^6\cdot 3^5\cdot 5^2\cdot 7=2721600$$

members. Each of the 1575 g is latent for 1728 of these.

The other classes than XXVIII whose members have latent g are XXXI, XXXII, LXII, LXIII; this is seen from the second column of Table III registering the cyclic decompositions of permutations of the 135 m under the subgroup A_m^+ of a stabiliser A_m. The dimension of the space latent for an operations of A^+ is always odd; if the space contains $3\ m$ it is either χ or g and, as found in § 34, χ occurs for classes XIV and XV. Members of class LXII are alluded to again in § 53 below; each g is latent for 1152 of them.

47. The spaces of even dimension containing t comprise

$$32P;\ 180E,\ 140H,\ 420F,\ 640J;\ 32g,\ 10f,\ 6h,\ 15e$$

so that the coset of A_t^+ consists of

$$32+720+840+840+1280+9728+2400+2400+4800=23040$$

operations. But the [7] itself and

$$75G,\ 256T,\ 80S;\ 45\varphi,\ 15\chi,\ 320\psi,\ 160\varkappa,\ 96\lambda,\ 5\gamma$$

only provide

$$1+75+256+160+540+660+6400+2560+2304+420=13376,$$

leaving a further 9664 operations of A_t to be accounted for. Table II discloses that classes XXVII, XXIX, XXXIII contribute; their contributions of 1920, 2304, 1920 can be found by constructing the normalisers of their respective members in A. The other classes which contribute the remaining 3520 operations are seen, on appealing again to the second column of Table III below, to be XXVI, XXXV, XXXVII.

The two latter classes will be encountered again in § 52; each of them includes 1440 operations of A_t. This leaves 640 operations to belong to class XXVI, which therefore has $640.3780=2419200$ members.

48. Consider now the stabiliser A_p of a point p; it includes the projection $\mathfrak{J}_p$ centred at p and has, indeed, the $\mathfrak{C}_2$ generated by $\mathfrak{J}_p$ for a direct factor. For let μ be any operation of A_p. Any point in the polar P of p is transformed by μ into a point also in P, so that every point of P is transformed by both $\mu\mathfrak{J}_p$ and $\mathfrak{J}_p\mu$ into the same point as that into which it is transformed by μ; thus $(\mu\mathfrak{J}_p)(\mathfrak{J}_p\mu)^{-1}$, imposing the identity projectivity in P, is either I or $\mathfrak{J}_p$.

But $\mu\mathfrak{J}_p$ and $\mathfrak{J}_p\mu$ both belong either to A^+ or to its coset, so that $(\mu\mathfrak{J}_p)(\mathfrak{J}_p\mu)^{-1}$ belongs to A^+ and cannot be $\mathfrak{J}_p$. Thus $\mathfrak{J}_p$ commutes with μ, and $A_p\cong\mathfrak{C}_2\times A_p^+$. This subgroup A_p^+ is the group "of the bitangents", and is well enough known (**9**, Table V; **7** *passim*) to be spared further comment here save for a brief mention of which further classes in A have representatives in A_p. Just as, in § 43, one encountered classes XLIII and XLV by permitting the transposition of the 2 m on c so one encounters further classes by permitting the transposition of any 2 of the 3 p on s. Thus one finds, as representatives of classes XLIX and LII,

$$\begin{bmatrix} . & 1 & . & . & . & . & . & . \\ 1 & 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & 1 \\ . & 1 & . & . & . & 1 & . & . \\ . & 1 & . & . & . & . & 1 & . \\ . & 1 & 1 & . & . & . & . & . \\ . & 1 & . & 1 & . & . & . & . \\ . & 1 & . & . & 1 & . & . & . \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} . & 1 & 1 & . & 1 & 1 & 1 & . \\ 1 & . & 1 & 1 & . & . & 1 & 1 \\ 1 & . & . & 1 & . & 1 & 1 & . \\ 1 & 1 & 1 & 1 & 1 & . & . & . \\ . & 1 & . & . & 1 & . & 1 & 1 \\ 1 & . & 1 & . & 1 & . & . & 1 \\ 1 & 1 & . & . & . & 1 & 1 & 1 \\ . & 1 & 1 & 1 & . & 1 & . & . \end{bmatrix}$$

whose squares are, respectively, (44.1) and (45.3). There also occur in A_p representatives of a class LI; they are of period 18 with their cubes belonging to class XLIX.

49. While A_p, as the direct product of $\mathfrak{C}_2$ and a well-known group, can be summarily dispatched A_m merits, as a group of low index in A, more formal recognition. Its order is $2^{13}\cdot 3^2\cdot 5\cdot 7$ and it has a subgroup A_m^+ of index 2; it does not share with A_p the prestige of being an exceptional Lie group and may not have been examined heretofore. So we display in tabular form the distribution of its operations among the conjugate classes of A. The number in any class is the quotient, by 135, of the product of the number of operations of A in the class and the number of m invariant under any one of these operations; it is therefore obtainable once this number of invariant m is known. The second and fifth columns of Table III show the cyclic decomposition of the permutation imposed on the 135 m by the operations of each class of A. For operations of prime period this decomposition is known, being determined by the latent space; for example, a projection in class II leaves invariant all 63 m in its latent P and transposes in pairs the 72 m outside P; a glide in class XIII leaves invariant all 39 m in its latent G and transposes in pairs the 96 m outside G; an operation of period 5 in class XII leaves invariant the 5 m in its latent λ and permutes the 130 m outside λ in cycles of 5, and so on. The decomposition of a permutation imposed by an operation μ whose period is not prime is then deduced from those of the permutations imposed by the powers of μ.

The left-hand half of the table appertains to A_m^+, the right-hand half to its coset which includes representatives of the classes XLVI, XLVII, LIV not encountered before. The third and sixth columns both sum to $1290240 = 2^{12}\cdot 3^{2}\cdot 5\cdot 7$.

TABLE III

The group A_m

I	1^{135}	1	II	$1^{63}2^{36}$	56
III	$1^{31}2^{52}$	868	V	$1^{15}2^{60}$	4200
IV	$1^{27}3^{36}$	448	VI	$1^{15}2^{6}3^{16}6^{10}$	8960
VIII	$1^{7}2^{64}$	2940	VII	$1^{11}2^{10}4^{26}$	3690
IX	$1^{7}2^{10}3^{8}6^{14}$	31360	XVI	$1^{3}2^{12}3^{4}6^{16}$	26880
X	$1^{9}3^{42}$	17920	XVII	$1^{3}2^{3}3^{20}6^{11}$	35840
XI	$1^{7}2^{12}4^{26}$	47040	XVIII	$1^{3}2^{14}4^{26}$	30240
XII	$1^{5}5^{26}$	21504	XIX	$1^{5}2^{1}3^{2}4^{5}6^{3}12^{7}$	134400
XIII	$1^{39}2^{48}$	455	XX	$1^{3}2^{1}5^{12}10^{7}$	129024
XIV	$1^{3}2^{12}3^{12}6^{12}$	2240	XXI	$1^{3}2^{18}4^{24}$	10080
XV	$1^{3}2^{18}4^{24}$	840	XXII	$1^{3}2^{3}3^{4}6^{19}$	107520
XXVI	$1^{1}2^{4}3^{10}6^{16}$	17920	XXIII	$1^{19}2^{6}4^{26}$	10640
XXVII	$1^{1}2^{3}3^{2}4^{5}6^{3}12^{7}$	53760	XXIV	$1^{1}2^{3}3^{6}4^{5}6^{1}12^{7}$	8960
XXVIII	$1^{3}2^{2}4^{32}$	60480	XXV	$1^{1}2^{1}4^{9}8^{12}$	13440
XXIX	$1^{1}2^{2}5^{6}10^{10}$	64512	XLIII	$2^{1}3^{1}5^{1}10^{1}15^{3}30^{2}$	—
XXX	$1^{2}3^{1}5^{5}15^{7}$	172032	XLIV	$1^{1}4^{1}5^{2}10^{2}20^{5}$	129024
XXXI	$1^{3}2^{3}3^{12}6^{15}$	35840	XLV	$2^{1}7^{9}14^{5}$	—
XXXII	$1^{3}4^{6}6^{6}12^{6}$	26880	XLVI	$1^{1}3^{6}4^{2}6^{2}12^{8}$	35840
XXXIII	$1^{1}2^{4}3^{2}6^{20}$	53760	XLVII	$1^{1}2^{1}8^{3}12^{3}24^{3}$	107520
XXXIV	$1^{2}7^{19}$	368640	XLVIII	$1^{1}3^{1}4^{1}8^{16}$	161280
XXXV	$1^{1}2^{1}4^{9}8^{12}$	40320	XLIX	$3^{21}6^{12}$	—
XXXVI	$1^{11}2^{14}4^{24}$	27720	L	$3^{5}6^{20}$	—
XXXVII	$1^{1}2^{5}4^{7}8^{12}$	40320	LI	$9^{7}18^{4}$	—
XXXVIII	3^{45}	—	LII	$3^{1}6^{6}12^{8}$	—
XXXIX	$1^{9}3^{10}6^{16}$	53760	LIII	$1^{7}2^{12}4^{26}$	23520
LXI	$1^{15}2^{60}$	840	LIV	$1^{1}3^{2}4^{2}6^{4}12^{8}$	107520
LXII	$1^{3}2^{6}4^{30}$	40320	LV	$1^{5}2^{3}4^{7}8^{12}$	201600
LXIII	$1^{3}2^{3}3^{4}6^{19}$	107520			

TABLE IV

The groups A_σ

The table shows, in addition to their orders, classes not occuring in Table II that have representatives in these groups.

σ		
[7]	1	
M	1	
P	2	
C	2	
G	2	XIII
T	4	
S	6	
D	8	
F	8	XXIII
J	12	
E	16	
H	24	
$\varkappa$	72	XXXIX
λ	120	
ω	64	LXI
φ	64	XXXVI
ψ	96	
γ	128	
χ	192	XIV, XV

σ		
d	1024	LIII
f	1152	LV
j	1440	
e	1536	XXI
h	1920	XXIV, XXV
g	36864	XXXI, XXXII, LXII, LXIII
c	40320	
t	46080	XXVI, XXXV, XXXVII
s	51840	XL, XLI, XLII
m	2580480	XLVI, XLVII, LIV
p	2903040	XLIII, XLIX, LI, LII

III

The Sylow 2-groups of A^+

50. The groups A_m are of order $2^{13}\cdot 3^2\cdot 5\cdot 7$. Any one of them acts as a transitive permutation group on the 35 g through m, each of which is invariant for $2^{13}\cdot 3^2$ of the permutations. These latter act as a transitive permutation group on the $9\,d$ through g, each of which is invariant for 2^{13} of them. Thus one comes upon the S_2. SYLOW subgroups of order 2^{13}, in A; each is associated with a flag $m<g<d$ on $\mathfrak{S}$. S_2 can, since the projections centred in the γ through d belong to it, transpose the two solids on $\mathfrak{S}$ through d; hence a complete flag $m<g<d<\omega$ - complete in that it cannot be extended without involving points off $\mathfrak{S}$ - is invariant for an S_2^+ of A^+. Since there are 2025 d, each containing 7 g on each of which lie 3 m, there are 42525 S_2 in A and S_2^+ in A^+.

Take $\mathfrak{S}$ to have the equation (10.4) and the spaces of the flag $\mathfrak{F}$, with their polars, to be

$$m_0:\ Y_1=Y_2=Y_3=Y_4=Z_1=Z_2=Z_3=0, \qquad M_0:\ Y_4=0,$$

$$g_0:\ Y_1=Y_2=Y_3=Y_4=Z_1=Z_2=0, \qquad G_0:\ Y_3=Y_4=0,$$

$$d_0:\ Y_1=Y_2=Y_3=Y_4=Z_1=0, \qquad D_0:\ Y_2=Y_3=Y_4=0$$

$$\omega_0:\ Y_1=Y_2=Y_3=Y_4=0.$$

Any matrix $\mathfrak{M}$ for which m_0, g_0, d_0, ω_0 are all invariant must have every element above its main diagonal in each of the last 4 columns zero. If $\mathfrak{S}$ is invariant too so must each of M_0, G_0, D_0 be; these requirements imply certain zeros below the main diagonal in the first three columns, and the presence of the zero in the fifth row and first column is then demanded by the invariance of

$$\omega'_0:\ Z_1=Y_2=Y_3=Y_4=0$$

or of

$$\gamma_0:\ Y_1+Z_1=Y_2=Y_3=Y_4=0.$$

Detailed examination then discloses that, if

$$Y_1Z_1+Y_2Z_2+Y_3Z_3+Y_4Z_4$$

is to be unchanged,

$$(50.1)\qquad \mathfrak{M} \equiv \begin{bmatrix} 1 & n & m & l' & . & . & . & . \\ . & 1 & l & m' & . & . & . & . \\ . & . & 1 & n' & . & . & . & . \\ . & . & . & 1 & . & . & . & . \\ . & c & b & a' & 1 & . & . & . \\ c & cn & a & b' & n & 1 & . & . \\ b+cl & a+A & al'+bm & c' & m+nl & l & 1 & . \\ a'+A' & b'+B' & c'+C' & R & l'+L' & m'+n'l & n' & 1 \end{bmatrix}$$

where

$$A = bn + c(m+nl),\quad A' = bn' + c(m'+n'l),$$
$$B' = a'n + c(l' + m'n) + n'(a+A),\qquad C' = am' + bl' + a'm + b'l,$$
$$R = a'l' + b'm' + c'n',\qquad L' = m'n + n'(m+nl).$$

Since each of
$$l,\ m,\ n,\ l',\ m',\ n',\ a,\ b,\ c,\ a',\ b',\ c'$$

can be either 0 or 1 there are 2^{12} $\mathfrak{M}$; they represent, and can be regarded as composing, an S_2^+ in A^+. They have the group property

$$(50.2)\qquad \begin{pmatrix} l_1, & m_1, & n_1; & l'_1, & m'_1, & n'_1 \\ a_1, & b_1, & c_1; & a'_1, & b'_1, & c'_1 \end{pmatrix}\begin{pmatrix} l_2, & m_2, & n_2: & l'_2, & m'_2, & n'_2 \\ a_2, & b_2, & c_2; & a'_2, & b'_2, & c'_2 \end{pmatrix} = \begin{pmatrix} \lambda, & \mu, & \nu; & \lambda', & \mu', & \nu' \\ \alpha, & \beta, & \gamma; & \alpha', & \beta', & \gamma' \end{pmatrix}$$

where

$$(50.3)\qquad \left\{\begin{array}{lll} \lambda = l_1 + l_2, & \mu = m_1 + m_2 + n_1 l_2 & \nu = n_1 + n_2; \\ \lambda' = l'_1 + l'_2 + m_1 n'_2 + n_1 m'_2, & \mu' = m'_1 + m'_2 + l_1 n'_2 & \nu' = n'_1 + n'_2; \\ \alpha = a_1 + a_2 + n_1 b_2 + c_1 m_2 + c_1 n_1 l_2, & \beta = b_1 + b_2 + c_1 l_2, & \gamma = c_1 + c_2; \\ \alpha' = a'_1 + a'_2 + b_1 n'_2 + c_1 m'_2, & \beta' = b'_1 + b'_2 + c_1(n_1 m'_2 + l'_2) + & \\ & \qquad + a_1 n'_2 + n_1 a'_2, & \\ \multicolumn{3}{l}{\gamma' = c'_1 + c'_2 + a_1(l_1 n'_2 + m'_2) + b_1(m_1 n'_2 + n_1 m'_2 + l'_2) +} \\ \multicolumn{3}{l}{\qquad\qquad + c_1(l_1 l'_2 + m_1 m'_2 + n_1 l_1 m'_2) + (n_1 l_1 + m_1)a'_2 + l_1 b'_2.} \end{array}\right.$$

In particular

$$\begin{pmatrix} l, & m, & n; & l', & m', & n' \\ a & b, & c; & a', & b', & c' \end{pmatrix}^2 =$$

(50.4)

$$= \begin{pmatrix} \cdot & , nl, & \cdot & ; & mn' + nm', & ln', & \cdot \\ nb + (m + nl)c, & cl, & \cdot & ; & bn' + cm', & c(nm' + l') + an' + na', & \Gamma \end{pmatrix}$$

where

$$\Gamma = a(ln' + m') + b(mn' + nm' + l') + c(ll' + mm' + nlm') + a'(nl + m) + b'l. \tag{50.5}$$

A repeated application of (50.4) shows that all save one of the 12 marks which determine the fourth power of any operation of S_2^+ are zero; the one that may not be zero corresponds to c' and is cl^2nn'. Hence there are 2^8 operations of period 8 in S_2^+, namely those having

$$c = l = n = n' = 1,$$

and no operation of S_2^+ has a higher period. All these 2^8 operations have the same fourth power - the gilde

$$Z_3 \to Y_4 + Z_3, \quad Z_4 \to Y_3 + Z_4 \tag{50.6}$$

associated with G_0. This glide is that operation of S_2^+ for which all 12 marks save c' are zero; the relations (50.3) show that it commutes with every operation of S_2^+. These relations also suffice to show, on scrutiny, that no operation save (50.6) and identity commutes with every operation of S_2^+; the centre of S_2^+ has order 2.

So far as c, l, n, n' are concerned the effect of multiplying two matrices (50.1) is to add the corresponding marks; hence those operations of S_2^+ for which any one of c, l, n, n' is zero form a maximal subgroup of index 2. These subgroups can be defined geometrically. For there are

(*i*) two points m_1, m_2 other than m_0 on g_0. These are transposed if $n' = 1$; the subgroup having $n' = 0$ leaves them both invariant. and is of index 9 in $A_{g_0}^+$.

(*ii*) two lines g_1, g_2 other than g_0 that pass through m_0 and lie in d_0. These are transposed if $l = 1$; the subgroup having $l = 0$ leaves them both invariant.

(iii) two planes d_1, d_2 other than d_0 that pass through g_0 and lie in ω_0. These are transposed if $n=1$: the subgroup having $n=0$ leaves them both invariant.

(iv) two solids ω_1, ω_2 other than ω_0, and belonging to the same system, that pass through g_0 and lie in G_0. These are transposed if $c=1$; the subgroup having $c=0$ leaves them both invariant.

To verify, say, *(iii)* one takes

$$d_1:\quad Y_1=Y_2=Y_3=Y_4=Z_2=0,\quad d_2:\ Y_1=Y_2=Y_3=Y_4=Z_1+Z_2=0;$$

when the column vector of coordinates of a point m in d_1 is premultiplied by $\mathfrak{M}$ that component which answers to Z_2 in the resulting vector is $n\zeta_1$, where ζ_1 is the Z_1-coordinate of m. To verify *(iv)* one takes

$$\omega_1:\ Y_3=Y_4=Z_1=Z_2=0,\quad \omega_2:\ Y_3=Y_4=Y_1+Z_2=Y_2+Z_1=0.$$

51. The 2^8 operations of period 8 in S_2^+ are precisely those not belonging to any of the above four maximal subgroups. No operation of period 8 in A can belong to more than one S_2^+. For suppose the contrary, that

$$\mathfrak{F}:\ m_0<g_0<d_0<\omega_0\quad\text{and}\quad \mathfrak{F}:\ m'<g'<d'<\omega'$$

both afford the same such operation μ. The glide μ^4 shows immediately that $g'\equiv g_0$, and as μ transposes two of the points on this line its invariant $m'\equiv m_0$. Were d' distinct from d_0 it could not, being invariant, be either d_1 or d_2 since μ transposes these; it could not, therefore, lie in ω_0 and so would lie in ω_1 or ω_2 - and so in both ω_1 and ω_2 because μ transposes them; this is impossible. Hence $d'\equiv d_0$, and $\omega'\equiv\omega_0$. As there are 2^8 operations of period 8 in each of 42525 S_2^+ 10886400 such operations in A^+ are hereby accounted for.

52. Any column vector that is latent for $\mathfrak{M}$ with $c=l=n=n'=1$ satisfies

$$Y_2=Y_3=Y_4=0$$

as well as

$$Z_1=Y_1,\ \ Z_2=(m+b)Y_1,\ \ Z_3=\{(m+b)(m+1)+a'+l'\}\,Y_1$$

These conditions determine a line, one of the 4 t that pass through m_0 and lie in D_0; each such t is latent for 2^6 operations of period 8 in S_2^+.

Table I tells that any t lies in $15\,D$; the polar d of any such D accompanies the m on t in 3 flags, one for each g in d through m. Thus t is latent for $2^6\times 45=2880$ operations of period 8 in A^+: they belong to classes XXXV and XXXVII, and indeed are equally distributed between them. For the latent space of the square of (50.1) is found, on using (50.4), to be, when $c=l=n=n'=1$, the solid

$$Y_3=Y_4=0,\ \ Z_1=Y_1+(b+m+1)Y_2$$

$$Z_2=(b+m)Y_1+(a+a'+l'+bm+bm'+mm')Y_2. \tag{52.1}$$

This is φ or χ, and so the square belongs to class XXXVI or class XV, according as the coefficient of Y_2 in (52.1) is 0 or 1.

53. There are in S_2^+ operations whose latent space is g_0; they must have $n'=0$ and, in order that no point off g_0 shall be invariant, satisfy (when $n'=0$) the further condition that $\mathfrak{M}$ has no latent column vector save those with

$$Y_1=Y_2=Y_3=Y_4=Z_1=Z_2=0.$$

The five top rows show that other latent vectors might occur unless

$$\begin{vmatrix} n & m & l' \\ \cdot & l & m' \\ c & b & a' \end{vmatrix}=1; \tag{53.1}$$

then, as this condition ensures that $Y_2=Y_3=Y_4=0$, the three bottom rows produce the same condition again to ensure that $Y_1=Z_1=Z_2=0$. Since, in (53.1), one has, in succession, 3 choices for the first, 6 for the central and 4 for the last column there are 72 non-vanishing determinants; and since a, b', c' are left free there are, in S_2^+, 576 operations with g_0 for their latent space. These, as is seen by squaring them, are not all of the same kind. (50.4) shows that, when $n'=0$, points invariant under the square of (50.1) satisfy

$$n(lY_3+m'Y_4)=c(lY_3+m'Y_4)=0$$

$$\{nb+(m+nl)c\}\,Y_3=\{c(nm'+l')+na'\}\,Y_4.$$

Since, by (53.1), n and c are not both zero, $lY_3 + m'Y_4 = 0$ and since, also by (53.1),

$$m'\{nb + (m + nl)c\} + l\{c(nm' + l') + na'\} = 1 \tag{53.2}$$

it must be that $Y_3 = Y_4 = 0$. The further conditions to be satisfied by invariant points are

$$clY_1 + \{nb + (m + nl)c\}Y_2 + nlZ_1 = 0,$$

$$cm'Y_1 + \{c(nm' + l') + na'\}Y_2 + nm'Z_1 = 0,$$

which, noting (53.2), combine to give

$$Y_2 = cY_1 + nZ_1 = 0$$

since, again by (53.1), it is impossible for l and m' to be both zero. Now the solids

$$cY_1 = nZ_1, \quad Y_2 = Y_3 = Y_4 = 0$$

are $\omega_0(c = 1,\ n = 0)$, $\omega'_0(c = 0,\ n = 1)$, and $\gamma_0(c = n = 1)$, so that those 576 operations of S_2^+ for which g_0 is latent fall into 3 sets of 192. The square of each operation has a latent solid, and the operation itself belongs to one or other set according as this is $(a)\omega_0$, $(b)\omega'_0$, $(c)\gamma_0$. The 192 operations (c) belong to class XXVIII; the others are in class LXII. It was proved in § 46 that each g is latent for 1728 members of class XXVIII; since g lies in 9 γ and 6 ω it follows that it is latent for 1152 members of class LXII. This last class therefore embraces $1152 \times 1575 = 1814400$ operations of A^+.

54. The subgroup Φ, of order 2^8, of S_2^+ that is common to the four maximal subgroups of § 50 belongs also to six other maximal subgroups, namely to those subgroups for whose operations some two of c, l, n, n' are equal. These maximal subgroups will have geometrical characterisations; for instance, if $n = c$, the two χ in G_0 are not transposed, for these solids in $Y_3 = Y_4 = 0$ are

$$\chi_0:\ Y_1 + Z_1 = Y_2 = Z_2, \quad \chi'_0:\ Y_1 = Z_1 = Y_2 + Z_2.$$

Φ and its cosets in S_2^+ are, as (50.3) shows, determined by c, l, n, n' in the sense that these marks are the same for all 2^8 operations in any coset. The group S_2^+/Φ of these cosets is thus elementary abelian - as always (**15**, 141) with the FRATTINI subgroup of a p-group - of order 16.

55. When $c = l = n = n' = 0$ the expression Γ of (50.5) reduces to $\eta \equiv am' + bl' + a'm$; those operations of Φ for which $\eta = 1$ have period 4 while all its other operations save identity have period 2. Since b', c' are not restricted, and since there are 28 points off a KLEIN quadric in [5], there are 112 operations of period 4 in Φ.

The top six rows of $\mathfrak{M}$ supply, when $c = l = n = n' = 0$, as equations of its latent space σ,

$$mY_3 + l'Y_4 = m'Y_4 = bY_3 + a'Y_4 = aY_3 + b'Y_4 = 0$$

of which two linear combinations are $\eta Y_3 = 0$ and $\eta Y_4 = 0$; so, if $\eta = 1$, σ is in G_0. The bottom rows of $\mathfrak{M}$ supply further equations of σ which reduce, when $Y_3 = Y_4 = 0$, to

$$(55.1) \qquad \left\{ \begin{aligned} bY_1 + aY_2 + mZ_1 \qquad\quad &= 0 \\ a'Y_1 + b'Y_2 + l'Z_1 + m'Z_2 &= 0 \end{aligned} \right.$$

Since

$$\begin{vmatrix} b & m \\ a' & l' \end{vmatrix} + \begin{vmatrix} a & \cdot \\ b' & m' \end{vmatrix} = \eta = 1$$

(55.1) can be solved *either* for Y_1 and Z_1 *or* for Y_2 and Z_2 as linear combinations of the others. The expression $Y_1Z_1 + Y_2Z_2$ thereby becomes a binary quadratic q and the solid σ is γ, χ, φ according as q is a perfect square, irreducible or factorisable over $\boldsymbol{F}$. But γ does not occur; the product term is always present in q. Consider now the alternatives.

(*i*) (55.1) can be solved for Y_1, Z_1; $a'm + bl' = 1$, $am' = 0$. This allows 6 tetrads of marks a', m, b, l' and 3 duads a, m'; 18 sets of marks in all. Thus, b', c' being free, 72 operations of period 4 come under this head. We have

$$q \equiv (al' + b'm)(aa' + bb')Y_2^2 + Y_2Z_2 + bmm'^2Z_2^2.$$

The only irreducible form is $Y_2^2 + Y_2Z_2 + Z_2^2$; hence, if q is irreducible,

$$(55.2) \qquad b = m = m' = 1, \;\; a = 0, \;\; b' = 1, \;\; a' + l' = 1.$$

This permits, c' being free, 4 operations.

(*ii*) (55.1) can be solved for Y_2, Z_2;

$$a = m' = 1, \;\; bl' = a'm.$$

This allows 10 tetrads of marks a', b, l', m; thus, b', c' being free. 40 operations of period 4 come under this head. We have

$$q \equiv b(a'+bb')Y_1^2 + Y_1Z_1 + m(l'+b'm)Z_1^2$$

which factorises unless $b=m=1$, $a'=l'=b'+1$. This permits, c' being free, 4 operations. It thus appears that, of the 112 operations of period 4 in Φ, 8 have a latent solid χ and so belong to class XV while the remaining 104 have a latent solid φ and belong to class XXXVI. These latent solids are all in G_0. If one substitutes from (55.2) in (55.1) the outcome is the set of equations for χ_0' which is therefore latent for the 4 corresponding operations, as χ_0 is, likewise, for the 4 operations under *(ii)*.

56. When all of c, l, n, n', whatever their suffix, are zero all the equations (50.3) except the last become symmetric in the suffixes; so, in order that operations in Φ commute, it is sufficient that

$$a_1m'_2 + b_1l'_2 + m_1a'_2 = a_2m'_1 + b_2l'_1 + m_2a'_1,$$

a polarised version of $\eta = 0$. Now all operations of Φ save those 4 for which $a=a'=b=l'=m=m'=0$ are mapped by the points of a [5]; each point, because of the non-occurrence in its coordinates of b' and c', maps 4 operations, and they are of period 2 or 4 according as their map is on or off a KLEIN quadric Ω. And operations commute or do not commute according as their maps are conjugate or not in the null polarity set up by $\mathbf{\Omega}$.

The operations that avoid the mapping, and only they, commute with every member of Φ; they form the centre of Φ, a 4-group z_2. But any of the other 252 operations commutes only with the members of z_2 and with the 124 mapped 4 by each point in the polar prime, with respect to Ω, of the point that maps the operation in question. Thus each operation of Φ outside z_2 has a normaliser of order 128 and so belongs, in Φ, to a conjugate class of only 2 members. The polar [4]'s with respect to Ω, of the 63 points in [5] correspond to 63 maximal subgroups (the above normalisers) of Φ. Each of the 30 planes on Ω corresponds to an elementary abelian subgroup, containing z_2 and of order 32.

The operations of z_2 must of course be closely linked to $\mathfrak{F}$; indeed one of them has already been noted, in (50.6), as the glide having G_0 latent. The other two involutions in z_2 are, as can easily be verified, those glides for which the other G, which contain D_0 and lie in M_0, are latent.

57. The 12 equations (50.3) provide, on stipulating that they are all symmetric in the suffixes, 8 conditions, for two operations in S_2^+ to commute. When these conditions are invoked to discover which operations of S_2^+ commute with every operation of the group it is found, as remarked in § 50, that all 12 marks save c' are zero; the centre z_1 of S_2^+ has order 2. One can now proceed with the upper, or ascending, central series (**15**, 50); the next stage is to identify the centre of S_2^+/z_1. Each element of S_2^+/z_1 is a coset of z_1 in S_2^+, i.e. a pair of operations differing only in their mark c'; the rule of multiplication in S_2^+/z_1 is therefore given by (50.3) without the last of its equations. Hence the centre of S_2^+/z_1 is found by insisting that, so far as (50.3) *minus* its last equation is concerned, an operation commutes with all others. There is no condition on b', but all other marks than b' and c' are found to be zero. Thus z_2, the subgroup of S_2^+ such that z_2/z_1 is the centre of S_2^+/z_1, is the 4-group whose operations have all their 12 marks zero save b' and c'; it has been encountered already as the centre of Φ. Next, in order to find the centre of S_2^+/z_2, one ignores the last two equations in (50.3); the remaining conditions for commutation are then satisfied so long as l, m, n, m', n', b, c all vanish.

Since a, a', l' now share the freedom of b', c' the centre z_3/z_2 of S_2^+/z_2 is of order 8 and z_3 is that subgroup of S_2^+ for which

$$l = m = n = m' = n' = b = c = 0.$$

It is, as (50.3) shows, elementary abelian and its order is 32. Next, in seeking the centre of S_2^+/z_3, only those equations for $\lambda, \mu, \nu, \mu', \nu', \beta, \gamma$ in (50.3) are relevant and the only conditions necessary to ensure their symmetry are

$$n_1 l_2 = n_2 l_1, \quad l_1 n'_2 = l_2 n'_1, \quad c_1 l_2 = c_2 l_1.$$

These only hold for all values of n_1, l_1, n'_1, c_1 if

$$n_2 = l_2 = n'_2 = c_2 = 0;$$

hence the subgroup z_4 of S_2^+ such that z_4/z_3 is the centre of S_2^+/z_3 is Φ. And as S_2^+/Φ is abelian it is its own centre. The ascending central series for S_2^+ is therefore

$$z_0 = I,\ z_1,\ z_2,\ z_3,\ z_4 = \Phi,\ z_5 = S_2^+. \tag{57.1}$$

One can derive the lower, or descending, central series (**15**, 156) too from (50.3). The 12 marks of the inverse of $\mathfrak{M}$ are found on replacing every Greek letter by 1 and solving the resulting 12 equations (50.3) for the marks with either suffix in terms of those with the other; the marks of $\mathfrak{M}_1^{-1}\mathfrak{M}_2^{-1}$ are those of the inverse of $\mathfrak{M}_1\mathfrak{M}_2$ with the two suffixes transposed; another appeal to (50.3) then yields, by a straightforward if if somewhat laboured process, those of

$$(\mathfrak{M}_1, \mathfrak{M}_2) = \mathfrak{M}_1^{-1}\mathfrak{M}_2^{-1}\mathfrak{M}_1\mathfrak{M}_2.$$

This commutator is known (**15**, 144) to belong to Φ; here it proves to be the whole of Φ. The mutual commutator group (Φ, S_2^+) then turns out to be z_3, and indeed the descending central series is simply (57.1) read from right to left. The coincidence of the two central series of a group is perhaps a consequence of such a distribution of zeros, in the matrices of some representation, that it may here be a consequence of (50.1).

The characteristic subgroup z_3 of S_2^+ can be identified by its effect on $\mathfrak{F}$. Those operations of S_2^+ that leave every point of d_0 invariant have

$$l = m' = n' = 0.$$

They permute the points of D_0 among themselves, and if one demands that every point in D_0 but outside d_0 is either invariant or transposed with the other point on its join to m_0 one finds the necessary conditions to be

$$m = n = b = c = 0.$$

58. If one disregards, for the moment, the centre z_1 of S_2^+ (it consists of identity and the glide (50.6)) the necessary and sufficient conditions for an operation to have period 2 are, by (50.4) and (50.5),

$$\text{(58.1)} \qquad \begin{cases} nl = cl = ln' = 0, \quad mn' = nm', \quad bn' = cm', \ nb = mc, \\ c(nm' + l') = an' + na', \\ am' + bl' + a'm + b'l = cmm'. \end{cases}$$

They are completely fulfilled if

$$\text{(58.2)} \qquad c = n = n' = 0 = am' + bl' + a'm + b'l,$$

equations which have 135 solutions - the number of points on STUDY'S quadric. Since, as will be explained immediately, the other solutions of (58.1) fall 16 under each of 7 heads there are $135 + 112 = 247$ in all, so that, since c' is not subjected to any constraint, there are, in addition to

(50.6), 494 operations of period 2 in S_2^+. The 7 heads are

(α) $c=n=l=0,\quad n'=1;\quad a=b=m=0.$

(β) $c=n'=l=0,\quad n=1;\quad a'=b=m'=0.$

(γ) $n=n'=l=0,\quad c=1;\quad l'=m=m'=0.$

(δ) $c=n=1,\quad n'=l=0;\quad m'=0,\quad m=b,\quad l'=a'.$

(ε) $c=n'=1,\quad n=l=0;\quad m=0,\quad m'=b,\quad l'=a.$

(ζ) $n=n'=1,\quad c=l=0;\quad b=0,\quad m=m',\quad a=a'.$

(η) $c=n=n'=1,\quad l=0;\quad m=m'=b,\quad l'=a+a'+b.$

Operations of Φ cannot occur under any of these heads, but only under (58.2) and then only when $l=0$. They are, excepting (50.6), mapped two by each point on the section of a Study quadric by one of its tangent primes; the contact B' of the prime maps those two members of the centre z_2 of Φ that do not belong to z_1, and the remaining 140 operations of period 2 are mapped two by each of 70 points collinear in 35 pairs with B'. Each of the 70 points maps a coset of z_1 in Φ, and the two cosets so mapped by points collinear with B' form a coset of z_2: the projection from B' gives the mapping already encountered in § 56.

59. Consider now the solutions of (58.2). When $c=n=n'=0$ the latent vectors of $\mathfrak{M}$ satisfy

(59.1) $$mY_3=l'Y_4,\quad lY_3=m'Y_4,\quad bY_3=a'Y_4,\quad aY_3=b'Y_4;$$

(59.2) $$\begin{cases} bY_1+aY_2+(al+bm)Y_3+c'Y_4=mZ_1+lZ_2\\ a'Y_1+b'Y_2+c'Y_3+(a'l'+b'm')Y_4=l'Z_1+m'Z_2\end{cases}$$

Since one is disregarding the special circumstance of all 8 marks being zero the rank r of

(59.3) $$\begin{bmatrix} b & a & m & l\\ a' & b' & l' & m'\end{bmatrix}$$

is either 1 or 2, and the discussion bifurcates accordingly.

Suppose that $r=1$

There are 3 mutually exclusive possibilities:

$$(i)\ a=b=l=m=0,\quad (ii)\ a'=b'=l'=m'=0,$$

(iii) Each column consists of equal marks, and in at least one column these are 1.

Take these possibilities in turn.

(i) The latent space is

$$Y_4=a'Y_1+b'Y_2+c'Y_3+l'Z_1+m'Z_2=0, \tag{59.4}$$

the polar of the join of m_0 to $(l', m', 0, 0, a', b', c', 0)$, a point on or off $\mathfrak{S}$ according as $a'l'+b'm'$ is 0 or 1, but in any event conjugate to m_0. So there occur 18 G and 12 T, the polars of those 18 g (other than g_0) and 12 t that pass through m_0 and lie in G_0; the associated 30 operations of S_2^+ all, since $l=0$, belong to Φ.

(ii) Here too there occur 18 G and 12 T, the polars of those 18 g (other than g_0) and 12 t that lie in G_0 and pass now through m_1. Those 18 operations having a latent G occur when $al+bm=0$; $l=0$ for 10 of these. Those 12 operations having a latent T occur when $al+bm=1$; $l=0$ for 4 of them. Hence 14 of the 30 operations belong to Φ; their latent spaces are the polars of those 10 g (other than g_0) and 4 t which pass through m_1 and lie not merely in G_0 but in D_0.

(iii) The latent space is

$$Y_3=Y_4,\quad bY_1+aY_2+(al+bm+c')Y_4=mZ_1+lZ_2.$$

The results are precisely as in *(ii)* - 30 operations of which 14 are in Φ - with m_1 replaced by m_2.

60. Suppose now that $r=2$: not all of

$$p_{14}=la'+bm' \qquad p_{24}=lb'+am' \qquad p_{34}=ll'+mm'$$

$$p_{23}=al'+mb' \qquad p_{13}=bl'+ma' \qquad p_{12}=aa'+bb'$$

are zero and, by (58.2), $p_{24}=p_{13}$. Thus one may take these p_{ij} to be PLÜCKER coordinates of the lines in a screw (4, § 8) or linear complex in a projective 3-space P over $\boldsymbol{F}$. Since a screw comprises 15 lines, and since a line over $\boldsymbol{F}$ provides 6 ordered pairs of points, one obtains 90 matrices

(59.3) and, c' being free as usual, 180 matrices $\mathfrak{M}$. These, with the 3 sets of 30 found when $r=1$, make up the whole set of 270 operations of period 2 that are permitted by (58.2).

The number of operations among the 180 that belong to Φ is found by requiring that $l=0$. If (ξ, η, ζ, τ) are coordinates in P one has to take a point-pair, on a line Λ of the screw, whose first member is the intersection of Λ with $\tau=0$ - except that, in the event of Λ lying wholly in $\tau=0$, any ordered pair on Λ is eligible. Since there are (4, § 9) 3 of the 15 lines of the screw in any plane the number of ordered point-pairs available is $12.2+3.6=42$; of the 180 $\mathfrak{M}$ 84 belong to Φ. And so we find again that, apart from (50.6), there are in Φ $30+14+14+84=142$ operations of period 2.

When $r=2$ (59.1) and (59.2) reduce to $Y_3=Y_4=0$ and

$$\text{(60.1)} \qquad \begin{cases} bY_1+aY_2=mZ_1+lZ_2 \\ a'Y_1+b'Y_2=l'Z_1+m'Z_2 \end{cases}$$

which, r being 2, can be solved for at least one pair of Y_1, Y_2, Z_1, Z_2 as linear combinations (possibly zero) of the others; each of the 180 operations has a latent solid σ in G_0. (60.1) is interpretable as equations of a line Λ^* in a projective 3-space P^*, and σ is ω, γ, φ or χ according as Λ^* lies on, touches, is a chord of or is skew to $\mathcal{H}$, the hyperboloid $Y_1Z_1=Y_2Z_2$ in P^*. But, by (58.2), Λ^* satisfies $p^*_{13}=p^*_{24}$ and the null polarity for this screw is also that for $\mathcal{H}$. The 15 lines Λ^* are therefore the 6 generators and 9 tangents of $\mathcal{H}$, and there are 6ω and 9γ - those passing through g_0 and lying in G_0 - each latent for 12 operations of S_2^+.

Among those 6ω and 9γ there occur, passing through d_0 and lying in D_0, ω_0, ω_0' and γ_0; they give $l=m'=0$ in (60.1) so that the 36 operations for which they are latent belong, all of them, to Φ. But any other of the 6ω and 9γ corresponds to a Λ^* not containing $Y_1=Y_2=Z_1=0$; one then takes the first equation (60.1) to be the plane joining Λ^* to this point. thereby ensuring that $l=0$, and the second to be either of the other two planes through Λ^*. Of the 12 operations of S_2^+ having the solid for latent space only 4 belong to Φ. And so one again obtains the $3.12+12.4=84$ operations of period 2 already found.

61. These discussions in §§ 58-60, involving as they do all the operations of period 2 in Φ, cover the whole of z_3 and enable one to define, in relation to $\mathfrak{F}$, all the operations of this important and characteristic subgroup of

S_2^+. That identity and the glides with axes g_0, g_1, g_2, compose z_2 is already known; those operations of z_3 that are not in z_2 and have latent spaces of dimension 5 are

4 involutions, the polars of whose latent T are those t which pass through m_0 and lie in γ_0;

8 glides, whose axes are those g through m_0 that lie in D_0 but not in d_0;

4 glides, whose axes are those g in d_0 that do not pass through m_0.

For such operations only occur when (59.3), here specialised to

$$\text{(61.1)} \qquad \begin{bmatrix} \cdot & a & \cdot & \cdot \\ a' & b' & l' & \cdot \end{bmatrix}$$

has rank 1. This is so when *(i)* $a=0$, and then the latent space is the polar of the join of m_0 to $(l', 0, 0, 0, a', b', c', 0)$. The join is in D_0; it is not in d_0 unless $a'=l'=0$, but this relegates the operation to z_2. The 4 t in γ_0 occur when $a'=l'=1$; otherwise the join is in $\omega_0(a'=1,\ l'=0)$ or $\omega_0'(a'=0,\ l'=1)$. If $a=1$ the rank of (61.1) is 1 when *(ii)* $a'=b'=l'=0$, giving those two glides whose axes lie in d_0, pass through m_1 and are distinct from g_0; and when *(iii)* $a'=l'=0$, $b'=1$, giving those two glides whose axes lie in d_0, pass through m_2 and are distinct from g_0.

When (61.1) has rank 2, $a=1$ and a', l' cannot both be zero; the latent solid is

γ_0 when $a'=l'=1$;

ω_0 when $a'=1$, $l'=0$; ω_0' when $a'=0$, $l'=1$.

Since b', c' both stay free 4 operations occur for each solid; the 12 complete z_3.

It remains, in order to describe z_3 fully, to explain how to identify, in each of the 3 instances, the 4 among those 28 operations in A^+ for which the solid is latent. That each of the 28 operations for which γ_0 is latent is the product of 4 commuting projections centred at the vertices of either of two tetrahedra was seen in § 25; the tetrahedra together account by their vertices for all 8 p in γ_0 and are in perspective from a point of d_0. Each of the 7 m in d_0 is the centre of perspective for 4 such pairs of tetrahedra, and it is when this m is m_0 that an operation in z_3 results.

62. The identification of operations of z_3 having ω_0 or ω_0' latent uses a mode of generating them, just as above a mode of generating operations having γ_0 latent was essential to their description. One must therefore first expound this elementary topic.

It was seen in § 24 that glides commute when their axes meet; but they commute also when their axes g_1, g_2 are skew and the solid $[g_1g_2]$ is an ω. Since G_1. G_2 both contain ω every point of ω is invariant for both glides.

Take any point A outside both G_1 and G_2. Let m_1, m_2 be the intersections of g_1, g_2 with the polar prime of A; A_1, A_2 the remaining points on Am_1, Am_2; A_{12} the intersection of A_1m_2 and A_2m_1. The first glide, as remarked in § 24, transforms A into A_1; since m_2 is conjugate both to m_1 and to A it is conjugate to A_1, so that the second glide transforms A_1 into A_{12}; the two glides combined (in this order) transform A into A_{12}. But so they do, by similar reasoning, when combined in the opposite order: the two products have the same effect on any point which lies either in both or in neither of G_1 and G_2. Indeed they have the same effect on every point in [7]. If A lies, say, in G_1 but not in ω its transform A_2 under the second glide lies in the plane Ag_2 and so, with A and g_2, in G_1; the first glide leaves both A and A_2 invariant and both products transform A into A_2.

A is distinct from A_{12}; it is only points of ω that are invariant under the product of the glides, and so the product of any two glides whose axes are skew lines in ω is an involution of class LXI. But whereas there are only 28 such involutions with ω latent there are

$$\frac{1}{2}.35.16 = 280$$

pairs of skew lines in ω; each involution is expressible as the product of glides with skew axes in 10 different ways. The explanation is that there are in ω (**4**, § 9) 28 screws each having 10 pairs of polar lines; products of pairs of glides with skew axes in ω are the same involution when the pairs of axes are polars in the same screw.

To return, now, to z_3. Of the 28 screws, whether in ω_0 or in ω_0', 4 provide involutions - products of glides whose axes are polars in the screw - that belong to z_3. The 4 screws are those to which g_0, g_1, g_2 belong; i.e. those in which the null plane of m_0 is d_0.

63. The preceding paragraphs do not merely calculate numbers of operations of period 2 in S_2^+ and Φ: they identify such operations by their latent space σ and its relation to $\mathfrak{F}$. Each of the 7 heads of § 58 provides

32 operations of period 2 that can be similarly identified, and these heads are now taken *seriatim*. None of these 224 operations belongs to Φ.

(α) The equations of σ are

$$Y_4 = a'Y_1 + b'Y_2 + c'Y_3 + l'Z_1 + m'Z_2 + Z_3 = 0;$$

it is the polar [5] of the line joining m_0 to $(l', m', 1, 0, a', b', c', 0)$. This is one of those 32 lines through m_0 which lie in M_0 but outside G_0: the line is g if $c' = a'l' + b'm'$, t otherwise. The 32 operations under this head have therefore as latent spaces those 16 G and 16 T in M_0 that contain m_0 but not g_0.

(β) The equations of σ are

$$\begin{aligned} Y_2 + mY_3 + l'Y_4 &= 0, \\ Z_1 \qquad + aY_3 + b'Y_4 &= 0, \\ mZ_1 + aY_2 \qquad + c'Y_4 &= 0, \\ l'Z_1 + b'Y_2 + c'Y_3 \qquad &+ 0, \end{aligned}$$

whose rank is 4 or 2 according as $al' + b'm$ is $c' + 1$ or c'. In the former instance σ is ω'_0, latent for 16 operations. In the latter σ is the [5]

$$Y_2 = mY_3 + l'Y_4, \quad Z_1 = aY_3 + b'Y_4.$$

This is one of those 16 G whose polar g lie in ω'_0 though not in d_0: moreover the intersection of g with d_0 is one of the 4 points not on g_0.

(γ The situation is as in (β). Of the 32 operations 16 have ω_0 for their latent space. The other 16 have each a latent G; the polars of these G are those 16 g, 4 through each point of d_0 not on g_0, that lie in ω_0 but not in d_0.

(δ) The equations of σ are

$$\begin{aligned} Y_2 + bY_3 + a'Y_4 &= 0, \\ Z_1 + Y_1 + Y_2 + aY_3 + b'Y_4 &= 0, \\ b(Z_1 + Y_1) + aY_2 + b'Y_3 + c'Y_4 &= 0, \\ a'(Z_1 + Y_1) + b'Y_2 + c'Y_3 + a'Y_4 &= 0. \end{aligned}$$

so that σ is the solid

$$Z_1 + Y_1 = Y_2 = Y_3 = Y_4 = 0,$$

i.e. γ_0, when $c' = aa' + bb' + a'b + 1$; this accounts for 16 operations. The others have each a latent [5]

$$Y_2 = bY_3 + a'Y_4, \quad Y_1 + Z_1 = (a+b)Y_3 + (a'+b')Y_4.$$

These 16 [5]'s are T, being polars of those 16 t in γ_0 that meet d_0 4 at each of the 4 points off g_0.

(ε) The equations of σ are

$$Y_1 = aY_3, \quad Y_2 = bY_3, \quad Y_4 = 0,$$

$$Z_3 = aZ_1 + bZ_2 + (aa' + bb' + c')Y_3,$$

a solid which is ω' or γ according as c' is $aa' + bb'$ or not; each solid is latent for 4 of the 32 operations. For there are 4 solids ω', namely those meeting ω_0 in the planes other than d_0 containing either g_1 or g_2, and 4 solids γ, one through each of these same planes.

(ζ) The situation is as in (ε); but now the 4 planes, other than d_0, through either g_1 or g_2 are in ω'_0 instead of in ω_0.

(η) The equations of σ are

$$Y_4 = 0, \quad Y_2 = bY_3, \quad Z_1 = Y_1 + (a+b)Y_3,$$

$$Z_3 + bZ_2 = (a+b)Z_1 + (aa' + ba' + bb' + b + c')Y_3;$$

these provide, where the marks take different values, 8 solids γ, each latent for 4 of the 32 operations. γ_0 and σ meet in the plane

$$Y_1 + Z_1 = Y_2 = Y_3 = Y_4 = 0, \quad Z_3 + bZ_2 = (a+b)Z_1;$$

this can be any of those 4 e, two ($b = 0$) through g_1 and two ($b = 1$) through g_2, that lie in γ_0. Since, in addition to γ_0, there are 2γ through each e the latent solids are identified.

64. All operations of S_2^+ belong to A^+ and have for their period some power of 2; they are distributed among the classes

I, III, VIII, XI, XIII, XV, XXVIII, XXXV, XXXVI, XXXVII, LXI, LXII with respective periods

$$1,\ 2,\ 2,\ 4,\ 2,\ 4,\ 4,\ 8,\ 4,\ 8,\ 2,\ 4$$

and latent spaces

$$[7],\ T,\ \gamma,\ \psi,\ G,\ \chi,\ g,\ t,\ \varphi,\ t,\ \omega,\ g.$$

Those of periods 1, 2, 8 have all been accounted for, the contributions to the classes being

1 to I, 68 to III, 188 to VIII, 103 to XIII, 128 to XXXV,
128 to XXXVII, 136 to LXI.

There remains a total of 3344 operations of period 4, distributed among classes XI, XV, XXVIII, XXXVI, LXII.

Some of these have been encountered too; for instance the 192 in class XXVIII and 384 in class LXII for which g_0 is latent. One might now enquire whether operations of S_2^+ have latent some g other than g_0 but, since $\mathfrak{F}$ is to be invariant, no space can be latent for any operation of S_2^+ unless it includes m_0. Take, then, either of those g other than g_0 which pass through m_0 and lie in d_0; say

$$g_1:\ Y_1 = Y_2 = Y_3 = Y_4 = Z_1 = Z_3 = 0,$$

and demand that a point on g_1 other than m_0 is invariant when the column vector of its coordinates is premultiplied by $\mathfrak{M}$. Then $l = m' = 0$. Furthermore, since m_1 and m_2 on g_0 have to be transposed, $n' = 1$. Next, in order that the latent space be g_1 itself and not a space of higher dimension, the rank of $\mathfrak{M} + \mathrm{I}$ must be 6, which it is seen to be if, and only if, with $l = m' = 0$ and $n' = 1$, $bn + cm = 1$. The remaining 5 marks are unrestricted, so that $6.2^5 = 192$ operations are obtained. An argument similar to, though rather shorter than, that of § 53 shows each of ω_0, ω'_0, γ_0 to be latent for the squares of 64 of these operations. Since g_2 could have served as latent space just as g_1 has done one accounts for 128 members of class XXVIII and 256 of class LXII.

The line

$$Y_1 = Y_2 = Y_3 = Y_4 = Z_2 = Z_3 = 0, \text{ passing through } m_0 \text{ and lying in } D_0,$$

does not lie in d_0; it is one of 8 such g, 4 in ω_0 and 4 in ω_0'. Its points are all invariant under $\mathfrak{M}$ when $l' = m = n = 0$ and then $\mathfrak{M} + \mathrm{I}$ has rank 6 if, and only if $n' = l = c = 1$. These restrictions allow 64 $\mathfrak{M}$ which, again by arguing as in § 53, belong 32 to each of the classes XXVIII and LXII; so one accounts for 256 more of each class.

65. This procedure, of using latent spaces and their relation to $\mathfrak{F}$ to determine operations in S_2^+, calls on the geometry to classify these operations and distinguish them from each other. The main outlines of some instances will now be given, although one does not probe matters down to the least detail.

Take the latent space to be ψ; it meets $\mathfrak{S}$ in 3 g concurrent at an *apex*. This apex may, or may not, be m_0; if it is not, its join to m_0 is one of the 3 g; if it is. g_0 may, or may not, lie in ψ. There are $75600/135 = 560\,\psi$ with a given m as apex; of these, $560 \times 3/35 = 48$ include a given g through m. There are (the dual statement is registered in Table I) 24 ψ through a plane e, and each of the 3 m in e is the apex of 8 of them; hence, as there pass through g_0 two e in γ_0, there are, among those 48 ψ having apex m_0 and containing g_0, 16 that meet γ_0 in planes. The other 32 meet γ_0 in g_0 only. Alternatively: take m_0 to be the apex, so that ψ lies in $Y_4 = 0$. Each g in ψ meets $Z_4 = 0$ and no two of these 3 intersections are conjugate. If it is also stipulated that g_0 belongs to ψ the solid is spanned by 4 points

$$(65.1)\qquad \left\{\begin{array}{cccccccc} . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & 1 & . \\ y_1 & y_2 & 1 & . & z_1 & z_2 & z_3 & . \\ y_1' & y_2' & 1 & . & z_1' & z_2' & z_3' & . \end{array}\right.$$

where

$$z_3 = y_1 z_1 + y_2 z_2, \quad z_3' = y_1' z_1' + y_2' z_2',$$

$$y_1 z_1' + y_1' z_1 + y_2 z_2' + y_3' z_2 = z_3 + z_3' + 1,$$

implying

$$(65.2)\qquad (y_1 + y_1')(z_1 + z_1') + (y_2 + y_2')(z_2 + z_2') = 1.$$

Thus y_1, y_2, z_1, z_2 can each be either 0 or 1 and z_3 is then known; (65.2) then allows 6 sets of marks for y_1', y_2', z_1', z_2' and so z_3' is determined. Hence there are $2^4\cdot 6 = 96$ ways of filling the two bottom rows in (65.1) and, as these can be transposed without changing the solid, there are 48 eligible ψ. There is a single e in ψ and containing g_0; it joins g_0 to

$$y_1 + y_1',\ y_2 + y_2',\ .\ .\ z_1 + z_1',\ z_2 + z_2',\ z_3 + z_3',\ .$$

which is not in γ_0 unless $y_2' = y_2$. It then follows from (65.2) that $y_1' = y_1 + 1$ and $z_1' = z_1 + 1$; only 2 of the 6 ways of satisfying (65.2), and so only 16 of the 48 ψ are now available. Two among these 16 ψ occur on putting

$$\alpha = 0,\ 1 \ \text{ in } \ Y_1 + Z_1 = Y_2 = Y_4 = 0, \quad Z_2 = \alpha Z_1 .$$

The requirement that every point of this solid be invariant demands

$$a = b = c' = l = m = n' = 0,$$

and the further requirement that no point outside the solid be invariant demands

$$c = n = m' = 1, \quad l' = a' + \alpha.$$

So one accounts, 4 $\mathfrak{M}$ being available for each ψ, for 64 operations of class XI.

There are 16 ψ having m_0 for apex which meet d_0 in g_1 and γ_0 in a plane; one is

$$Y_1 + Z_1 = Y_3 = Z_3 = Y_4 = 0.$$

The conditions for it to be the latent solid of an operation (50.1) of S_2^+ are

$$a = b' = c = l = m' = n = 0, \quad b = m = n' = 1, \quad a' = l',$$

so that 4 $\mathfrak{M}$ are permissible. Thus 64 operations occur, and there is another set of 64 related to g_2 as this set is to g_1

Next, still retaining m_0 for apex, take any of the 4 g outside d_0 in ω_0 and any of the 4 g outside d_0 in ω_0'. A further g, joined to both the selected ones by planes f yet lying in G_0, can be chosen in 2 ways; for example, if the g in ω_0 and ω_0' join m_0 to (. . . . 1 . . .) and (1 . . . .) the further g joins m_0 to (1 1 . . 1 1 α .) where α can be either 0 or 1. These g lie in

$$Y_2 + Z_2 = Y_3 = Y_4 = 0, \quad Z_2 = \alpha Z_1;$$

the conditions on $\mathfrak{M}$ are

$$a' = b = c = l' = m = n = 0, \quad a = l = n' = 1, \quad m' = b' + \alpha$$

so that 4 matrices are permitted. This type of ψ therefore accounts for $4.4.2.4 = 128$ operations of S_2^+.

If m_1 is the apex ψ necessarily includes g_0; among such ψ are 16 whose intersection with γ_0 is a plane. Two of these ψ occur on putting $\alpha = 0,1$ in

$$Y_1 + Z_1 = Y_2 = Y_3 = 0, \quad Z_2 = \alpha Z_1;$$

this solid is found to be latent when, in $\mathfrak{M}$,

$$a' = b' = c' = l' = m' = n' = 0, \quad c = n = l = 1, \quad b = \alpha l + m;$$

since a and m stay free there are 4 such $\mathfrak{M}$. Since m_2 could have filled the role of apex just as m_1 has done the number of operations of S_2^+ that fall under this head is $2.16.4 = 128$.

This discussion, occupying § 65, shows there to be (at least, though in fact the number is exact) 448 members of class XI in S_2^+.

66. It was shown in § 55 that each of χ_0, χ_0' is latent for 4 operations of Φ. In order to find how many operations of S_2^+ have χ_0 for their latent space one notes that

$$(\xi,\ \xi+\eta,\ 0,\ 0,\ \eta,\ \xi+\eta,\ \zeta,\ \tau),$$

which traces χ_0 as ξ, η, ζ, τ vary, has to be invariant, under premultiplication by $\mathfrak{M}$, for all ξ, η, ζ, τ in $\boldsymbol{F}$; this is not so unless

$$c = n = n' = 0, \quad l = a + b, \quad l' = a', \quad m = b, \quad m' = n' + b'.$$

Further, in order that no point outside χ_0 be invariant it is necessary that $aa' + bb' = 1$. As this equation has 6 solutions, and c' is free, 12 operations of S_2^+ have χ_0 for their latent space. Those for which $l = 0$, and so $a = b$, are the ones in Φ. The same holds for χ_0'.

The solid $Y_1 + Z_1 = Y_3 = Z_3$, $Y_2 = Y_4 = 0$ is one of the two χ through g_1; it provides 4 operations of S_2^+ for which it is latent, namely those with

$$b = l = m = m' = 0, \quad a = c = n = n' = 1, \quad a' = l' = c' + 1,$$

both b' and c' staying free. Since 4 operations are so associated with each χ through g_1 and each χ through g_2 there are 16 operations of class XV whose effect on $\mathfrak{F}$ is different from that of the 24 having χ_0 and χ_0' as their latent solids.

67. The class to afford, geometrically speaking, the most diverse operations of S_2^+ is XXXVI. The latent solid φ meets $\mathfrak{S}$ in 2 planes whose common line V can be called the *hinge*; if m_0 is not on V its join to V lies on $\mathfrak{S}$ and is one of the two planes constituting the section. If m_0 does lie on V this hinge may be g_0, or one of g_1 and g_2, or one of those $8\,g$ in D_0 but not in d_0, and so on; and after V is chosen there remains some freedom of choice for the two planes. To span φ one requires two points on V and then two further points on $\mathfrak{S}$ not coniugate to one another but both conjugate to both points on V.

Suppose – an obvious first choice – V to be g_0. One sees, on observing the section $\mathcal{H}$ of $\mathfrak{S}$ by a solid $\varkappa$ in G_0 skew to g_0, that φ can belong to any of 4 categories. For $\mathcal{H}$ consists of 6 lines, 3 in each of its complementary reguli; those passing through the intersection of $\varkappa$ with d_0 belong one to ω_0 and the other to ω_0', and the φ hinged on g_0 join it to the 18 chords of $\mathcal{H}$. They are distributed as

(*a*) 4 which contain d_0;

(*b*) 4 containing one plane in ω_0 and another in ω'_0;

(*c*) 8 whose 2 planes of intersection with $\mathfrak{S}$ lie one in either ω_0 or ω_0', the other in neither;

(*d*) 2 meeting $\mathfrak{S}$ in planes neither of which is in either ω_0 or ω_0'.

Each of these 18 φ is found to be latent for 12 operations; it will appear that these 216 operations of S_2^+ include the 104 that were seen in § 55 to belong to Φ, the categories respectively contributing 48, 16, 32, 8. Instances are as follows.

(*a*) $Y_1 = Y_3 = Y_4 = Z_1 = 0$, latent for those 12 $\mathfrak{M}$ wherein

$$a = c = l = n = b' = m' = n' = 0, \quad a'm + bl' = 1$$

(*b*) $Y_2 = Y_3 = Y_4 = Z_2 = 0$, latent for those 12 $\mathfrak{M}$ wherein

$$a' = b = c = l' = m = n = n' = 0, \quad am' + b'l = 1.$$

(*c*) $Y_1 = Y_2,\ Y_3 = Y_4 = Z_2 = 0$, latent for those 12 $\mathfrak{M}$ wherein

$$c = m = n = l' = n' = 0, \quad a = b, \quad a' = b', \quad a'l + bm' = 1.$$

(*d*) $Y_1 + Y_2 = Z_1,\ Y_3 = Y_4 = Z_2 = 0$, latent for those 12 $\mathfrak{M}$ wherein

$$c = n = n' = 0, \quad a = b = m, \quad a' = b' = l', \quad ll' + mm' = 1.$$

Every operation under (*a*) belongs to Φ; of the other sets of 12 those 4 members of the set having $l = 0$ belong to Φ.

The same 4 categories apply to those 18 φ hinged either on g_1 or on g_2; but now, while under (*a*) φ is again latent for 12 operations, it is latent for only 4 under the other headings. Instances, with g_1 as hinge, as as follows.

(*a*) $Y_1 = Y_2 = Y_4 = Z_1 = 0$, latent for those 12 $\mathfrak{M}$ wherein

$$a = b = c' = l = m = m' = n' = 0, \quad na' + cl' = 1.$$

(*b*) $Y_2 = Y_3 = Y_4 = Z_3 = 0$, latent for those 4 $\mathfrak{M}$ wherein

$$a' = b = c = l = m = n = l' = m' = 0, \quad a = n' = 1.$$

(*c*) $Y_1 = Y_3$, $Y_2 = Z_1 = Z_3 = 0$, latent for those 4 $\mathfrak{M}$ wherein

$$b = l = m = n = l' = m' = 0, \quad a = n' = 1, \quad c = a, \quad c' = a'.$$

(*d*) $Y_1 + Y_3 = Z_1$, $Y_2 = Y_4 = Z_3 = 0$, latent for those 4 $\mathfrak{M}$ wherein

$$b = l = m = m' = 0, \quad c = a = n = n' = 1, \quad c' = a' = l'.$$

The total number of operations of S_2^+ with a latent φ hinged on g_1 is therefore

$$4.12 + 4.4 + 8.4 + 2.4 = 104,$$

and, since there is an equal number with a latent φ hinged on g_2, 208 operations of S_2^+ are accounted for.

One of those 8 g through m_0 which lie in D_0 though outside d_0 is

$$Y_1 = Y_2 = Y_3 = Y_4 = Z_2 = Z_3 = 0;$$

this, which is in ω_0, can serve as a hinge. If one of the two planes through it that compose the section of $\mathfrak{S}$ by φ lies in ω_0 this plane meets d_0 in one of g_0, g_1, g_2; in any event there are 4 choices for the companion plane, and each φ is latent for 4 operations of S_2^+. If φ contains g_0 it can be any of the 4 solids which arise on taking either 0 or 1 for α, β in

$$Y_1 = Y_4 = 0, \quad Y_2 = \alpha Y_3, \quad Z_2 = \beta Y_3,$$

a solid which is latent for those 4 $\mathfrak{M}$ having

$$a = b = l' = m = n = n' = 0, \quad c = m' = 1, \quad b = \alpha, \quad c' = \alpha b' + \beta.$$

If φ contains g_1 it can be any of

$$Y_1 = Y_4 = 0, \quad Y_3 = \alpha Y_2, \quad Z_3 = \beta Y_2,$$

latent for those 4 $\mathfrak{M}$ having

$$a = l = m = n = l' = m' = 0, \quad b = n' = 1, \quad c = \alpha, \quad b' = \alpha c' + \beta.$$

And analogously if φ contains g_2. As there are 8 choices for the hinge one accounts for $8.3.4.4 = 384$ operations of S_2^+.

Given any of the $8\,g$ through m_0 that lie in G_0 but outside D_0 there are, among the $18\,\varphi$ having g for hinge, 4 such that, of the two planes constituting the section of $\mathfrak{S}$ by φ, one meets ω_0 and the other meets ω_0' in a line. As each such φ is latent for 4 operations of S_2^+ a further 128 operations are accounted for. If the hinge is

$$Y_1 = Y_3 = Y_4 = Z_1 = Z_2 = Z_3 = 0$$

the 4 relevant φ have equations

$$Y_3 = Y_4 = Z_2 = 0, \quad Z_3 = \alpha Y_1 + \beta Z_1;$$

and this is latent for those 4 $\mathfrak{M}$ having

$$a = b = c = m = n = b' = 0, \quad n' = l = 1, \quad a' = \alpha, \quad l' = \beta.$$

68. Suppose now that the hinge does not pass through m_0, but that it is one of those $4\,g$ not passing through m_0 that lie in d_0. Then φ must contain d_0, and there are 4 possibilities. If the hinge is

$$Y_1 = Y_2 = Y_3 = Y_4 = Z_1 = Z_4 = 0$$

the 4 φ are found by taking α, β to be either 0 or 1 in

$$Y_2 = Y_3 = 0, \quad Y_1 = \alpha Y_4, \quad Z_1 = \beta Y_4.$$

This solid is latent for the 12 $\mathfrak{M}$ having

$$a' = l = l' = m' = n' = 0, \quad b' = \alpha c + \beta n, \quad c' = \alpha b + \beta m, \quad bn + cm = 1.$$

As there are 4 eligible hinges 192 operations of S_2^+ are accounted for.

Among g which neither pass through m_0 nor lie in d_0 are 16, 8 through each of m_1 and m_2, that meet g_0 and lie in D_0. One of them is

$$Y_1 = Y_2 = Y_3 = Y_4 = Z_2 = Z_4 = 0$$

and if φ has this hinge and contains m_0 it is one of the 4 solids

$$Y_1 = Y_3 = 0, \quad Y_2 = \alpha Y_4, \quad Z_2 = \beta Y_4.$$

This is latent for those 4 $\mathfrak{M}$ having

$$m = n = l' = m' = n' = b' = 0, \quad c = l = 1, \quad a' = \alpha, \quad c' = \alpha a + \beta,$$

and one thereby accounts for 256 operations of S_2^+. The aggregate of operations in S_2^+ that have, in these §§ 67, 68, been recognised as belonging to class XXXVI is

$$216 + 208 + 384 + 128 + 192 + 256 = 1384.$$

The 3344 operations of period 4 that, as remarked in § 64, belong to S_2^+ have all been found; the classes

XI, XV, XXVIII, XXXVI, LXII

contribute, respectively,

448, 40, 576, 1384, 896.

69. The 4096 operation of S_2^+ can now, when the relation of each latent space to $\mathfrak{F}$ is known, be partitioned into sets: not necessarily, at a first essay, into the conjugate classes of S_2^+ but certainly into sets each of which is a union of conjugate classes, no two operations being conjugate in S_2^+ unless they belong to the same set. Should any further refinement of these sets, separating conjugate classes from each other, be desired (50.3) enables one to find the normaliser in S_2^+ of any chosen operation; the index of this normaliser is the number of operations in the conjugate class to which the chosen operation belongs. The number in any conjugate class must, of course, be a power of 2.

It is natural to carry out the partitioning by ascending the upper central series. Each of the two members of z_1 is itself a conjugate class, and a third set consists of the two members of z_2 outside z_1. As for the 28 members of z_3 outside z_2, the discussion in §§ 61, 62 partitions them into 7 sets of 4; a glide whose axis is in ω_0 but not in d_0 cannot be conjugate in S_2^+ to one whose axis is in ω_0'.

There are 112 operations of period 2 in Φ that are not in z_3. Of those with a latent [5] 16, 8 of them glides, come under (*i*) of § 59 while 12, including 4 glides with axes in ω_0 and 4 with axes in ω_0', come under each of (*ii*) and (*iii*); as these latter heads include operations that are conjugate in S_2^+ the 40 operations of Φ, of period 2, outside z_3 and with latent [5]'s, fall into 5 sets of 8. Next: each of ω_0, ω_0', γ_0 is latent for 8 operations of Φ not in z_3, and the argument at the end of § 60 serves to partition the remainder as $8 + 8$ with latent ω and $8 + 8 + 16$ with latent γ. So the 112 operations fall into 12 sets of 8 and one of 16. As for the operations of period 4 in Φ 8 of them have a latent χ and the other 104 fall, by § 67 and the fact that ω_0 and ω_0' cannot be transformed either into the other under S_2^+, into sets of

$$48,\ 8,\ 8.\ 16,\ 16,\ 8.$$

It can then be shown, using (50 3), that one of the set of 48 has a normaliser of order 2^8, so that the set is the union of 3 conjugate classes with 16 members in each. And so on.

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