A SPECIAL POLYHEDRAL NET OF QUADRICS

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1.

Among specialised forms of nets of quadrics in $[n]$ (that is, in projective space of n dimensions) is one that may appropriately be called *polyhedral*, this because it is linked to G, a g_{n+2}^1 on a rational normal curve C of order n. The osculating primes, or hyperplanes, of C at the $n+2$ points of a set of G form an $(n+2)$-hedron $\mathscr{H}$; as the set varies in G the locus ϑ of the vertices of $\mathscr{H}$ is the Jacobian curve of the net, of order $\frac{1}{2}n(n+1) = \binom{n+1}{2}$; the locus of the edges of $\mathscr{H}$ is a ruled surface R_2 of order $2\binom{n+1}{3}$ on which ϑ has multiplicity $n = \binom{n}{1}$; the locus of the plane faces of $\mathscr{H}$ is a threefold R_3 of order $3\binom{n+1}{4}$ on which ϑ has multiplicity $\binom{n}{2}$; and so on; for these matters see [**4**]. In order to be polyhedral a net has to satisfy $\frac{1}{2}(n^2+n-10)$ conditions [**4**; p. 188].

The pentahedral net in [3], obtainable from the general one by imposing a single condition—the vanishing of a combinant—was first regarded as the net of polar quadrics of points of a plane with respect to a cubic surface. But the cubic surface is not unique; nor is the plane, which can be any osculating plane of a twisted cubic uniquely determined by the net; this cubic is indeed the one osculated by all the faces of the pentahedra. It seems, therefore, as remarked on an earlier occasion [**4**; p. 186], preferable to define the net by the twisted cubic and g_5^1 thereon. The hexahedral net in [4] was investigated more than 40 years ago [**3**; pp. 275–315], and other specialised nets in [4] were studied in the same paper. But it was there decided [**3**; p. 255] not to press specialisation so far as to force ϑ to be composite.

Now G has a Jacobian set J of $2n+2$ points; if, on the other hand, $2n+2$ points are chosen arbitrarily on C then there is a finite number of linear series G of which they are the Jacobian set [**6**]. The set of G which includes a point P of J consists of n points together with P reckoned twice. One way of specialising the polyhedral net is to specialise J; if J is such that P accounts for k of its $2n+2$ members then P accounts for $k+1$ members of the set of G to which it belongs. An obvious specialisation is to require J to consist of a pair of points X_0, X_n each counted $n+1$ times; then the only sets of G whose members are not all distinct are X_0^{n+2} and X_n^{n+2}. For $n = 3$ the geometry of this special net of quadrics was described on pp. 471–480 of [**5**]; for $n = 4$ a contribution to the geometry is submitted below. But first a few paragraphs should be written about the geometry in n dimensions.

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2.

If θ is a parameter on C then take the parameters of a set of G to be the zeros of polynomials linearly dependent on

$$f(\theta) \equiv \theta^{n+2}-\alpha^{n+2}, \qquad g(\theta) \equiv \theta^{n+2}-\beta^{n+2}.$$

The sets with multiple members correspond to $f(\theta)-g(\theta)$, whose $n+2$ zeros are all $\theta = \infty$, and $\beta^{n+2}f(\theta)-\alpha^{n+2}g(\theta)$, whose $n+2$ zeros are all $\theta = 0$.

The zeros of $f(\theta)$ are $\theta_j = \alpha\eta^j$ where η is any primitive $(n+2)$-th root of unity and j runs from 1 to $n+2$ inclusive. Since

$$f'(\theta_j) = (n+2)\theta_j^{n+1} = (n+2)\alpha^{n+1}\eta^{-j}, \quad \text{and} \quad g(\theta_j) = \alpha^{n+2}-\beta^{n+2}$$

the expressions

$$\theta_j^k P_j^2/f'(\theta_j)g(\theta_j)$$

appearing, with $k = 0, 1, 2$, on p. 195 of [4] may, ignoring a factor not involving j or k and common to all $n+2$ denominators, be replaced by

$$\alpha^k\eta^{jk}(x_0+\alpha\eta^j x_1+\alpha^2\eta^{2j}x_2+\ldots+\alpha^n\eta^{nj}x_n)^2\eta^j\,; \tag{2.1}$$

the linear form here indicates an osculating prime of C with θ_j the parameter of the point of osculation. When (2.1) is summed over $1 \leqslant j \leqslant n+2$ all terms sum to zero save those in which the power of η^j is zero or a multiple of $n+2$. But the zero power does not occur nor, with $k < 3$, does any higher than $2n+3$; so the non-zero surviving terms of the sum all involve α to the same power. Hence α can be dropped when the sum is equated to zero. So one obtains three linearly independent quadrics corresponding to $k = 0, 1, 2$, the products (or the square) appearing in the respective quadratic forms having $n+1, n, n-1$ for sums of suffixes.

The quadratic forms are

$$Q_0 \equiv \sum_{j=1}^{n} x_j x_{n+1-j}, \qquad Q_1 \equiv \sum_{j=0}^{n} x_j x_{n-j}, \qquad Q_2 \equiv \sum_{j=0}^{n-1} x_j x_{n-1-j}$$

and one may be allowed to speak simply of Q_i when meaning the quadric $Q_i = 0$. Any product with unequal suffixes occurs twice in a sum, but a square only once. The special polyhedral net N consists of the quadrics

$$\lambda Q_0+\mu Q_1+\nu Q_2 = 0. \tag{2.2}$$

The quadric Q_0, since x_0 is absent from it, is a cone with vertex X_0; Q_2 is a cone with vertex X_n; here X_j means, as is customary, that vertex of the simplex of reference opposite to the face $x_j = 0$. Since neither x_0^2 nor x_n^2 appears in any Q_k, the $(n-3)$-fold B common to all the quadrics of N contains X_0 and X_n; the Jacobian curve has these points in common with B. Whenever the Jacobian curve and the base $(n-3)$-fold of a net of quadrics have a common point one may expect it to be multiple on the curve.

3.

The discriminant of (2.2) is

$$D_n = \begin{vmatrix} & & & \cdot & \nu & \mu \\ & & \nu & \mu & \lambda & \\ & & \mu & \lambda & \cdot & \\ & \iddots & \iddots & \iddots & & \\ \cdot & \nu & & & & \\ \nu & \mu & & & & \\ \mu & \lambda & & & & \end{vmatrix} = (-1)^n \mu D_{n-1} + \nu\lambda D_{n-2},$$

with initial conditions

$$D_0 = \mu, \qquad D_1 = \nu\lambda - \mu^2 .$$

The upshot is that each D_n is a product of factors $c\nu\lambda - \mu^2$, with an additional factor μ when n is even. Also, since

$$D_{n+1} = (\nu\lambda - \mu^2)D_{n-1} + (-1)^{n+1}\lambda\mu\nu D_{n-2},$$

one of the factors of D_n has $c = 1$ whenever $n \equiv 1 \pmod 3$. Thus, as will be exploited later, D_4 has both μ and $\nu\lambda - \mu^2$ for factors.

The Jacobian curve ϑ of N therefore splits. There is, for $\mu = 0$, a pencil of cones when n is even; apart from this the cones fall into systems of index 2; all these systems include Q_0 and Q_2. It is a property of, and indeed it is sufficient to define, ϑ that the polar primes of a point on it with respect to the quadrics (2.2) intersect not in an $[n-3]$ but in an $[n-2]$; as ϑ splits so, correspondingly, does the primal R_{n-1}, of order n^2-1, generated by these spaces [4; p. 205]. A point of ϑ cannot lie in the corresponding $[n-3]$ unless it is self-conjugate for every quadric of N, that is, unless it belongs to B.

Suppose, momentarily, that $n = 2m$ so that $Q_0 = uQ_2$ is a cone whatever u. The form of D_{2m}, with $\lambda = 1$, $\mu = 0$, $\nu = -u$ shows, reading by rows from the bottom upwards, that the vertex satisfies

$$-x_1 = 0, \qquad ux_j - x_{j+2} = 0 \; (j = 0, 1, \ldots, 2m-2), \qquad ux_{2m-1} = 0,$$

so that its coordinates are

$$(1, 0, u, 0, u^2, \ldots, 0, u^m); \tag{3.1}$$

it traces, as u varies, a rational normal curve Γ in the $[m]$ $X_0X_2X_4\ldots X_{2m}$. This point (3.1) has the same polar with respect to Q_0 and Q_2, namely

$$x_{2m-1} + ux_{2m-3} + \ldots + u^{m-1}x_1 = 0$$

while its polar prime with respect to Q_1 is

$$x_{2m} + ux_{2m-2} + \ldots + u^m x_0 = 0 .$$

The equation of the primal generated by the $[2m-2]$ common to these polar primes is obtained by eliminating u between the two equations; Sylvester's dialytic process applied to these two polynomials of degrees $m-1$ and m in u at once shows the primal to be of order $2m-1$.

Suppose now, again momentarily, that $n = 3p+1$. Then $Q_0+vQ_1+v^2Q_2$ is singular whatever v and the form of D_{3p+1} with $\lambda = 1, \mu = v, \nu = v^2$ shows, reading by rows from the bottom upwards, that the vertex satisfies

$$vx_0+x_1 = 0, \qquad v^2x_{j-1}+vx_j+x_{j+1} = 0 \quad (j = 1, 2, \ldots, 3p),\ v^2x_{3p}+vx_{3p+1} = 0.$$

Put $x_j = \xi_j v^j$ so that

$$\xi_0+\xi_1 = 0, \qquad \xi_{j-1}+\xi_j+\xi_{j+1} = 0 \quad (j = 1, 2, \ldots, 3p),\ \xi_{3p}+\xi_{3p+1} = 0.$$

Taking $\xi_0 = 1$ these equations give

$$\xi_0 = 1, \quad \xi_1 = -1, \quad \xi_2 = 0, \quad \xi_3 = 1, \quad \xi_4 = -1, \xi_5 = 0, \quad \ldots, \quad \xi_{3p} = 1,$$
$$\xi_{3p+1} = -1$$

and the vertex has coordinates

$$(1, -v, 0, v^3, -v^4, 0, \ldots, 0, v^{3p}, -v^{3p+1}). \tag{3.2}$$

Its locus as v varies is a rational curve K of order $3p+1$ in the $[2p+1]$ $x_2 = x_5 = \ldots = x_{3p-1} = 0$.

The curve K has trisecants. If ω is a complex cube root of unity then the points with parameters $v, \omega v, \omega^2 v$ are manifestly collinear. In particular, for $v = 0, \infty$, two trisecants have all three intersections with K coincident; K has inflections at X_0 $(v = 0)$ and X_{3p+1} $(v = \infty)$.

It remains to verify that the polar primes of (3.2) with respect to the quadrics of N have a common $[3p-1]$. Since every x_j with suffix one less than a multiple of 3 in (3.2) is zero, the polar primes with respect to Q_0, Q_1, Q_2 have equations from which, respectively, all of $x_{3j}, x_{3j-1}, x_{3j+1}$ are absent. The equations are

$$\left.\begin{array}{l} -vx_{3p+1} + v^3x_{3p-1} - v^4x_{3p-2} + v^6x_{3p-4} \ldots\ldots - v^{3p+1}x_1 = 0, \\ x_{3p+1} - vx_{3p} + v^3x_{3p-2} - v^4x_{3p-3} \ldots\ldots\ldots\ldots + v^{3p}x_1 - v^{3p+1}x_0 = 0, \\ x_{3p} - vx_{3p-1} + v^3x_{3p-3} - v^4x_{3p-4} \ldots\ldots + v^{3p}x_0 = 0. \end{array}\right\} \tag{3.3}$$

Since (3.2) is the vertex of $Q_0+vQ_1+v^2Q_2 = 0$ it is predestined that (3.3) should be linearly dependent with multipliers $1, v, v^2$; but explicit forms of two of the equations are wanted in order to determine the degree of the eliminant. Since, by cancelling v from the first and taking the result with the third equation, one can use two polynomials of degree $3p$ in v, the eliminant is of degree $6p$ in the x_j. This is the contribution that K makes to the order $3p(3p+2)$ of R_{3p}.

4. *The special hexahedral net*

We now commence a study of the specialised hexahedral net N in [**4**]. Now N is based on

$$Q_0 \equiv 2(x_1x_4+x_2x_3), \qquad Q_1 \equiv 2(x_0x_4+x_1x_3)+x_2^2, \qquad Q_2 \equiv 2(x_0x_3+x_1x_2).$$

The cones Q_0 and Q_2 both contain the plane $x_1 = x_3 = 0$; apart from this their common points satisfy

$$\frac{x_0}{x_2} = \frac{x_2}{x_4} = -\frac{x_1}{x_3} \tag{4.1}$$

which are [**1**; p. 362] a canonical form for the equations of a cubic scroll S in [4]. The octavic base curve B of N is therefore composite, consisting of the conic in which Q_1 meets $x_1 = x_3 = 0$ and the curve in which it meets S. But this latter curve is itself composite; the generators of S are, for varying p, the lines

$$x_0 = px_2, \qquad x_2 = px_4, \qquad x_1 = -px_3$$

and those for $p = 0, \infty$, that is, X_3X_4 and X_0X_1 are both on Q_1. The residue, bisecant to the generators, is at once recognised as a rational normal quartic. For a parametric form of (4.1) is

$$(p^2, -pq, p, q, 1) \tag{4.2}$$

and this point is on Q_1 when $3p = 2q^2$. Replacing p by $2q^2/3$ in (4.2) gives the point

$$(4q^4, -6q^3, 6q^2, 9q, 9).$$

The generators X_3X_4 and X_0X_1 of S are the tangents to this quartic at X_4 and X_0.

Since $D_4 \equiv \mu(\nu\lambda-\mu^2)(3\nu\lambda-\mu^2)$ the Jacobian curve of N is tripartite.

Case (i), *when* $\mu = 0$. The pencil of cones has vertices (see 3.1)

$$(1, 0, u, 0, u^2)$$

on a conic Γ in the plane $x_1 = x_3 = 0$. A vertex has the same polar $ux_1+x_3 = 0$ with respect to Q_0 and Q_2, while its polar with respect to Q_1 is $u^2x_0+ux_2+x_4 = 0$. The common plane of these polar solids generates the cubic

$$x_0x_3^2-x_1x_2x_3+x_1^2x_4 = 0$$

having the plane of Γ for a double plane.

Case (ii), *when* $\nu\lambda = \mu^2$. The system, of index 2, of cones has (see 3.2) vertices

$$(1, -v, 0, v^3, -v^4)$$

on a rational quartic K in $x_2 = 0$. The trisecants of K are the members of a regulus on the only quadric $x_0 x_4 = x_1 x_3$ containing K; two of them are inflectional tangents, their contacts X_0 and X_4.

The plane common to the polar solids of a point on K with respect to all the quadrics of N is, by (3.3) with $p = 1$,

$$\begin{aligned} v^3 x_0 \qquad\quad & -v x_2 + x_3 = 0, \\ v^3 x_1 - v^2 x_2 \qquad & \qquad + x_4 = 0 \end{aligned}$$

and the equation of the sextic primal generated by this plane is, by Sylvester's dialytic process,

$$\begin{vmatrix} x_0 & \cdot & -x_2 & x_3 & \cdot & \cdot \\ \cdot & x_0 & \cdot & -x_2 & x_3 & \cdot \\ \cdot & \cdot & x_0 & \cdot & -x_2 & x_3 \\ \cdot & \cdot & x_1 & -x_2 & \cdot & x_4 \\ \cdot & x_1 & -x_2 & \cdot & x_4 & \cdot \\ x_1 & -x_2 & \cdot & x_4 & \cdot & \cdot \end{vmatrix} = 0.$$

It has been deemed worthwhile to write this determinant down because it shows that the primal meets $x_2 = 0$ in the quadric $x_0 x_4 = x_1 x_3$ three times over. Also its intersection with $x_1 = x_3 = 0$ is $x_0 x_4 (x_0 x_4 - x_2^2)^2 = 0$.

Case (iii), *when* $3\nu\lambda = \mu^2$. This is the only one of the three families of cones not covered by the work in §3. Any cone belonging to it can be identified by $\lambda : \mu : \nu = t^{-2} : -t^{-1}\sqrt{3} : 1$ and the structure of D_4 shows, taking the rows *seriatim* downwards, that the vertex of the cone satisfies

$$\begin{aligned} x_3 - t^{-1}\sqrt{3}x_4 = x_2 - t^{-1}\sqrt{3}x_3 + t^{-2}x_4 &= x_1 - t^{-1}\sqrt{3}x_2 + t^{-2}x_3 \\ = x_0 - t^{-1}\sqrt{3}x_1 + t^{-2}x_2 &= -t^{-1}\sqrt{3}x_0 + t^{-2}x_1 = 0, \end{aligned}$$

so that it is the point

$$(1, t\sqrt{3}, 2t^2, t^3\sqrt{3}, t^4) \tag{4.3}$$

and traces, as the cone varies in the family, a rational normal quartic Δ.

The polar solids of (4.3) have a common plane, so that it is enough to define this as the plane of intersection of the polar solids with respect to Q_0 and Q_2. These are

$$\begin{aligned} t^3 x_1 + t^2\sqrt{3}x_2 + 2t x_3 + \sqrt{3}x_4 &= 0, \\ t^3\sqrt{3}x_0 + 2t^2 x_1 + t\sqrt{3}x_2 + x_3 &= 0 \end{aligned} \tag{4.4}$$

and the dialytic elimination of t shows that this plane generates a sextic primal $\mathfrak{S}$ when t varies.

The fact of ϑ being tripartite for N forces R_3^{15} to be tripartite too, and the above discussion has shown that R_3^{15} consists of a cubic and two sextics.

5.

It may be recalled that if P is a point on the Jacobian curve ϑ of a general net of quadrics in [4] its polar solids with respect to the quadrics of the net all contain the same plane, a *secant plane* meeting ϑ in six points [**2**; p. 197] and called the plane *conjugate to* P. When the net satisfies the necessary conditions to be hexahedral the six points are the vertices of a quadrilateral of trisecants of ϑ [**3**; p. 273]; whereas the Jacobian curve of a general net has but 20 trisecants [**2**; p. 205], that of the hexahedral net has an infinity generating a scroll R_2^{20} [**3**; p. 284]. But the net now being studied is further specialised, ϑ being composed of Γ, K, Δ, so that R_2^{20} breaks up.

Now Γ and Δ are both rational normal curves so that neither has any trisecants; but K has, and these contribute a quadric to R_2^{20}. Any other trisecants of ϑ are either transversal to all three of its components or else are chords of one and meet another. The chords of Γ all lie in its plane, and those of K in $x_2 = 0$; but those of Δ generate [**7**; p. 10] the cubic

$$\begin{vmatrix} x_0 & x_1/\sqrt{3} & x_2/2 \\ x_1/\sqrt{3} & x_2/2 & x_3/\sqrt{3} \\ x_2/2 & x_3/\sqrt{3} & x_4 \end{vmatrix} = 0 \tag{5.1}$$

meeting $x_2 = 0$ in the cubic scroll $x_0 x_3^2 + x_1^2 x_4 = 0$ which contains K. So one anticipates that R_2^{20} has three components, viz.

(1) trisecants of K;

(2) transversal trisecants;

(3) chords of Δ meeting K.

It is a straightforward matter to see how these lines are distributed in the secant planes according as the points to which these are conjugate are on Γ, K, or Δ; one simply takes the plane, given by a pair of already available linear equations, and notes its intersections with Γ, K, Δ. For instance, (4.4) meets Δ in the points with parameters $-t, \omega t, \omega^2 t$, that is, t multiplied by the non-primitive sixth roots of unity other than unity itself. The same plane meets K where

$$t^3\sqrt{3} - 2t^2 v + v^3 = -t^3 v + 2tv^3 - \sqrt{3}v^4 = 0,$$

and so where

$$t^2 - tv\sqrt{3} + v^2 = 0,$$

$$t^2 + i(\omega - \omega^2)tv + v^2 = 0,$$

$$(t + i\omega v)(t - i\omega^2 v) = 0;$$

and it is found to meet Γ where $u = -t^2$. Hence the secant plane conjugate to a point of Δ is trisecant to Δ, bisecant to K, unisecant to Γ.

Similar routine proceedings show that if P is on K with v as parameter the conjugate secant plane is trisecant to K at $-v$, $-\omega v$, $-\omega^2 v$, bisecant to Δ at $t = -i\omega v, i\omega^2 v$ and unisecant to Γ at $u = v^2$; while if P is on Γ with u as parameter the corresponding plane is bisecant to all of Γ, K, Δ, the intersections having parameters ωu, $\omega^2 u$; $v = u^{1/2}$, $-u^{1/2}$; $t = iu^{1/2}$, $-iu^{1/2}$. We just give this one example in detail: the six points have coordinates

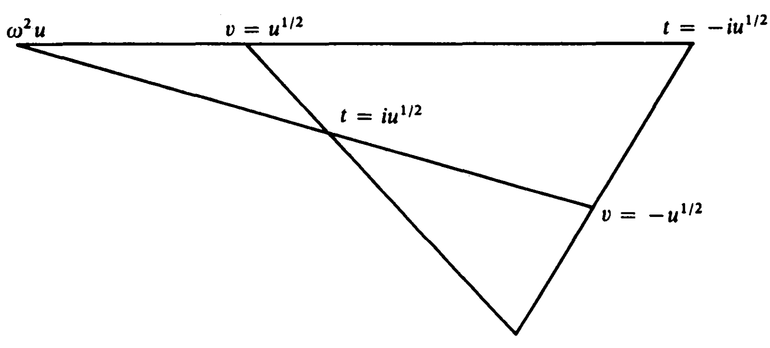
$$\begin{aligned} &(1, 0, \omega u, 0, \omega^2 u^2) \quad \text{and} \quad (1, 0, \omega^2 u, 0, \omega u^2) \quad \text{on } \Gamma, \\ &(1, -u^{1/2}, 0, u^{3/2}, -u^2) \quad \text{and} \quad (1, u^{1/2}, 0, -u^{3/2}, -u^2) \quad \text{on } K, \\ &(1, iu^{1/2}\sqrt{3}, -2u, -iu^{3/2}\sqrt{3}, u^2) \quad \text{and} \quad (1, -iu^{1/2}\sqrt{3}, -2u, iu^{3/2}\sqrt{3}, u^2) \quad \text{on } \Delta. \end{aligned}$$

The general theory requires that these six points lie by threes on four lines; that they do so is shown by identities like

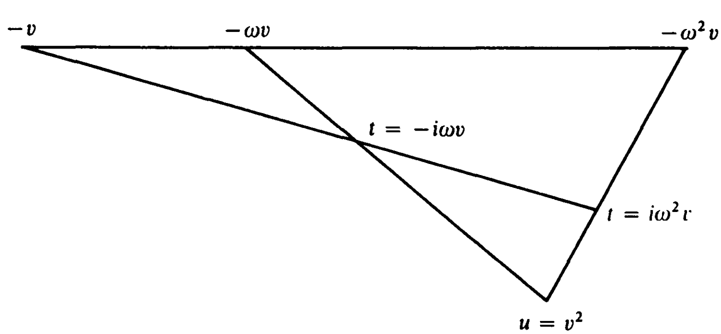
$$[1, (\omega-\omega^2)u^{1/2}, -2u, (\omega^2-\omega)u^{3/2}, u^2] + (\omega-\omega^2)[1, -u^{1/2}, 0, u^{3/2}, -u^2] + 2\omega^2[1, 0, \omega u, 0, \omega^2 u^2] \equiv 0$$

wherein $i\sqrt{3}$ has been given its value $\omega - \omega^2$.

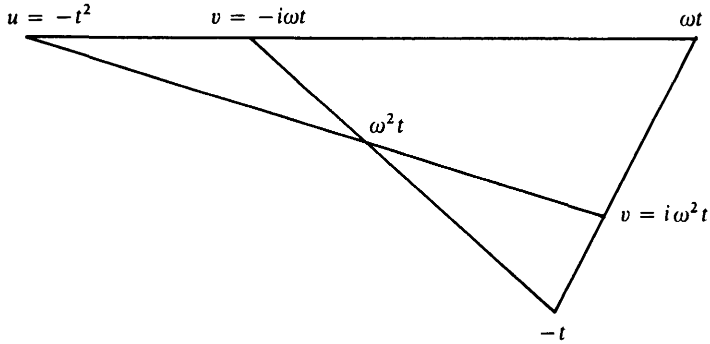
The collinearities in the various secant planes are indicated in the accompanying diagrams which note the parameters, of the six intersections of each plane with the



Γ



K



Δ

composite curve, in terms of that of the point on Γ, K or Δ, to which the plane is conjugate.

6.

The two chords of Δ through the point with parameter t which meet K have $-\omega t$ and $-\omega^2 t$ for the parameters of their further intersections with Δ. This multiplication by the two primitive sixth roots of unity suggests ranging the chords of Δ in closed hexagons Ψ; the sides of all these Ψ all meet K and therefore do so at their intersections with $x_2 = 0$. A move round Ψ from one side to the next multiplies the two parameters on Δ, and so the parameter of the intersection of the side with K, by $-\omega$; repetition multiplies the parameter of the intersection with K by ω^2. Hence alternate sides of Ψ meet K in the collinear points of one of its trisecants, this being the unique line meeting the three sides of Ψ involved in the alternation.

The planes spanned by *alternate* vertices of Ψ contain X_2; (4.3) and the two rows derived from it by replacing t in succession by ωt and $\omega^2 t$ are three rows of which $(0, 0, 1, 0, 0)$ is a linear combination.

The planes spanned by three *consecutive* vertices of Ψ generate $\mathfrak{S}$; such planes occur on varying t in (4.4). Now $\mathfrak{S}$ meets Π, the cubic primal (5.1) generated by all the chords of Δ, in two scrolls; one of these consists of the joins of alternate vertices of Ψ, and these joins are all in trisecant planes of Δ containing X_2. These planes generate [7; p. 10] a quadric cone and the joins a scroll of order 6, the surface common to this cone and Π. A residue of order 12 remains to complete the surface common to $\mathfrak{S}$ and Π. But the sides of Ψ are double lines on $\mathfrak{S}$; if A, B, C, D are consecutive vertices in this order BC is in both planes ABC and BCD. Hence the sides of Ψ generate a sextic scroll; its intersection with $x_2 = 0$ includes K and is completed by X_0X_1 and X_3X_4.

The scroll R_2^{20}, mentioned in §5, of trisecants of the Jacobian curve of a hexahedral net is now tripartite, and two of its three constituents have been identified, namely the sextic scroll immediately above and the quadric through K. There remains a scroll T of order 12 which must consist of the transversals of Γ, K, Δ. If u, v, t are the parameters of the intersections of such a transversal then the matrix

$$\begin{bmatrix} 1 & 0 & u & 0 & u^2 \\ 1 & -v & 0 & v^3 & -v^4 \\ 1 & t\sqrt{3} & 2t^2 & t^3\sqrt{3} & t^4 \end{bmatrix}$$

has rank 2: the conditions

$$v^2+t^2 = 0, \quad t^4-ut^2+u^2 = 0, \quad u^2+uv^2+v^4 = 0$$

for this are seen to hold for all the 4, 2, 2 transversals in the diagrams. If v is given then there are two admissible values $\pm iv$ for t; the two quadratic conditions on u are then consistent. Also K is a double curve on T and is included twice in the duodecimic intersection of T with $x_2 = 0$. The remaining quartic will consist of X_0X_1 and X_3X_4, both reckoned twice.

7.

The rational normal quartic C osculated by the faces of the hexahedra has meanwhile been off stage; but the fact that the plane faces of these hexahedra generate three primals while their vertices trace three algebraically distinct curves is a consequence of there being three ways of relating pairs of six objects running in a single cycle; two places in the cycle may be (a) opposite, (b) alternative, (c) consecutive.

An osculating solid of C is (see 2.1)

$$x_0+\theta x_1+\theta^2 x_2+\theta^3 x_3+\theta^4 x_4 = 0;$$

those solids whose contacts have parameters $\phi_1, \phi_2, \phi_3, \phi_4$ intersect [7; p. 8] at

$$(e_4, -e_3, e_2, -e_1, 1)$$

where e_j is the elementary symmetric function of degree j in $\phi_1, \phi_2, \phi_3, \phi_4$. Take the hexahedron whose bounding solids osculate C at those points with the six sixth roots of the complex number z^6 for their parameters.

Case (a). If $\phi_5 = z$ and $\phi_6 = -z$ then, since

$$\theta^6-z^6 \equiv (\theta^2-z^2)(\theta^4+\theta^2 z^2+z^4),$$

the four solids other than the opposite pair meet at

$$(z^4, 0, z^2, 0, 1),$$

which is the point on Γ with $u = z^{-2}$.

Case (b). If $\phi_5 = z$ and $\phi_6 = \omega z$ then, since

$$\theta^6-z^6 \equiv (\theta^2+\omega^2\theta z+\omega z^2)(\theta^4-\omega^2\theta^3 z+\theta z^3-\omega^2 z^4),$$

the four solids other than the alternate pair meet at

$$(-\omega^2 z^4, z^3, 0, -\omega^2 z, 1),$$

which is the point on K with $v = \omega z^{-1}$.

Case (c). If $\phi_5 = z$ and $\phi_6 = -\omega z$ then, since

$$\theta^6 - z^6 \equiv \{\theta^2 + (\omega-1)\theta z - \omega z^2\}\{\theta^4 + (1-\omega)\theta^3 z - 2\omega\theta^2 z^2 + (\omega^2-\omega)\theta z^3 + \omega^2 z^4\},$$

the four solids other than the consecutive pair meet at

$$[\omega^2 z^4, (\omega^2-\omega)z^3, -2\omega z^2, (1-\omega)z, 1],$$

which is the point on Δ with $t = -i\omega z^{-1}$.

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