NOTES ON A NET OF QUADRIC SURFACES: (IV) COMBINANTAL COVARIANTS OF LOW ORDER

By W. L. EDGE.

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1. Some of the combinantal covariants of a net $\mathfrak{N}$ of quadric surfaces have already been discussed in this series of Notes*; they are all of order eight or more, those actually of order eight being

R^8: the scroll of trisecants of the Jacobian curve;

E^8: the locus of equianharmonic base curves of pencils of quadrics belonging to $\mathfrak{N}$;

and ρ^8: a scroll generated by quadritangent lines of R^8.

To these we may add, from an earlier paper†, a fourth combinant,

Π^8: the product of the eight tritangent planes of R^8.

There seems to be no record, for a general net of quadrics, of any combinantal covariant whose order is less than eight; in the present Note we obtain combinantal covariants of orders four and six. It is not intended to give a complete list of combinants, but just to register those which present themselves most readily.

The Note falls into four sections, or rather into three sections to which an appendix (§§ 19–23) has been added; in this appendix the opportunity has been taken of applying some of the considerations of the

* The three preceding Notes appeared as follows: (I) *Proc. London Math. Soc.* (2), 43 (1937), 302–315; (II) *Journal London Math. Soc.*, 12 (1937), 276–280; (III) *Proc. London Math. Soc.* (2), 44 (1938), 466–480.

† *Proc. London Math. Soc.* (2), 34 (1932), 492–525.

Note proper to the particular net of quadrics whose eight base points form a pair of Möbius tetrads. Most of the results for the sake of which the Note was written appear in the third section (§§ 14–18). Before these can be obtained, however, the ground has to be prepared; hence the first section (§§ 2–6) is, except for § 4, mostly concerned with explaining and co-ordinating those results obtained by other writers which are essential for the prosecution of the work. § 4 itself is in the nature of a digression, in the course of which some remarks are made about combinantal invariants of $\mathfrak{N}$.

One of the quartic surfaces which arises as a combinant of $\mathfrak{N}$ has the property of having all the twenty-eight lines of intersection of the pairs of eight associated planes as bitangents. The second section of the Note is concerned with establishing this property, the recognition of which is surely overdue.

I.

2. No systematic search for the combinants of three quadrics has yet been made. Turnbull* and, after him, Williamson†, have studied the complete system of concomitants of three quadrics; but the combinantal forms cannot be selected forthwith from the lists of concomitants that they obtain, because the algebraic standpoint of these authors gives an utterly different view of the system from that given by the geometrical standpoint adopted here‡. We obtain combinantal covariants as surfaces defined by some geometrical relation to the net of quadrics as a whole; for us the order to which the coefficients in the equation of a quadric of the net enter in the equation of such a surface is not so important, though often enough it could easily be found. Yet, even when we do know the degrees of the equation of such a surface both in the variables and in these coefficients, we cannot identify the corresponding combinant with a concomitant found by Turnbull or Williamson for which these degrees are the same. In their work, concomitants which differ only by reducible terms (reducible, that is, in the usual sense of the theory of algebraic forms) are considered as equivalent; whereas for us the combinantal

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* *Proc. London Math. Soc.* (2), 20 (1921), 465–489; and 23 (1924), 423–427.

† *Proc. Edin. Math. Soc.*, 43 (1925), 2–16.

‡ The remarks of Grace and Young in § 229 of their *Algebra of invariants* (Cambridge, 1903) are pertinent

property is vital, and the rejection of reducible terms may well destroy the combinantal character of a concomitant.

3. There is no doubt that, from a geometrical standpoint, it is the combinants which are the more important; we wish to consider surfaces related to the net of quadrics as a whole, and not to some particular set of three quadrics on which the net happens to be based. To borrow Sylvester's phrase, used when he first wrote about and invented the name of combinants*, we consider surfaces which are related to the net *in its corporate capacity*.

This idea of regarding the net as a whole, though advocated by Sylvester so long ago, has not been used to sufficient purpose since; and but for this neglect of combinants it is probable that the net of quadrics would have been studied to much better advantage than it has been hitherto. The neglect is doubtless partly due to the manner in which pencils of conics and quadrics are treated in the text-books on two- and three-dimensional analytical geometry. These pencils are simple enough to allow of a large number of concomitants being obtained, and their geometrical interpretation found, without regard to any combinantal forms. The influence of this treatment of the concomitants of two conics is seen, for example, in Salmon's treatise where†, when he obtains a set of invariants of three conics, he does not stop to enquire which of them, or what functions of them, are combinants; indeed the term "combinant" does not appear in the index of the book! This influence has persisted; though, whatever may have been the outlook of those who followed him, Salmon himself did realise that combinants deserved special attention; indeed he gave‡, and interpreted geometrically, three of the combinantal invariants of a net of quadrics.

4. Before proceeding, let us add two more combinantal invariants to the three given by Salmon. The quadrics of $\mathfrak{N}$ have equations of the form

$$\xi Q_0+\eta Q_1+\zeta Q_2=0,$$

and we denote by I_w a combinantal invariant which is of degree w in the coefficients of Q_i; the three invariants given by Salmon may then be called I_2, I_{10}, I_{16}.

* *Cambridge and Dublin Mathematical Journal*, 8 (1853), 256–269; *Mathematical Papers* I, 411–422.

† *Conic sections* (Dublin, 1879), §389b.

‡ *Analytical geometry of three dimensions* (Fourth Edition, Dublin, 1882), 208–212.

We now do for $\mathfrak{N}$ what was done by W. S. Burnside* for a net of conics; but, whereas Burnside only had to express the combinants which he found in terms of the two known combinants T and M (see § 5 below), the combinants which are found by the analogous procedure for quadrics are new. The discriminant of the quadratic form $\xi Q_0+\eta Q_1+\zeta Q_2$ is a ternary quartic in ξ, η, ζ whose coefficients are all invariants, but not combinants, of the three quadrics. But the invariants of this ternary quartic, except for being multiplied by a power of the determinant of the transformation, are unchanged when ξ, η, ζ are subjected to a non-singular linear transformation, and so *are* combinants; the degree of such a combinant in the coefficients of Q_t is the weight of the invariant of the ternary quartic. Now two invariants of a ternary quartic are well known. The first, denoted by $(abc)^4$ in Aronhold's symbolism and written out *in extenso* in Salmon's *Higher plane curves*, is of weight four, and so furnishes a combinant I_4. The second is the determinant of the coefficients in the six second polars of the ternary quartic, its vanishing being the condition that the quartic should be expressible as a sum of five fourth powers; it is of weight eight, and furnishes a combinant I_8.

We may therefore add I_4 and I_8 to the three combinants given by Salmon.

Let us now take the discriminant of the ternary quartic, not because it furnishes a new invariant but, on the contrary, because of its relation to the known invariants; it is of weight thirty-six, and so gives a combinant I_{36}. If I_{36} vanishes, $\mathfrak{N}$ must be such that the ternary quartic, when equated to zero, gives a quartic curve with a node; this node corresponds to a cone Q of $\mathfrak{N}$ which counts for two among the four cones of any pencil of quadrics which includes Q and belongs to $\mathfrak{N}$. Now this can happen in two ways.

(*a*) $\mathfrak{N}$ includes a plane-pair Q. For such a net, $I_{10}=0$.

(*b*) $\mathfrak{N}$ has two coincident base points; it then includes a cone Q with this base point for vertex. For such a net, $I_{16}=0$.

Thus I_{36} contains both I_{10} and I_{16} as factors†; and indeed I_{36} is a numerical multiple of the product $I_{10}{}^2 I_{16}$.

This factorisation of I_{36} may also be seen by the following argument,

* *Quart. J. of Math.*, 10 (1870), 242.

† For the special case when the net has self-polar pentahedra, see Salmon, *Analytical geometry of three dimensions* (Fourth Edition, Dublin, 1882), 212–3.

which is, in a manner, the converse of Clebsch's principle and was often used by Salmon.

Consider not a net but a web of quadrics

$$\xi S_0 + \eta S_1 + \zeta S_2 + \tau S_3 = 0.$$

The discriminant of the quadratic form on the left of this equation is a quaternary quartic in ξ, η, ζ, τ which, when equated to zero, gives the quartic surface known as the Cayley symmetroid. This surface has ten nodes, corresponding to the ten plane-pairs of the web. Any net belonging to the web is given by a linear relation between ξ, η, ζ, τ, and so corresponds to a plane. If the net is such that $I_{36} = 0$, the curve in which the plane meets the symmetroid has a node. Thus the nets for which $I_{36} = 0$ are found simply by taking the tangential equation of the symmetroid, and it is well known that this equation contains the *squares* of the equations of the nodes. The degree of the residual factor is the "proper" class of the symmetroid, namely 16.

5. A plane cuts the net $\mathfrak{N}$ of quadrics in a net of conics; if the plane is such that this net of conics has some special property, expressed by some single relation between its invariants, then, by Clebsch's well-known principle of transference*, the envelope of the plane is a corresponding contravariant of $\mathfrak{N}$.

It was shown by Gundelfinger† that all combinantal invariants of a net of conics are expressible as polynomials in two of them. The first of these two invariants we denote by T, and call Sylvester's invariant; it was first obtained by him, in the paper already cited, as a commutant of the line-equation of a general conic of the net. If T vanishes the net of conics has the property, given by Salmon and Gundelfinger, that its Jacobian curve is apolar to the envelope of lines which are cut in involution by the net. The second invariant we call M; its vanishing is a necessary and sufficient condition for the net to be such that one of its members is a repeated line.

The actual expressions for T and M are found in Salmon's *Conic sections*; the coefficients in the equation of a conic of the net occur to

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* *Journal für Math.*, 59 (1861), 28. The principle is also described by Turnbull, *The theory of determinants, matrices and invariants* (Blackie, 1929), 287–289, as well as by Grace and Young, *Algebra of invariants* (Cambridge, 1903), 265 and 271.

† *Journal für Math.*, 80 (1875), 73–85.

degree two in T and to degree four in M. It follows that, when T is expressed in Aronhold's symbolic notation, it must be a sum of products each of *four* determinantal factors, since each determinant is of degree three in the symbols, and the total number of symbols in a term must be twelve, to account for the second degree in the coefficients of each of three conics on which the net is based. Hence the envelope of those planes which cut $\mathfrak{N}$ in nets of conics for which T vanishes is a contravariant of class *four*. This will be denoted by ϕ^4 and called Gundelfinger's contravariant, since its existence was first established by him*; it is important for our work.

Similarly it follows that the envelope of those planes which cut $\mathfrak{N}$ in nets of conics for which M vanishes is a contravariant of class eight. From the geometrical interpretation of the vanishing of M which has already been given it follows, by Clebsch's principle, that this contravariant is simply the envelope of the cones which belong to $\mathfrak{N}$; its existence was first pointed out by Sturm†, who defined it as the envelope of the cones and obtained its class. It will be denoted here by σ^8.

6. The resultant of three conics is a linear combination of T^2 and M; it was indeed by obtaining the resultant in this form that Sylvester discovered these invariants‡. Gundelfinger, by applying Clebsch's principle, deduced that the resultant of three quadrics and a plane is a linear combination of $(\phi^4)^2$ and σ^8. This resultant, however, consists of the eight base points of the net determined by the three quadrics, so that there is an identity

$$\pi^8 \equiv (\phi^4)^2 + \lambda\sigma^8, \tag{G}$$

where $\pi^8 = 0$ is the equation of the eight base points.

This identity was given, possibly in consequence of Gundelfinger's paper, by Fiedler in his German translation§ of Salmon's *Analytic geometry of three dimensions*; it is to be regretted that, through some misplaced

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* *Loc. cit.*, 83.

† *Journal für Math.*, 70 (1869), 236.

‡ Sylvester, *loc. cit.* The first explicit method of finding the resultant of three conics was given earlier by Sylvester, who applied his dialytic process of elimination to the three conics and their Jacobian, *Cambridge and Dublin Mathematical Journal*, 2 (1841), 232–6; *Mathematical Papers* I, 64–65.

§ Salmon-Fiedler, *Analytische Geometrie des Raumes* I (*Dritte Auflage*), (Leipzig, 1879), 319.

expurgatory zeal, it has been omitted from later German editions of this work.

II.

7. Consider now those planes, eight in all if they are finite in number, which belong* to the envelope σ^8 and which pass through the join $B_i B_j$ of two of the eight base points B_i and B_j of the net $\mathfrak{N}$. Since every cone of $\mathfrak{N}$ passes through both B_i and B_j, no tangent plane of such a cone can pass through $B_i B_j$ unless $B_i B_j$ lies entirely on the cone. Now $B_i B_j$ is a chord of ϑ, the Jacobian curve of $\mathfrak{N}$, and is known to lie on two of the cones; each of these two cones has one of its tangent planes passing through $B_i B_j$, and these are the only planes through $B_i B_j$ which can belong to σ^8. Wherefore the eight planes of σ^8 which pass through $B_i B_j$ must consist of these two, each counted four times. Now every plane through $B_i B_j$ belongs (doubly in fact) to the envelope π^8; it follows from (G) that any plane of ϕ^4 which passes through $B_i B_j$ must, since its coordinates cause π^8 and ϕ^4 both to vanish, also be a plane of σ^8. Hence the four planes of ϕ^4 which pass through $B_i B_j$ must consist of the above two planes, each counted twice. Every one of the twenty-eight joins of pairs of base points has this property.

Just as we speak of a line, two of whose intersections with a surface coincide, as a *tangent* of a surface, so we may speak, in the dual sense, of a line which is so related to an envelope that two of the planes of the envelope which pass through it coincide as a *tangent* of the envelope; and, dual to a bitangent or *double tangent* of a surface, we have a *double tangent* of an envelope. *Gundelfinger's contravariant ϕ^4 then has the property of having the twenty-eight joins of the pairs of base points all as double tangents.* This property of ϕ^4 does not appear to have been noticed before; indeed no instance seems yet to have been met with of a quartic envelope having as double tangents the twenty-eight joins of the pairs of eight associated points, nor has even the *a priori* possibility of this happening been considered.

8. This property of ϕ^4 followed from the result that those planes of ϕ^4 which pass through $B_i B_j$ are also planes of σ^8; this result was itself

* Instead of saying in full "a plane which belongs to the envelope whose equation is $\sigma^8 = 0$" we shall, when there is no possibility of ambiguity or misunderstanding, say, more briefly, "a plane which belongs to the envelope σ^8", or "a plane of the envelope σ^8", or "a plane of σ^8". A corresponding procedure will be adopted, as the occasion serves, for other envelopes and, dually, for surfaces.

obtained as an immediate corollary of (G). Let us now give an alternative way of obtaining it, without appealing to (G). We first use the fact that ϕ^4 is the envelope of those planes which meet $\mathfrak{N}$ in nets of conics for which T vanishes, and then deduce the result by using the form of T given by Salmon in § 389 of his *Conic sections*.

We desire, then, to obtain those planes which belong to ϕ^4 and pass through $B_i B_j$. Such a plane meets $\mathfrak{N}$ in a net of conics which has two base points and for which Sylvester's invariant vanishes. Now, when a net of conics has two base points it may, by taking these as two of the three vertices of the triangle of reference, be based on three conics whose equations have the forms

$$a_1 x^2 + 2f_1 yz + 2g_1 zx + 2h_1 xy = 0,$$

$$a_2 x^2 + 2f_2 yz + 2g_2 zx + 2h_2 xy = 0,$$

$$a_3 x^2 + 2f_3 yx + 2g_3 zx + 2h_3 xy = 0.$$

For such a net of conics, all the terms in Salmon's expression for T vanish except the last; and the condition $T = 0$ reduces, for this special net, to the vanishing of the determinant $(f_1 g_2 h_3)$. This, however, is precisely the condition for the net to include the conic $x^2 = 0$, which is the repeated line joining the base points. Hence the plane must touch a cone of $\mathfrak{N}$ along $B_i B_j$, and we have again the result that any plane through $B_i B_j$ which belongs to ϕ^4 must belong also to σ^8. It may be remarked in passing that the invariant T for the above net of conics is a numerical multiple of the *square* of $(f_1 g_2 h_3)$, and this implies that the two planes of σ^8 which pass through $B_i B_j$ count *doubly* as planes of ϕ^4.

9. There are, of course, properties dual to the above for a net of quadric envelopes, and a corresponding identity

$$\Pi^8 \equiv (F^4)^2 + \lambda S^8, \qquad (\Gamma)$$

where Π^8 is the product of the eight base planes,

S^8 is the surface generated by the conics—quadric envelopes with vanishing discriminants—belonging to the net,

F^4 is a quartic surface, dual to Gundelfinger's contravariant.

This quartic surface has the twenty-eight lines of intersection of the pairs of base planes all as double tangents.

An immediate consequence of (Γ) is that S^8 touches each of the eight base planes along the quartic curve in which this plane meets F^4; these quartic curves are the loci of the points of contact of the respective base planes with the conics belonging to the net of quadric envelopes.

10. The above derivation of the property of ϕ^4 and F^4 of having twenty-eight bitangent lines which are, for ϕ^4, joins of pairs of eight associated points, and, for F^4, intersections of pairs of eight associated planes, rests primarily on the work of Sylvester and Gundelfinger. This work is algebraic; we now give an alternative derivation of the property, which rests on some work, of a more geometrical character, of Aronhold and Frobenius. It is a matter of choice whether we prove the property for ϕ^4 or for F^4; once it is proved for either, the principle of duality establishes it for the other. We shall, however, prove the property for F^4, as we are thereby able to appeal more directly to known results.

11. The six points of contact of three bitangents of a non-singular plane quartic curve may lie on a conic, in which case the three bitangents are said to be syzygetic; or they may not. If any set of eight, or more, of the twenty-eight bitangents is taken, it always includes at least one syzygetic triplet; the maximum number of bitangents which can be chosen in such a way that it does *not* include a syzygetic triplet is seven. Such a set of seven bitangents is called an *Aronhold set*. If seven lines of general position are given in a plane, it is known that there is a unique quartic curve for which these seven lines are an Aronhold set of bitangents*.

Take now eight associated planes $\Pi_1, \Pi_2, \ldots, \Pi_8$; there is a net of quadric envelopes to which they belong, and this net includes ∞^1 envelopes which degenerate into conics. We quote, concerning this configuration, two results due to Frobenius†:

(i) The locus of the points of contact of the conics with the plane Π_i is a quartic curve C_i. This is the curve which has the seven lines in which Π_i is met by the remaining seven planes as an Aronhold set of bitangents.

(ii) C_i and C_j touch π_{ij}, the line common to Π_i and Π_j, in the same two points.

* Aronhold, *Berliner Monatsberichte* (1864), 499.

† *Journal für Math.*, 99 (1886), 278–280.

12. We now prove that *the eight quartic curves C_i lie on a quartic surface.* One way of proving this would be to take a general quartic surface, and so restrict the coefficients in its equation by linear conditions that it contains first one, then another, of the curves C_i. Proceeding in this way we should find that, at a certain stage, all those curves C_i through which the surface has not already been made to pass would lie on the surface automatically. We will, however, marshal the argument somewhat differently; we take four of the curves and prove that they lie on a quartic surface *other* than that formed by the four planes in which they lie. From this the desired result easily follows.

Take the four curves C_1, C_2, C_3, C_4 lying in the planes Π_1, Π_2, Π_3, Π_4. The line π_{12} is touched by C_1 and C_2 in the same two points; in order that a quartic surface should touch π_{12} at these points it must be subjected to four linear conditions. Similarly four linear conditions must be imposed to ensure the two proper contacts with π_{23}, and four more to ensure them with π_{31}. The quartic surface, now subjected to twelve linear conditions, has the three concurrent bitangents π_{12}, π_{23}, π_{31}; we next proceed to make the surface touch π_{14}, π_{24}, π_{34} in the requisite pairs of points. A new feature here appears in the argument; for, if a plane quartic has each of three lines for bitangents, any quartic curve which passes through the six points of contact and touches the given quartic at five of them must also touch it at the sixth—this being a particular instance of the theorem that the twelve points in which a quartic meets a cubic curve are determined when eleven of them are assigned. Hence the quartic surface, already having π_{12}, π_{23}, π_{31} as bitangents, need only be subjected to three further linear conditions in order that it should touch, say, π_{34} at the two requisite points; similarly three further linear conditions are sufficient to ensure the proper contacts with π_{14}, and three more to ensure them with π_{24}.

The general quartic surface has now, by the imposition of twenty-one linear conditions been caused to have as bitangents the six edges of the tetrahedron whose faces are Π_1, Π_2, Π_3, Π_4; the two points of contact with each edge are the points where this edge is touched by two of the curves C_i. We now impose further linear conditions on the surface so as to make it contain the curves C_1, C_2, C_3, C_4 entirely, and three conditions are required for each of the four curves. For, when a quartic curve passes through the twelve intersections of a second quartic with a cubic, its four remaining intersections with this second quartic are collinear; therefore, if it is constrained to pass through three arbitrary points of this second quartic it must contain it completely. Hence a general quartic surface can, by the imposition of thirty-three linear conditions, be made to contain the four curves C_1, C_2, C_3, C_4. This leaves one free constant in the equation of the surface, and so a pencil of quartic surfaces through the four curves, if the thirty-three conditions are independent. Were they not independent, there would be a linear system, of freedom greater than one, of quartic surfaces containing the four curves; but the total order of the four curves is 16, so that this latter contingency cannot arise. Thus C_1, C_2, C_3, C_4 together constitute the base of a pencil of quartic surfaces*, one surface of the pencil of course being the product of the four planes Π_1, Π_2, Π_3, Π_4.

There is one quartic surface belonging to this pencil which contains the four remaining curves C_5, C_6, C_7, C_8. For the base curve of the pencil meets C_5, say, in sixteen points, namely the contacts with its four bitangents π_{15}, π_{25}, π_{35}, π_{45}; through any further point of C_5 there passes one surface of the pencil, and this contains C_5 entirely. This particular surface also contains C_6, C_7, C_8; for it already has twenty intersections with each of them at their points of contact with five of their bitangents.

13. We have now obtained a quartic surface F^4 which contains the eight curves C_i. It has therefore the twenty-eight lines π_{ij} for bitangents. It is the same surface F^4 as that which appears in the identity (Γ); for both these quartic surfaces F^4 contain the eight curves C_i, and so must be one and the same surface. We are not, however, going to appeal to the identity (Γ), but to obtain it anew by continuing the present argument.

The surface S^8 which is generated by the conics which belong to the net of quadric envelopes touches, by Frobenius' result, Π_i along C_i. Hence, if F^4 is the surface, just obtained, on which all the curves C_i lie, every octavic surface which belongs to the pencil $(F^4)^2+\lambda S^8$ touches each of the planes Π_i along the corresponding curve C_i. There is one surface of this pencil which contains the whole of the plane Π_i; this surface not only touches any other, Π_j say, of the eight planes along C_j, but it meets it further in the line π_{ij}; hence it contains the whole of the plane Π_j. It must therefore be the product of the eight planes, so that we have an identity

$$\Pi^8 \equiv (F^4)^2+\lambda S^8. \qquad (\Gamma)$$

* The same argument, with the few necessary verbal alterations, establishes the same result for four plane quartics which are such that any two meet the line of intersection of their planes in the same four (distinct) points.

III.

14. It has, then, been known, since the publication of Gundelfinger's paper, that a net of quadric surfaces possesses a combinantal contravariant of class four. It will now be shown that it also possesses a combinantal covariant of order four. The existence of such a covariant, indeed the existence of several, follows immediately from the theory of algebraic forms; this will be explained below. But let us obtain one geometrically first.

The net $\mathfrak{N}$ has a Jacobian curve ϑ, whose trisecants generate a scroll R^8; this scroll has eight tritangent planes. It was shown by W. P. Milne* that *these eight planes are associated.* Denote by ν the net of quadric envelopes to which these eight planes belong. Then any combinantal concomitant of either of the nets $\mathfrak{N}$ or ν must be a combinantal concomitant of $\mathfrak{N}$; in particular, any covariant† of ν must also be be a covariant of $\mathfrak{N}$. But ν certainly has a combinantal covariant F^4, namely the dual of Gundelfinger's contravariant. So we have obtained, in the surface F^4, a quartic surface which is a combinantal covariant of $\mathfrak{N}$.

15. Milne, starting from a canonical form for $\mathfrak{N}$, obtains equations for ν and observes that this "net of quadric envelopes is obviously combinantal in character" with respect to $\mathfrak{N}$. Obvious geometrically it certainly is; but algebraically it is perhaps not quite so obvious as Milne appears to believe. The coefficients in any combinant of three quadrics must be functions of the 120 determinants, of three rows and columns, of the 3×10 matrix formed by the coefficients in the equations of the three quadrics; the coefficients in Milne's equations (19) have not this property as they stand‡. They do, however, acquire the property if they are multiplied throughout, in Milne's notation, by k^3. This, at first sight small, criticism is important algebraically; for it means that the equation

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* *Journal London Math. Soc.*, 8 (1933), 211–216.

† The quadrics of $\mathfrak{N}$ are quaternary forms in the *point* coordinates, those of ν quaternary forms in the *plane* coordinates. Here the term *covariant* will denote always a concomitant which is a quaternary form in the *point* coordinates. Thus we speak below of forms as covariants of ϕ^4 although the variables (namely the point coordinates) which they contain are contragredient to those in ϕ^4.

‡ Incidentally, in the third of these three equations, the first term should have a *minus* sign and, in the last term, n is a misprint for m.

of a quadric envelope of ν contains the coefficients of the equations of the quadrics of $\mathfrak{N}$ each to degree *two*, not merely to degree one.

16. We now have, as combinantal concomitants of $\mathfrak{N}$, the covariant F^4, which is the dual of Gundelfinger's contravariant for ν, and the contravariant ϕ^4, which is Gundelfinger's contravariant for $\mathfrak{N}$. The whole of the simultaneous system of concomitants of F^4 and ϕ^4 must consist of combinantal concomitants of $\mathfrak{N}$. This system includes all the covariants of ϕ^4 and so, when once the existence of ϕ^4 is known, we can obtain a quartic covariant of $\mathfrak{N}$ forthwith. For every covariant of ϕ^4 is necessarily a covariant of $\mathfrak{N}$, and ϕ^4 certainly has a quartic covariant, namely the form which, when ϕ^4 is written symbolically as α_u^4, is given by $(\alpha\beta\gamma x)^4$. This is all that needed to be said, at any moment since the publication of Gundelfinger's paper sixty-five years ago, to establish the existence of a quartic surface combinantally covariant for a net of quadric surfaces.

But there is more to be said; for ϕ^4 has other covariants of the fourth order. The best known of these is obtained as follows*. Take the ten second polars of ϕ^4, which are quadric functions of the four plane coördinates, and the matrix of their coefficients. The determinant of this matrix is an invariant of ϕ^4, whose vanishing is a necessary and sufficient condition for ϕ^4 to have an outpolar quadric, or to be expressible as a sum of nine fourth powers. [If ϕ^4 was expressible as a sum of $k \leqslant 9$ fourth powers, it would have a linear system of ∞^{9-k} outpolar quadrics, and the determinant would be of rank k.] If, now, this matrix is bordered by a row and column whose elements are the squares and products of the point coordinates, the determinant of this bordered matrix is the quartic covariant in question. It is of degree nine in the coefficients of ϕ^4, and would vanish identically if ϕ^4 happened to be expressible as the sum of eight (or less) fourth powers.

We have thus found three quartic surfaces which are combinantal covariants of $\mathfrak{N}$; namely F^4 and the two quartic covariants that we have obtained from ϕ^4. We also have, by dual reasoning, three quartic envelopes which are combinantal contravariants; namely ϕ^4 and the two quartic contravariants that are correspondingly derived from F^4.

17. The processes by which the two quartic covariants of ϕ^4 were derived also furnish, when generalized, for any form of even degree $2q$, two concomitants, also of degree $2q$, in the contragredient variables; if

* Reye, *Journal für Math.*, 78 (1874), 123–129.

there are p variables*, the degrees of these concomitants in the coefficients of the original form are $p-1$ and $\{(q+p-1)!/q!\,(p-1)!\}-1$ respectively. In the special case when $q=1$, however, the two concomitants coincide, and we have the well-known process of obtaining the tangential equation of a quadric by bordering the matrix of its discriminant with a row and column of prime coordinates. Another known example occurs when the original form is a ternary quartic in the point co-ordinates, for the second concomitant is then Clebsch's contravariant Ω. Though Clebsch did not actually obtain Ω by bordering the invariant determinant, the expression which he gives for it† shows clearly that it is obtainable in this way. Geometrical interpretations of Ω have been given by other writers‡; by dualizing these, and generalizing them so that they refer to quaternary instead of to ternary forms, geometrical interpretations of the corresponding covariant of ϕ^4 can be found.

18. When a combinantal covariant C^p, of order p, of $\mathfrak{N}$ has been found, another one, of order $p+2$, is at once obtained as the Jacobian of C^p and three linearly independent quadrics of $\mathfrak{N}$—at least unless this Jacobian happens to vanish identically, as it would if C^p was a polynomial

* Binary forms ($p=2$) are exceptional, in that the variables contragredient to (x_1, x_2) are $(-x_2, x_1)$. The two processes do, of course, furnish covariants, but the first of them is merely a multiple of the binary form itself. The second, however, is a genuine covariant, of degree q in the coefficients, whenever $q>1$, and is the penultimate member of a scale of covariants found by Sylvester, *Mathematical Papers* I, 297. For the binary quartic $(a, b, c, d, e\between x_1, x_2)^4$ it is

$$\begin{vmatrix} a & b & c & x_2^2 \\ b & c & d & -x_1x_2 \\ c & d & e & x_1^2 \\ x_2^2 & -x_1x_2 & x_1^2 & 0 \end{vmatrix},$$

the Hessian; for the binary sextic it is, with us,

$$\begin{vmatrix} a & b & c & d & -x_2^3 \\ b & c & d & e & x_2^2x_1 \\ c & d & e & f & -x_2x_1^2 \\ d & e & f & g & x_1^3 \\ -x_2^3 & x_2^2x_1 & -x_2x_1^2 & x_1^3 & 0 \end{vmatrix},$$

and, with Sylvester,

$$\begin{vmatrix} ax_1^2+2bx_1x_2+cx_2^2 & bx_1^2+2cx_1x_2+dx_2^2 & cx_1^2+2dx_1x_2+ex_2^2 \\ bx_1^2+2cx_1x_2+dx_2^2 & cx_1^2+2dx_1x_2+ex_2^2 & dx_1^2+2ex_1x_2+fx_2^2 \\ cx_1^2+2dx_1x_2+ex_2^2 & dx_1^2+2ex_1x_2+fx_2^2 & ex_1^2+2fx_1x_2+gx_2^2 \end{vmatrix}.$$

† *Journal für Math*, 59 (1861), 139.

‡ *Encyklopädie der Math. Wissenschaften*, III, C 5, 524.

of degree $\frac{1}{2}p$ in the three quadrics. The covariant C^{p+2} so obtained will be spoken of simply as the Jacobian of C^p. In virtue of its character as a Jacobian, the surface C^{p+2} always passes through ϑ, the Jacobian curve of $\mathfrak{N}$.

Thus there arises a sextic covariant F^6, combinantal for $\mathfrak{N}$, which is the Jacobian of F^4 and which contains ϑ. This surface is the locus of points whose polar planes with respect to F^4 and all the quadrics of $\mathfrak{N}$ are concurrent; or, expressing the same statement slightly differently, it is the locus of a point O which is such that its polar plane with respect to F^4 passes through O', the point which, in the manner fully described in Note I, is conjugate to O with respect to $\mathfrak{N}$. The curve of intersection of F^4 and F^6 is the locus of points where quadrics of $\mathfrak{N}$ touch F^4.

Similarly, the other two quartic covariants give rise to sextic covariants as their Jacobians.

Consider, next, combinantal covariants of order eight. It was mentioned at the outset that these include R^8, E^8, ρ^8 and Π^8; we may add to these S^8, the locus of conics which belong to ν. Further, we also have the Hessians of the three quartic covariants and the Jacobians of the three sextic covariants, making eleven octavic covariants in all. These may not all be distinct; and it would certainly be of interest to investigate what linear relations connect them with one another and with the squares of the quartic covariants. One such relation is given by the identity (Γ).

For covariants of order ten we may take the Jacobians of covariants of order eight; though the Jacobian of E^8 vanishes identically. Having found all the non-vanishing Jacobians, we may add those surfaces which are conjugate, in the sense of Note I, to the three sextic covariants; these three surfaces of order ten have ϑ for a nodal curve. And then it may be investigated what linear relations exist between these covariants and between the nine products, such as $F^4 F^6$, of a quartic and a sextic covariant. And so on.

IV.—*The Möbius Net as an example.*

19. When properties of a general net of quadrics have been found, it is of interest to verify them for some particular net that has been studied previously; though it must be borne in mind that a special net will have, merely in consequence of the specialization, properties which are not possessed by the general net. Suppose, then, that $\mathfrak{N}$ is a net whose eight base points are a pair of Möbius tetrads T, T'; it may fittingly be called a *Möbius Net*. We obtain the corresponding net ν of quadric envelopes, and the identity (Γ). The Möbius Net has been described in detail in a

previous paper*; we shall refer to this paper as M.T., and use here the same symbols as there.

If T is taken as tetrahedron of reference for a system of homogeneous co-ordinates x, y, z, t, the faces of T' are found (M.T. p. 338) by equating to zero the linear forms x', y', z', t' given by

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} . & -n & m & -l \\ n & . & -l & -m \\ -m & l & . & -n \\ l & m & n & . \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

Here l, m, n are any three numbers, subject only to the condition that the above square matrix is non-singular; thus $l^2+m^2+n^2$ must not be zero. We describe the eight vertices of the two tetrahedra as the *base points*, and of the eight faces of the two tetrahedra as the *base planes* of the Möbius configuration. The base points of the configuration are also the base points of $\mathfrak{N}$; it will be seen that the base planes of the configuration are also the base planes of ν.

A general net of quadrics may be based on any three of its members which are linearly independent; save for this condition of linear independence it is immaterial which three are chosen, the net having no peculiarity that enables us to pick out one set of quadrics rather than another. But when the net is specialised this may no longer be so, as the specialization may well throw into prominence certain particular quadrics of the net. This happens for the Möbius Net; and, moreover, it happens very conveniently for the algebraical treatment, for the net can be based symmetrically on three particular quadrics. These are (M.T., p. 341):

$$Q_1 \equiv yy'+zz' \equiv -xx'-tt' = 0,$$

$$Q_2 \equiv zz'+xx' \equiv -yy'-tt' = 0,$$

$$Q_3 \equiv xx'+yy' \equiv -zz'-tt' = 0.$$

20. In order to obtain the net ν of quadric envelopes which corresponds to the Möbius Net $\mathfrak{N}$, we have to discover its eight base planes; these are, as was said in § 14, tritangent planes of R^8, the scroll generated by the trisecants of the Jacobian curve of $\mathfrak{N}$. The Möbius Net, however, includes the four plane-pairs

$$xx' = 0, \quad yy' = 0, \quad zz' = 0, \quad tt' = 0,$$

* "The net of quadric surfaces associated with a pair of Möbius tetrads", *Proc. London Math. Soc.* (2), 41 (1936), 337–360.

whose vertices are, respectively, four skew lines a, b, c, d; the Jacobian curve of the net consists (*cf.* M.T., p. 343) of these four lines and their two transversals e, f. Hence, if the regulus generated by the transversals of a, b, c is denoted by $\{abc\}$, the scroll R^8 now consists of the four quadrics on which lie, respectively, $\{abc\}$, $\{bcd\}$, $\{cda\}$ and $\{dab\}$.

Any plane through a meets b, c, d in points β, γ, δ; the lines $\gamma\delta, \delta\beta, \beta\gamma$ belong respectively to $\{cda\}$, $\{dab\}$, $\{abc\}$; hence the plane contains three generators of R^8. Similarly, any plane through b, or c, or d contains three generators of R^8. Thus, for the Möbius Net, R^8 has not merely eight but an infinite number of tritangent planes, and so the base planes of ν cannot be found merely by using their property of being tritangent planes of R^8. Some further property which they possess must be invoked in order to identify them. Such a property is known*: those trisecants of of the Jacobian curve of a general net of quadrics which lie in tritangent planes of R^8 are also generators of cones belonging to the net. Now none of $\gamma\delta, \delta\beta, \beta\gamma$ can meet either e or f; hence they cannot possibly be generators of cones of the net with point-vertices. The only possibility of their being generators of cones of the net with line-vertices is for their plane to be one of the two planes which make up the plane-pair with vertex a. So it is seen that the eight base planes of ν are precisely the eight base planes of the Möbius configuration. Incidentally, Milne's result that, for a general net of quadrics, the eight tritangent planes of R^8 are associated, includes as a special case the well-known result that the eight base planes of a Möbius configuration are associated.

21. Consider now the identity (Γ); it contains three terms. Of these Π^8 is the product of the eight base planes of ν, and so is a multiple of $16\,xx'yy'zz'tt'$, *i.e.* of

$$Q_1^4+Q_2^4+Q_3^4-2Q_2^2Q_3^2-2Q_3^2Q_1^2-2Q_1^2Q_2^2.$$

As for S^8, it was explained in M.T. (pp. 344–5) that the locus of the conics which touch the eight base planes of a Möbius configuration consists of a pair of Plücker surfaces U and V; the equation of U was given (p. 343) and that of V is obtained from it by changing the sign of p. From this it is found that S^8 is a multiple of

$$\Sigma\{(l^2+m^2)^2(l^2+n^2)^2Q_1^4+2(m^2+n^2)^2(m^2n^2+l^2p^2)Q_2^2Q_3^2\},$$

* *Proc. London Math. Soc.* (2), 34 (1932), 513–514.

where Σ denotes the sum of the three terms obtained by permuting cyclically, and simultaneously, the letters l, m, n and the suffixes 1, 2, 3; and

$$l^2+m^2+n^2+p^2=0.$$

The expressions for both Π^8 and S^8 being now known, that for F^4 is found from the fact that its square must be of the form $\Pi^8-\lambda S^8$. Since Π^8 and S^8 are both homogeneous quartic polynomials in Q_1, Q_2, Q_3 it is already clear, before the actual expression for F^4 is found, that it must be a homogeneous quadratic polynomial in Q_1, Q_2, Q_3. Hence, for the Möbius Net, F^4 is an octadic surface with nodes at the eight base points; this property of F^4 is due to the specialization of the net, and does not hold for a general net of quadrics. An incidental consequence is that, for the Möbius Net, the Jacobian F^6 of F^4 vanishes identically. The actual identity (Γ) is found to be

$$\begin{aligned}4l^2m^2n^2p^2\{Q_1^4+Q_2^4+Q_3^4-2Q_2^2Q_3^2-2Q_3^2Q_1^2-2Q_1^2Q_2^2\}&\\ \equiv \{(m^2n^2+l^2p^2)\,Q_1^2+(n^2l^2+m^2p^2)\,Q_2^2+(l^2m^2+n^2p^2)\,Q_3^2\}^2&\\ -\Sigma\{(l^2+m^2)^2(l^2+n^2)^2\,Q_1^4+2(m^2+n^2)^2(m^2n^2+l^2p^2)\,Q_2^2\,Q_3^2\}&.\end{aligned}$$

22. The equation of the quartic surface F^4 is therefore

$$(m^2n^2+l^2p^2)\,Q_1^2+(n^2l^2+m^2p^2)\,Q_2^2+(l^2m^2+n^2p^2)\,Q_3^2=0;$$

like any other octadic surface it can be generated by base curves of pencils of quadrics belonging to the net, and a geometrical interpretation of the surface can be given. For the conic whose line equation is

$$(m^2n^2+l^2p^2)\,\lambda_1^2+(n^2l^2+m^2p^2)\,\lambda_2^2+(l^2m^2+n^2p^2)\,\lambda_3^2=0$$

is seen to be the Φ-conic of the pair of conics u and v whose point equations (*cf.* M.T. p. 342) are

$$(m^2+n^2)^2k_1^2+(lm+np)^2k_2^2+(nl-mp)^2k_3^2=0$$

and

$$(m^2+n^2)^2k_1^2+(lm-np)^2k_2^2+(nl+mp)^2k_3^2=0.$$

Hence we have the following. Any pencil of quadrics belonging to the net includes two pairs of cones; the cones of one pair have their vertices on e and are enveloping cones of U, those of the other pair have their vertices on f and are enveloping cones of V. If these two pairs of cones are harmonically conjugate to one another within the pencil, then the base curve of the pencil lies on F^4, and F^4 is generated by such base curves*.

23. We conclude by verifying, for the Möbius Net, that F^4 has all the lines of intersection of pairs of base planes of ν as bitangents. It is a known property of the Möbius configuration that, of the 28 intersections of pairs of base planes, 24 are also joins of pairs of base points; but the base points are all nodes of F^4, so that these 24 lines may certainly be reckoned as bitangents. It only remains to verify that those four lines are bitangents of F^4 which are intersections of pairs of base planes but *not* joins of pairs of base points. These are the lines denoted above by a, b, c, d; and it is easily shown, using the equation of F^4 and the algebra of the first three pages of M.T., that each of these four lines touches F^4 at a point on e and at a point on f.

Mathematical Institute,
16 Chambers Street,
Edinburgh, 1.

* It was shown in Note II that the locus of harmonic base curves for a general net of quadrics is a surface of order 12 with sextuple points at the base points. For the Möbius Net this surface consists of F^4 and a surface F^8 having quadruple points at the base points.