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Non-singular models of specialized Weddle surfaces

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INTRODUCTION

(1) There are, in projective space Σ of three dimensions, two famous quartic surfaces: W, the Weddle surface with six nodes ((13), p. 69, footnote) and K, the Kummer surface with sixteen ((8), p. 246, (7) *passim*). They are in birational correspondence and have the same non-singular model: the octavic base surface F of the net of quadrics in [5], which contain a given line λ and for which a given simplex S is self-polar ((5); (6)). One naturally takes S, with vertices $X_0, X_1, X_2, X_3, X_4, X_5$ as simplex of reference for homogeneous coordinates $x_0, x_1, x_2, x_3, x_4, x_5$; F is invariant under the harmonic inversions h_j in the vertices X_j and opposite bounding primes $x_j = 0$ of S. These six h_j, mutually commutative and having identity for their product, generate an elementary abelian group $\mathscr{E}$ of order 32. This representation of $\mathscr{E}$ throws into prominence what may, in this context, be called its positive subgroup $\mathscr{E}^+$, of order 16, consisting of identity and the 15 products $h_j h_k = h_k h_j$; these are harmonic inversions in the edges $X_j X_k$ and opposite bounding solids $x_j = x_k = 0$ of S. The coset of $\mathscr{E}^+$ consists of the six h_j and their ten products in threes, complementary products being the same $(h_0 h_1 h_2 \equiv h_3 h_4 h_5)$ because of the product of all six h_j being identity. These ten products are harmonic inversions in the ten pairs of opposite plane faces of S.

λ, which is presumed skew to every solid $x_j = x_k = 0$, is thus one of a set of 32 lines

$$\lambda, \quad \lambda_j, \quad \lambda_{jk}, \quad \lambda_{ijk};$$

it meets, at its intersections with the primes $x_j = 0$, its six images in the h_j but is skew to the other 25 lines. The 32 lines are a closed set under $\mathscr{E}$, and are equivalent; absence of a suffix confers no privilege, each line meeting its six images in the h_j. For example,

$$\lambda_{01} \quad \text{meets} \quad \lambda_1, \quad \lambda_0, \quad \lambda_{012}, \quad \lambda_{013}, \quad \lambda_{014}, \quad \lambda_{015};$$
$$\lambda_{012} \quad \text{meets} \quad \lambda_{12}, \quad \lambda_{02}, \quad \lambda_{01}, \quad \lambda_{45}, \quad \lambda_{35}, \quad \lambda_{34};$$

and so on.

The projection of F from λ onto a solid Σ skew to λ is W; this is explained in (5). The plane joining λ to a point P of F meets Σ at a point p and, as P traces F, p traces W. But the plane λX_j contains λ_j so that its intersection n_j with Σ is the projection of every point on λ_j; n_j is a node of W. Since λ_{jk} meets both λ_j and λ_k its projection is $n_j n_k$. Since the projection of λ_{012} must meet the projections of any lines on F that meet λ_{012} this projection is the line $l_{012} = l_{345}$ common to the planes $n_0 n_1 n_2$ and $n_3 n_4 n_5$.

(2) In this communication some description is offered of specialized surfaces F_* whose projections are specialized surfaces W_*; they possess properties not possessed

by F. This endowment with additional properties is familiar for specialized Kummer surfaces K_*. The best known example is the tetrahedroid K_1 recognized by Cayley(3) in 1846 as the projective generalization of the Wave Surface in Euclidean space; K_1 can be parametrized by elliptic functions, not so K. And whereas every curve on K is its curve of contact with some other surface there are curves on K_1 that are not curves of contact: indeed there is a tetrahedron T (hence the name tetrahedroid) each of whose faces meets K_1 in a pair of conics. The corresponding non-singular model F_1 was mentioned towards the close of (6); it contains eight conics, whereas there are no conics on F. Hence there are eight curves on W_1 that are not present on W; these are lines or conics according as the conics on F_1 of which they are projections meet, or do not meet, λ.

Cayley ((3), p. 303) unhappily says that there are, in addition to the faces of T, 48 other planes meeting K_1 in pairs of conics; this is not so, nor is the dual statement on p. 303 concerning pairs of quadric tangent cones true. The slip is a consequence of the unwarranted assumption that six concurrent lines are joined in pairs by 15 planes, whereas the lines in question happen to be intersections in pairs of only four planes! The 48 planes imagined by Cayley consist in fact of four sets of four concurrent tropes, each trope reckoned thrice. It was only too easy to be so deceived when the geometry, indeed the very existence, of the general Kummer surface was unknown; but regret that the mistake was not corrected when the volume of collected papers was published more than 40 years later is perhaps permissible.

The specialized surfaces W_* admit groups of self-projectivities; these, of course, operate on Σ whereas the field of operations of the non-projective involutions discovered by Baker (1) so long ago is confined to W_* itself. But our main purpose is to give some description of the non-singular models F_* and thereby of the nets of quadrics whose base surfaces they are.

THE ESSENTIAL FEATURES OF THE NON-SPECIALIZED FIGURE

(3) It is necessary, before imposing any specialization, to give details of certain formalities concerning F, and the algebra involved when projecting F into W.

The equations determining F are

$$\Omega_0 = \Omega_1 = \Omega_2 = 0,$$

where
$$\Omega_k \equiv \Sigma a_j^k x_j^2,$$

summations being for $j = 0, 1, 2, 3, 4, 5$. It is presumed throughout that no two a_j are equal. Write

$$f(\theta) \equiv (\theta - a_0)(\theta - a_1)(\theta - a_2)(\theta - a_3)(\theta - a_4)(\theta - a_5)$$
$$\equiv \theta^6 - e_1\theta^5 + e_2\theta^4 - e_3\theta^3 + e_4\theta^2 - e_5\theta + e_6;$$

also
$$s_k = \Sigma a_j^k / f'(a_j).$$

Then
$$s_0 = s_1 = s_2 = s_3 = s_4 = 0, \quad s_5 = 1, \tag{3·1}$$

and any s_n can be found in terms of the e_j from these initial conditions and the recurrence relation

$$s_{k+6}-e_1 s_{k+5}+e_2 s_{k+4}-e_3 s_{k+3}+e_4 s_{k+2}-e_5 s_{k+1}+e_6 s_k = 0, \tag{3·2}$$

the last $5-k$ terms here being zero if $k<5$.

The relations (3·1) show that λ, the line

$$x_j\sqrt{f'(a_j)} = \theta + a_j, \tag{3·3}$$

where θ is a parameter, lies on F; 32 lines λ, λ_j, λ_{jk}, λ_{ijk} are obtained from the different signing of the six square roots. Every point on λ satisfies

$$X_0 = X_1 = X_2 = X_3 = 0,$$

where

$$X_k = \Sigma a_j^k x_j/\sqrt{f'(a_j)}; \tag{3·4}$$

λ is skew to Σ, the solid $X_4 = X_5 = 0$, because the Vandermonde determinant $|a_j^i|$ is not zero. So F may be projected from λ onto Σ, wherein W then appears.

The tetrahedron of reference in Σ is to be that whose faces $x=0$, $y=0$, $z=0$, $t=0$ are its planes of intersection with the [4]'s $X_0=0$, $X_1=0$, $X_2=0$, $X_3=0$. One therefore seeks to know the coordinates of its vertices in terms of the x_j. With this in view note that, when x_j is replaced by $1/\sqrt{f'(a_j)}$, X_k becomes s_k which, by (3·1), makes each X_k zero save $X_5=1$. Next, replacing x_j by $(a_j-e_1)/\sqrt{f'(a_j)}$, X_k becomes $s_{k+1}-e_1 s_k$ which, by (3·1), is zero for $k=0,1,2,3$ and 1 for $k=4$; it is also zero for $k=5$ by (3·2). Proceeding thus one finally replaces x_j by

$$(a_j^5-e_1 a_j^4+e_2 a_j^3-e_3 a_j^2+e_4 a_j-e_5)/\sqrt{f'(a_j)}$$

and so X_k becomes

$$s_{k+5}-e_1 s_{k+4}+e_2 s_{k+3}-e_3 s_{k+2}+e_4 s_{k+1}-e_5 s_k,$$

which is $-e_6 s_{k-1}=0$ for $k=1,2,3,4,5$ and $s_5=1$ for $k=0$. The upshot is that any point in [5] can be labelled by

$$\begin{aligned}x_j\sqrt{f'(a_j)} &= x(a_j^5-e_1 a_j^4+e_2 a_j^3-e_3 a_j^2+e_4 a_j-e_5)\\ &\quad +y(a_j^4-e_1 a_j^3+e_2 a_j^2-e_3 a_j+e_4)\\ &\quad +z(a_j^3-e_1 a_j^2+e_2 a_j-e_3)+t(a_j^2-e_1 a_j+e_2)+p(a_j-e_1)+q,\end{aligned} \tag{3·5}$$

where x, y, z, t can now be used as coordinates in Σ; p, q as coordinates on λ.

Now substitute these expressions for x_j in $\Omega_0=0$, $\Omega_1=0$, $\Omega_2=0$; all three substitutions produce linear relations in p and q which may then be eliminated determinantally. The first two columns of the determinant are linear, the third quadratic, in x, y, z, t and so the equation (3·6) below of the quartic surface W emerges. That this form of equation for a surface mentioned so long ago as 1849 is new is because the projection from Σ has not been used before. But however new the form may be the equation itself, as a referee has kindly said, is not; it was found by Cayley in 1869 ((4), p. 179; (7), p. 172). One may therefore be excused from detailing the calculations of the various entries in the determinant; those in the first two columns are dealt with more briefly than those in the third, but constantly repeated appeals to (3·1) and (3·2) bring one to

$$\begin{vmatrix} x & y & 2zt - e_1(z^2+2yt) + 2e_2(yz+xt) - e_3(y^2+2zx) + 2e_4xy - e_5x^2 \\ y & z & t^2 - 2e_1zt + e_2(z^2+2yt) - 2e_3yz + e_4y^2 - e_6x^2 \\ z & t & -e_1t^2 + 2e_2zt - e_3z^2 + e_5y^2 - 2e_6xy \end{vmatrix} = 0. \qquad (3{\cdot}6)$$

Incidentally the tangent plane to F at a point of λ, being common to the tangent primes of the quadrics containing F, is

$$\theta X_0 + X_1 = \theta X_1 + X_2 = \theta X_2 + X_3 = 0,$$

so that these tangent planes generate

$$X_0/X_1 = X_1/X_2 = X_2/X_3,$$

a line-cone meeting Σ in the twisted cubic γ;

$$x/y = y/z = z/t.$$

On γ the first two columns of the determinant are proportional, so that γ is on W which is readily seen to have nodes at the points $\phi = a_j$ when γ is parametrized as

$$x:y:z:t = 1:\phi:\phi^2:\phi^3.$$

(4) Several writers have submitted equations for W, Baker proferring no less than eleven in (1); but they seem, with the notable exception of Cayley, always to select a special tetrahedron of reference. Nothing could be more natural: but equations so obtained, unlike Cayley's, are not symmetrically related to the six nodes.

Cayley's equation appears at the very end of a long memoir, possibly as an afterthought and his method of obtaining it is perhaps not the simplest one. For all one needs is the Jacobian of the web of quadrics through the nodes, a web spanned by the net containing γ and one other quadric meeting γ only at the nodes. As the net is spanned by

$$xz = y^2, \quad xt = yz, \quad yt = z^2$$

the web is spanned by these and, say, any of the ten plane-pairs containing the six nodes. But in order to obtain an equation symmetric in the a_j one must use a quadric that itself has this symmetry; for example

$$e_6x^2 - e_5xy + e_4zx - e_3yz + e_2yt - e_1zt + t^2 = 0.$$

The Jacobian W of this web is

$$\begin{vmatrix} z & t & . & 2e_6x - e_5y + e_4z \\ -2y & -z & t & -e_5x - e_3z + e_2t \\ x & -y & -2z & e_4x - e_3y - e_1t \\ . & x & y & e_2y - e_1z + 2t \end{vmatrix} = 0.$$

This determinant, not given by Cayley, is that in (3·6) multiplied by -2.

(5) The merit of Cayley's equation, in whichever form, is to be available even when the situation of the nodes on γ is projectively specialized. Suppose, to take a single example, that the a_j are the six sixth roots of unity. Then

$$e_1 = e_2 = e_3 = e_4 = e_5 = 0, \quad e_6 = -1$$

and (3·6) collapses to

$$\begin{vmatrix} x & y & 2zt \\ y & z & x^2+t^2 \\ z & t & 2xy \end{vmatrix} = 0,$$

$$2(xy^3+z^3t) = (x^2+t^2)(3yz-xt). \tag{5·1}$$

This surface W_4 is in birational correspondence with the quadruple tetrahedroid K_4 ((6), p. 966).

F_1 AND ITS DOUBLE-FOUR OF CONICS

(6) F is invariant under the harmonic inversions in pairs of opposite plane faces of S, and under no other biplanar inversions than these ten. But since specializations F_* of F will be invariant under further such inversions it will best serve our purpose if the following lemma is ready to hand. The notation used chimes in with later developments, and although the lemma is stated with $m = 3$ it has a clear analogue in $[2m-1]$.

LEMMA. *The harmonic inversion in the two planes*

$$\alpha x_0/y_0 = \beta x_2/y_2 = \gamma x_4/y_4 = \pm k \tag{6·1}$$

transposes the points whose coordinate vectors are

$$x_0, \quad y_0, \quad x_2, \quad y_2, \quad x_4, \quad y_4 \tag{6·2}$$

and

$$ky_0/\alpha, \quad \alpha x_0/k, \quad ky_2/\beta, \quad \beta x_2/k, \quad ky_4/\gamma, \quad \gamma x_4/k. \tag{6·3}$$

This is because the sum of these two vectors provides a point in one of the planes (6·1), their difference a point in the other. One may add, what is readily proved, that the pairs (6·1) with $\pm k$ and $\pm k'$ afford commutative inversions if, and only if, $k' = \pm ik$; and, further, that the product of any such commutative pair is harmonic inversion in the planes $x_0 = x_2 = x_4 = 0 (k = 0)$ and $y_0 = y_2 = y_4 = 0 (k = \infty)$.

(7) The Kummer surface K becomes a tetrahedroid K_1 when the six nodes on the conic in any trope can be partitioned as three pairs of an involution J. Assign to the foci of J the parameters 0 and ∞; then the parameters of any pair of J have zero sum. The non-singular model of K_1 is, as remarked in (6), F_1, the surface $\Omega_0 = \Omega_1 = \Omega_2 = 0$ where now

$$\Omega_0 \equiv x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2,$$
$$\Omega_1 \equiv a(x_0^2-x_1^2)+b(x_2^2-x_3^2)+c(x_4^2-x_5^2),$$
$$\Omega_2 \equiv a^2(x_0^2+x_1^2)+b^2(x_2^2+x_3^2)+c^2(x_4^2+x_5^2).$$

F_1, though not F, is unchanged by the triple transposition $(x_0x_1)(x_2x_3)(x_4x_5)$ since this changes neither Ω_0 nor Ω_2 and merely multiplies Ω_1 by -1. This operation is, by (6·3), the harmonic inversion in the planes

$$x_0/x_1 = x_2/x_3 = x_4/x_5 = \pm 1 \tag{7·1}$$

and so commutes with H, the harmonic inversion in

$$x_0/x_1 = x_2/x_3 = x_4/x_5 = \pm i; \tag{7·2}$$

the product of these two inversions is $h_0h_2h_4 \equiv h_1h_3h_5$, inversion in the opposite plane faces $X_0X_2X_4$ and $X_1X_3X_5$ of S; so H and $h_0h_2h_4$ commute. Since H permutes the vertices of S as

$$(X_0X_1)(X_2X_3)(X_4X_5)$$

it transforms each member of any pair

$$h_0, h_1; \quad h_2, h_3; \quad h_4, h_5$$

into the other, and so commutes with h_0h_1, h_2h_3, h_4h_5. Thus H commutes with all of

$$\left.\begin{matrix} I & h_0h_1 & h_2h_3 & h_4h_5 \\ h_0h_2h_4 & h_1h_2h_4 & h_0h_3h_4 & h_0h_2h_5 \end{matrix}\right\} \tag{7·3}$$

which compose the centre of the non-abelian group of self-projectivities of F_1.

The eight planes

$$x_0^2+x_1^2 = x_2^2+x_3^2 = x_4^2+x_5^2 = 0 \tag{$\mathscr{D}$}$$

are, as noted in (6), on both $\Omega_0 = 0$ and $\Omega_2 = 0$, so that the conics in which they meet $\Omega_1 = 0$ are on F_1. They consist of four pairs of skew planes, one such pair being

$$-x_0/x_1 = x_2/x_3 = x_4/x_5 = \pm i; \tag{7·4}$$

harmonic inversion in this pair is the transform of H by h_0 (or by h_1). Inversions in two other pairs of skew planes in $\mathscr{D}$ are the transforms of H by h_2 and by h_4; the four inversions in opposite pairs of planes in $\mathscr{D}$ are

$$H, \quad Hh_0h_1, \quad Hh_2h_3, \quad Hh_4h_5.$$

These are the coset of the first line of (7·3) in an elementary abelian group of order 8.

(8) The eight planes $\mathscr{D}$ form the base surface of a net of quadrics whose singular members include three pencils of line-cones with respective vertices X_0X_1, X_2X_3, X_4X_5; the common member of any two of these pencils is a pair of primes. The eight planes belong four to each of the two systems on any non-singular quadric of the net. The figure has been aptly called ((2), p. 238) a double-four of planes and occurs when the lines of a tetrahedral complex in [3] are mapped on a non-singular quadric in [5], the vertices and faces of the tetrahedron providing the four planes of each of the two systems.

Each plane of $\mathscr{D}$ is transversal to X_0X_1, X_2X_3, X_4X_5; it is spanned by three points, one on each of these edges of S. Each of the 12 joins of two of these points not on the same edge is, as lying in two of the planes, on both $\Omega_0 = 0$ and $\Omega_2 = 0$ so that its two intersections with $\Omega_1 = 0$ are on F_1; the conics in which the two planes meet F_1 both pass through these two points on the line.

Through each of the six points, two on each of X_0X_1, X_2X_3, X_4X_5, pass four planes of $\mathscr{D}$; they lie in the [4] joining the point to the solid spanned by those two of the three edges on which the point does not lie; this [4] meets F_1 in four conics, any one of these being met twice by each of two others. There are six such composite sections of F_1.

(9) The cardinal feature on which the relation between F_1 and its projection W_1 hinges is that each line on F_1 *meets one of the four skew pairs of* $\mathscr{D}$. Since

$$\begin{matrix} a_0, & a_1, & a_2, & a_3, & a_4, & a_5 \end{matrix}$$

are now replaced by

$$\begin{matrix} a, & -a, & b, & -b, & c, & -c \end{matrix}$$

$f(\theta)$ becomes an even and so $f'(\theta)$ an odd function. The first two of the six relations (3·3) become

$$x_0\sqrt{f'(a)} = \theta + a, \quad x_1\sqrt{f'(-a)} = \theta - a$$

so that $(x_0^2 + x_1^2) f'(a) = 4a\theta$, a 'quadratic' in θ whose roots are 0 and ∞; one of these roots makes $x_0 + ix_1$, the other $x_0 - ix_1$, zero. Analogous statements hold for the present forms of the other two pairs of equations in (3·3). The upshot is that the two foci of that involution now existing on λ three of whose pairs are its intersections with λ_0 and λ_1, λ_2 and λ_3, λ_4 and λ_5 are one in each of a skew pair of $\mathscr{D}$.

Every individual plane of $\mathscr{D}$ is invariant under the four projectivities in the top line of (7·3) while those in the bottom line transpose the planes of each skew pair. If, therefore, a skew pair of $\mathscr{D}$ meets λ it meets all of

$$\left.\begin{matrix} \lambda & \lambda_{01} & \lambda_{23} & \lambda_{45} \\ \lambda_{024} & \lambda_{124} & \lambda_{034} & \lambda_{025}. \end{matrix}\right\} \tag{9·1}$$

A plane of $\mathscr{D}$ that meets λ meets λ_{01}, λ_{23}, λ_{45} in points with the same parameter, whether 0 or ∞, since $h_0 h_1$, $h_2 h_3$, $h_4 h_5$ leave both plane and parameter unchanged; but the same plane meets λ_{024}, λ_{124}, λ_{034}, λ_{025} in the points with the opposite parameter because the inversions in the lower line on (7·3), while not changing the parameter, transpose the plane with its opposite member in $\mathscr{D}$.

The three sets of eight lines that meet the other skew pairs are found by operating on this set with h_0, h_2 and h_4 since these inversions transform any opposite pair of $\mathscr{D}$ into the other three pairs. One such set is

$$\left.\begin{matrix} \lambda_0 & \lambda_1 & \lambda_{023} & \lambda_{045} \\ \lambda_{24} & \lambda_{35} & \lambda_{34} & \lambda_{25}. \end{matrix}\right\} \tag{9·2}$$

λ, meeting two conics on F_1, is skew to the six others and their planes. Consider, then the [4] spanned by λ and σ, one of these six conics. The opposite pair of planes of $\mathscr{D}$ which includes the plane of σ is obtained from the pair which meet λ by operating with an h_j, say, to be definite, with h_0 and so with h_1 too. Then σ meets both $h_0\lambda = \lambda_0$ and $h_0\lambda_{01} = \lambda_1$; the [4] therefore contains λ, λ_0 and λ_1 as meeting both σ and λ, as well as λ_{01} meeting both λ_0 and λ_1. But X_0 and X_1 are one on each diagonal of the quadrilateral $\lambda\lambda_0\lambda_{01}\lambda_1$ so that the [4] is invariant under h_0 and h_1 and contains the conic

$$\tau = h_0\sigma = h_1\sigma.$$

Its complete intersection with F_1 consists of a quadrilateral and a pair of conics, each conic meeting an opposite pair of sides of the quadrilateral.

(10) The projection W_1 of F_1 from λ onto Σ is now seen to contain two lines, projections of the conics on F_1 that meet λ; these lines ((11), p. 359) μ, μ' pass one through

each focus of that involution now existing on γ three of whose pairs are n_0 and n_1, n_2 and n_3, n_4 and n_5. Both μ and μ' meet all seven of the lines (cf. (9·1))

$$n_0 n_1, \quad n_2 n_3, \quad n_4 n_5; \quad l_{024}, \quad l_{124}, \quad l_{034}, \quad l_{025}.$$

There are also six conics on W_1; these lie one in each of the planes joining μ and μ' to $n_0 n_1$, $n_2 n_3$, $n_4 n_5$.

When the involution on γ is $\phi + \phi' = 0$ the chord joining $n_0(\phi = a)$ to $n_1(\phi = -a)$ is

$$t - a^2 y = z - a^2 x = 0.$$

The lines μ, μ' are $y = t = 0$ and $z = x = 0$. Since, with

$$f(\theta) = (\theta^2 - a^2)(\theta^2 - b^2)(\theta^2 - c^2),$$
$$e_1 = e_3 = e_5 = 0$$

the equation of W_1 is

$$\begin{vmatrix} x & y & 2zt + 2e_2(yz + xt) + 2e_4 xy \\ y & z & t^2 + e_2(z^2 + 2yt) + e_4 y^2 - e_6 x^2 \\ z & t & 2e_2 zt - 2e_6 xy \end{vmatrix} = 0,$$

and the presence of the lines on the surface can be verified. The expanded form, to be available for reference, is

$$t(3yzt - 2z^3 - xt^2) + e_2 z(2y^2 t - yz^2 - xzt) + e_4 y(y^2 z + xyt - 2xz^2) + e_6 x(x^2 t + 2y^3 - 3xyz) = 0. \quad (10·1)$$

Points in [5], neither of them on λ, that are harmonic inverses in the pair of planes of $\mathscr{D}$ that meet λ are projected into points in Σ that are harmonic inverses in μ and μ'. With the present coordinate system this is effected by multiplying either x and z or y and t by -1, a procedure which merely multiplies the left-hand side of (10·1) by -1 and so leaves W_1 unchanged.

F_2

(11) If $a_0, a_1, a_2, a_3, a_4, a_5$ can be paired not only in one but in two involutions their single common pair may, or may not, include one and therefore two of the a_j. Suppose that it does not, and assign to its two members the parameters 0 and ∞. Any involution which includes this pair has the same product of the two parameters of all its pairs so that there are, say, relations

$$a_0 a_1 = a_2 a_3 = a_4 a_5 = k_1,$$
$$a_3 a_4 = a_0 a_5 = a_1 a_2 = k_2.$$

But these relations imply that

$$a_2 a_5 = a_1 a_4 = a_0 a_3 = k_3,$$

where

$$k_1 k_2 k_3 = a_0 a_1 a_2 a_3 a_4 a_5.$$

Hence the a_j can be paired also in a third involution as proved geometrically long ago on page 12 of (9). This situation will be explored below; one may then presume, for the moment, that the common pair of the two involutions consists of a_4 and a_5 and assign to these the parameters 1 and -1. Any involution that includes this pair has an equation

$$\theta\theta' + \kappa(\theta+\theta') + 1 = 0.$$

Use, therefore, the same label J for the involution $\theta+\theta' = 0$ as above and call the involution $\theta\theta'+1 = 0$ J'; take

$$a_0 + a_1 = a_2 + a_3 = a_4 + a_5 = 0,$$
$$a_0 a_3 = a_1 a_2 = a_4 a_5 = -1$$

so that the a_j are

$$\alpha, \quad -\alpha, \quad \alpha^{-1}, \quad -\alpha^{-1}, \quad 1, \quad -1$$

and

$$f(\theta) \equiv (\theta^2-\alpha^2)(\theta^2-\alpha^{-2})(\theta^2-1). \tag{11·1}$$

This may be the proper place to note, with an eye to any calculations involving points on λ, that

$$f'(\alpha), \quad f'(-\alpha), \quad f'(\alpha^{-1}), \quad f'(-\alpha^{-1}), \quad f'(1), \quad f'(-1)$$

are proportional to

$$\alpha^2, \quad -\alpha^2, \quad \alpha^{-2}, \quad -\alpha^{-2}, \quad -1/(\alpha+\alpha^{-1}), \quad 1/(\alpha+\alpha^{-1}). \tag{11·2}$$

(12) F_2 is the surface common to the quadrics

$$\left.\begin{aligned}\Omega_0 &\equiv x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 = 0,\\ \Omega_1 &\equiv \alpha(x_0^2-x_1^2)+\alpha^{-1}(x_2^2-x_3^2)+x_4^2-x_5^2 = 0,\\ \Omega_2 &\equiv \alpha^2(x_0^2+x_1^2)+\alpha^{-2}(x_2^2+x_3^2)+x_4^2+x_5^2 = 0.\end{aligned}\right\}$$

As with F_1 so with its specialization F_2: the planes of the double-four

$$x_0^2+x_1^2 = x_2^2+x_3^2 = x_4^2+x_5^2 = 0 \tag{$\mathscr{D}$}$$

meet the surface in conics. But now there appears a second double-four

$$\alpha^2x_0^2-x_3^2 = \alpha^2x_1^2-x_2^2 = x_4^2-x_5^2 = 0, \tag{$\mathscr{D}'$}$$

on both $\Omega_1 = 0$ and $\Omega_0 = \Omega_2$, whose eight planes therefore also meet F_2 in conics.

Any of the 32 lines λ on F_2 meets two skew planes of $\mathscr{D}$ and two skew planes of $\mathscr{D}'$, the intersections being the pairs of foci of the two involutions in which the six intersections of λ with other lines on F_2 are paired. The projection W_2 of F_2 from λ onto Σ contains, in consequence of the specialization, four lines and twelve conics.

It was seen in section 9 that λ meets planes of $\mathscr{D}$ where $\theta = 0$ and $\theta = \infty$; (3·3) and (11·2) show that it meets

$$\alpha x_0 + x_3 = \alpha x_1 + x_2 = x_4 - x_5 = 0 \quad \text{where} \quad \theta = i,$$
$$\alpha x_0 - x_3 = \alpha x_1 - x_2 = x_4 + x_5 = 0 \quad \text{where} \quad \theta = -i.$$

The inversion H' in this pair of planes of $\mathscr{D}'$, whose equations are

$$\alpha x_0/x_3 = \alpha^{-1}x_2/x_1 = -x_4/x_5 = \pm 1,$$

commutes with

$$\begin{matrix} I, & h_0h_3, & h_1h_2, & h_4h_5 \\ h_0h_2h_4, & h_2h_3h_4, & h_0h_1h_4, & h_0h_2h_5. \end{matrix}$$

It replaces $x_0, x_1, x_2, x_3, x_4, x_5$ by $\alpha^{-1}x_3, \alpha^{-1}x_2, \alpha x_1, \alpha x_0, -x_5, -x_4$ and so is seen to commute with H that replaces them by

$$ix_1, \quad -ix_0, \quad ix_3, \quad -ix_2, \quad ix_5, \quad -ix_4.$$

Both H and H' commute with

$$I, \quad h_4h_5, \quad h_0h_2h_4, \quad h_0h_2h_5,$$

which form the centre of the group of 128 self-projectivities of F_2; this group consists of its normal subgroup $\mathscr{E}$ and three cosets $H\mathscr{E}$, $H'\mathscr{E}$, $HH'\mathscr{E}$.

The commutativity of H and H' is also apparent from geometrical considerations. For the four planes concerned, skew pairs of $\mathscr{D}$ and $\mathscr{D}'$, all meet the four skew lines $\lambda, \lambda_{45}, \lambda_{024}, \lambda_{025}$ and do so in related ranges: the parameters on λ and λ_{45} cut by any of the four planes are equal while the parameter on either λ or λ_{45} is related to that on either λ_{024} or λ_{025} by the involution $\theta \leftrightarrow \theta^{-1}$. This shows the four planes to be among the ∞^1 generating planes of a cubic threefold V, a family fittingly described ((12), p. 12) as a regulus $\mathscr{R}$ of planes, and parametrized by their intersections with any of the ∞^2 directrix lines of V. The two pairs of $\mathscr{D}$ and $\mathscr{D}'$ are, as their parameters show, harmonic in $\mathscr{R}$ and this not only explains why H and H' commute but discloses that HH' is also a biplanar inversion, its fundamental planes completing in $\mathscr{R}$ a regular sextuple with the two harmonic pairs.

When V is projected from λ, one of its own directrices, onto Σ the planes on V become lines in Σ. Any transversal of three of these lines is joined to λ by a solid meeting three planes of $\mathscr{R}$ each in lines: these lines have a regulus of transversals which, meeting three planes of V, meet them all ((12), p. 13, with $n = 3$). The regulus of planes on V is thus projected from λ into an ordinary regulus of lines in Σ, and the inversions H, H', HH' induce three mutually commutative biaxial inversions in Σ whose axes form a regular sextuple in a regulus.

(13) The projection W_2 of F_2 from λ onto Σ contains four lines μ, μ', ν, ν' belonging to a regulus, and twelve conics. Just as μ, μ' are the transversals $y = t = 0$ and $z = x = 0$ from the foci of J on γ to the chords joining its pairs so are ν, ν' related to J'. But if $\phi\phi' = -1$ the chords

$$\phi\phi' x - (\phi+\phi')y + z = \phi\phi' y - (\phi+\phi')z + t = 0$$

of γ are members of the regulus

$$z - x = py, \quad t - y = pz$$

and so have for transversals the lines of the complementary regulus

$$q(z-x) = t-y, \quad qy = z.$$

This line meets γ where $\phi = q$; nowhere else. So ν, ν' occur when $q = \pm i$ and are

$$\nu: \quad y = iz, \quad x = -it; \quad \nu': \quad y = -iz, \quad x = it.$$

They must, with μ and μ', lie on W_2 and are quickly seen to do so. For now, by (11·1), $e_6 = -1$ and $e_4 = -e_2$ so that (10·1) may be written

$$(x^2+t^2)(3yz-xt) - 2(xy^3+z^3t) = e_2\{(y^2+z^2)(yz+xt) - 2yz(xz+yt)\}. \qquad (13·1)$$

The harmonic inversion in ν and ν' transposes (x, y, z, t) and $(it, -iz, iy, -ix)$; this merely multiplies each side of (13·1) by -1.

It is apparent that μ, μ', ν, ν', all lie on Q, the quadric $xy = zt$, occurring when $p = \infty$, 0, $-i$, i in the regulus $x/t = z/y = p$. They are harmonic pairs therein, and the regular sextuple is completed by $x/t = z/y = \pm 1$. Harmonic inversion in these two lines transposes x with t and y with z, an operation which, by (13·1), leaves W_2 unchanged.

When $e_2 = 0$ (13·1) collapses to (5·1); this occurs when $\alpha^2 + 1 + \alpha^{-2} = 0$, so that α is a sixth root of unity other than 1 and -1. This special case will be encountered later.

Now as with Q, so with W_2; a plane through n_4n_5 has only a single contact, namely the intersection, other than n_4 and n_5, of n_4n_5 with the cubic curve residual to n_4n_5 in which the plane meets W_2. So the points of n_4n_5 are in $(1, 1)$ correspondence with the contacts of planes through n_4n_5 either with Q or with W_2: the sets of contacts with the two surfaces are projectively related. Hence if Q and W_2 have the same tangent plane at three (or more) points on n_4n_5 they touch all along the line. And this they do, having the same tangent plane at the four intersections of n_4n_5 with μ, μ', ν, ν'.

The twelve conics on W_2 are one in each of the planes joining n_0n_1, n_2n_3, n_4n_5 to μ and μ' or joining n_0n_3, n_1n_2, n_4n_5 to ν and ν'.

Since λ_{024} and λ_{025} meet those pairs of $\mathscr{D}$ and $\mathscr{D}'$ which meet λ the lines l_{024} and l_{025} meet all of μ, μ', ν, ν' and so lie on Q. The complete intersection of Q and W_2 is thus accounted for, consisting of (if the line of contact is reckoned twice) four lines in each regulus.

F_3

(14) That there are on conics hexads whose points may be paired in three different involutions is apparent from the figure of two equilateral triangles with the same circumcircle; their six vertices are joined by three parallel chords in three different directions and, incidentally, the three involutions have a common pair 'at infinity'. This Euclidean example also indicates that the six points may be assigned parameters

$$a, \quad \omega a, \quad \omega^2 a, \quad b, \quad \omega b, \quad \omega^2 b,$$

zeros of $f(\theta) \equiv (\theta^3 - a^3)(\theta^3 - b^3)$. The pairings in the three involutions are

I_0	I_2	I_4
a, b	$a, \omega b$	$a, \omega^2 b$
$\omega a, \omega^2 b$	$\omega a, b$	$\omega a, \omega b$
$\omega^2 a, \omega b$	$\omega^2 a, \omega^2 b$	$\omega^2 a, b.$

The respective quadratics whose roots are the foci of these involutions are

$$\theta^2 = ab, \quad \theta^2 = \omega ab, \quad \theta^2 = \omega^2 ab.$$

Take, then, in [5] the quadrics

$$\left.\begin{aligned} \Omega_0 &\equiv x_0^2 + x_2^2 + x_4^2 \qquad\quad + x_1^2 + x_3^2 + x_5^2 \qquad\quad = 0, \\ \Omega_1 &\equiv a(x_0^2 + \omega x_2^2 + \omega^2 x_4^2) \;+\; b(x_1^2 + \omega x_3^2 + \omega^2 x_5^2) = 0, \\ \Omega_2 &\equiv a^2(x_0^2 + \omega^2 x_2^2 + \omega x_4^2) + b^2(x_1^2 + \omega^2 x_3^2 + \omega x_5^2) = 0. \end{aligned}\right\} \qquad (14\cdot1)$$

F_3, the surface $\Omega_0 = \Omega_1 = \Omega_2 = 0$, is invariant under $\wp$, the projectivity of period 3 that replaces

$$x_0, \quad x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5,$$

by

$$\omega^2 x_2, \quad \omega^2 x_3, \quad \omega^2 x_4, \quad \omega^2 x_5, \quad \omega^2 x_0, \quad \omega^2 x_1$$

and so multiplies Ω_0, Ω_1, Ω_2 by ω, 1, ω^2. One expects that F_3 will contain 24 conics, and so it does; they lie in the planes of the three double-fours

$$ax_0^2 + bx_1^2 = \omega a x_2^2 + \omega^2 b x_5^2 = \omega^2 a x_4^2 + \omega b x_3^2 = 0, \qquad (\mathscr{D}_0)$$

$$ax_0^2 + \omega b x_3^2 = \omega a x_2^2 + bx_1^2 = \omega^2 a x_4^2 + \omega^2 b x_5^2 = 0, \qquad (\mathscr{D}_2)$$

$$ax_0^2 + \omega^2 b x_5^2 = \omega a x_2^2 + \omega b x_3^2 = \omega^2 a x_4^2 + bx_1^2 = 0. \qquad (\mathscr{D}_4)$$

Indeed all these 24 planes are on $\Omega_1 = 0$ while those of $\mathscr{D}_0$, $\mathscr{D}_2$, $\mathscr{D}_4$ are, respectively, on $ab\Omega_0 + \Omega_2 = 0$, $\omega ab\Omega_0 + \Omega_2 = 0$, $\omega^2 ab\Omega_0 + \Omega_2 = 0$.

Since, now,

$$f'(\theta) \equiv 3\theta^2(2\theta^3 - a^3 - b^3)$$

the six square roots of

$$f'(a), \quad f'(\omega a), \quad f'(\omega^2 a), \quad f'(b), \quad f'(\omega b), \quad f'(\omega^2 b)$$

are proportional to

$$a, \quad \omega a, \quad \omega^2 a, \quad ib, \quad i\omega b, \quad i\omega^2 b$$

so that λ, given in terms of a parameter θ by

$$\left.\begin{aligned} x_0 &= ib(\theta + a), & x_2 &= ib(\omega^2\theta + a), & x_4 &= ib(\omega\theta + a), \\ x_1 &= a(\theta + b), & x_3 &= a(\omega^2\theta + b), & x_5 &= a(\omega\theta + b), \end{aligned}\right\} \qquad (14{\cdot}2)$$

is on F_3, as are all the other lines which occur on prefixing *minus* signs to any of these x_j. Each of the 32 lines meets six others, the intersections being one in each of the primes $x_j = 0$ and so having parameters

$$-a, \quad -\omega a, \quad -\omega^2 a, \quad -b, \quad -\omega b, \quad -\omega^2 b.$$

These six points are also paired in I_0, in I_2 and in I_4.

Whatever line among the 32 is chosen it meets two planes of each double-four, the parameters of the intersections satisfying

$$\theta^2 = ab, \quad \theta^2 = \omega ab, \quad \theta^2 = \omega^2 ab$$

according to which double-four (but *not* which line) is in question. These are the three pairs of foci of I_0, I_2, I_4. The 192 intersections, six on each of 32 lines, lie eight in each of the 24 planes.

(15) If the non-singular $\Omega_1 = 0$ is cast in the role of Grassmannian of lines in [3] so that the planes, four in each system on $\Omega_1 = 0$, of a double-four map the vertices and faces of a tetrahedron, then $\mathscr{D}_0$, $\mathscr{D}_2$, $\mathscr{D}_4$ correspond to tetrahedra of a desmic triad: for if two planes are taken, one from $\mathscr{D}_0$ and one from $\mathscr{D}_2$, in the same system on $\Omega_1 = 0$ and so with a single common point, it is found that a plane of $\mathscr{D}_4$ also passes through this point. Such a point is (15·2) below. The 16 such points afforded by one system of planes map the 16 lines that contain three vertices, one of each tetrahedron; the 16 afforded by the other system map the 16 lines that lie in three faces, one of each tetrahedron.

Three planes, one belonging to each of $\mathscr{D}_0$, $\mathscr{D}_2$, $\mathscr{D}_4$, that meet λ are

$$\begin{aligned} x_0\sqrt{a}+\ \ ix_1\sqrt{b} &= \omega^2x_2\sqrt{a}+\ \ i\omega x_5\sqrt{b} = \omega x_4\sqrt{a}+i\omega^2x_3\sqrt{b} = 0,\\ x_0\sqrt{a}+i\omega^2x_3\sqrt{b} &= \omega^2x_2\sqrt{a}+\ \ ix_1\sqrt{b} = \omega x_4\sqrt{a}+\ \ i\omega x_5\sqrt{b} = 0,\\ x_0\sqrt{a}+\ \ i\omega x_5\sqrt{b} &= \omega^2x_2\sqrt{a}+i\omega^2x_3\sqrt{b} = \omega x_4\sqrt{a}+\ \ ix_1\sqrt{b} = 0; \end{aligned}$$

here one may take either square root of a and either square root of b, but the same roots are used throughout. These three planes all lie in the prime

$$(x_0+\omega^2x_2+\omega x_4)\sqrt{a}+i(x_1+\omega^2x_3+\omega x_5)\sqrt{b} = 0 \tag{15·1}$$

and so, since they are all on $\Omega_1 = 0$, all contain the pole

$$(\sqrt{b},\ i\sqrt{a},\ \omega\sqrt{b},\ i\omega\sqrt{a},\ \omega^2\sqrt{b},\ i\omega^2\sqrt{a}) \tag{15·2}$$

of this prime with respect to $\Omega_1 = 0$.

The opposite planes of the double-fours occur on multiplying (either $\sqrt{a}$ or) $\sqrt{b}$ by -1 and lie in the prime

$$(x_0+\omega^2x_2+\omega x_4)\sqrt{a}-i(x_1+\omega^2x_3+\omega x_5)\sqrt{b} = 0. \tag{15·3}$$

These primes are transposed by the biplanar inversion $h_0h_2h_4$ and so both contain λ_{024} as well as λ. They both meet F_3 in λ, λ_{024} and three concurrent conics. Each pair of the 32 lines that are transposed by this inversion, for example λ_3 and λ_{15}, spans a solid through which pass two primes each meeting F_3 further in three concurrent conics, so that there are 32 such composite sections of F_3. Since there are 32 points each common to three of the 24 conics there will be four of the points on each conic.

(16) If r is a square root of $-b/a$ F_3 is invariant under the biplanar inversions

$$\begin{aligned} &H_0 \text{ in the planes} \quad x_0/x_1 = \omega x_2/x_5 = \omega^2x_4/x_3 = \pm r,\\ &H_2 \text{ in the planes} \quad \omega x_0/x_3 = \omega^2x_2/x_1 = x_4/x_5 = \pm r,\\ &H_4 \text{ in the planes} \quad \omega^2x_0/x_5 = x_2/x_3 = \omega x_4/x_1 = \pm r, \end{aligned}$$

which, by section 6, transpose x_0, x_1, x_2, x_3, x_4, x_5 with, respectively

$$\left.\begin{matrix} rx_1, & r^{-1}x_0, & \omega^2rx_5, & \omega^2r^{-1}x_4, & \omega rx_3, & \omega r^{-1}x_2;\\ \omega^2rx_3, & \omega^2r^{-1}x_2, & \omega rx_1, & \omega r^{-1}x_0, & rx_5, & r^{-1}x_4;\\ \omega rx_5, & \omega r^{-1}x_4, & rx_3, & r^{-1}x_2, & \omega^2rx_1, & \omega^2r^{-1}x_0. \end{matrix}\right\} \tag{16·1}$$

Matrices imposing H_0, H_2, H_4 are

$$\begin{bmatrix} R & . & . \\ . & . & \omega^2R \\ . & \omega R & . \end{bmatrix},\quad \begin{bmatrix} . & \omega^2R & . \\ \omega R & . & . \\ . & . & R \end{bmatrix},\quad \begin{bmatrix} . & . & \omega R \\ . & R & . \\ \omega^2R & . & . \end{bmatrix},$$

where
$$R = \begin{bmatrix} . & r \\ r^{-1} & . \end{bmatrix}.$$

These matrices are all, with R, of period 2 and, as their form shows, generate a symmetric group of degree 3. Their products in pairs impose $\wp$ and $\wp^2$.

H_0, H_2, H_4 subject the vertices of S to the permutations

$$(X_0 X_1)(X_2 X_5)(X_4 X_3), \quad (X_0 X_3)(X_2 X_1)(X_4 X_5), \quad (X_0 X_5)(X_2 X_3)(X_4 X_1).$$

The group so generated is imprimitive, the planes $X_0 X_2 X_4$ and $X_1 X_3 X_5$ being either invariant or transposed. $\mathscr{E}$, under which every vertex of S is stable, is a normal subgroup of the group of 192 self-projectivities of F_3; $h_0 h_2 h_4$ is still central for this larger group.

(17) The projection W_3 of F_3 from λ onto Σ contains six lines

$$\mu_0, \mu_2, \mu_4; \quad \mu_0', \mu_2', \mu_4'$$

and 18 conics. The μ_j are coplanar and concurrent, as are the μ_j'. Moreover, since $x_0 + \omega^2 x_2 + \omega x_4 = 0$ joins λ to $X_1 X_3 X_5$ and $x_1 + \omega^2 x_3 + \omega x_5 = 0$ joins λ to $X_0 X_2 X_4$ the two planes in Σ both contain $l_{024} \equiv l_{135}$ and are harmonic to $n_0 n_2 n_4$ and $n_1 n_3 n_5$. μ_j and μ_j' meet λ one at each focus of I_j, and are transversals to the three joins of nodes of W_3 that are paired in I_j on γ; the planes which contain either μ_j or μ_j' and one of these joins meet W_3 further in its 18 conics.

The plane containing μ_0, μ_2, μ_4 is the intersection of Σ with (15·1) and this opens one way to finding its equation. Since, with F_3, three of the relations (3·5) may be taken as

$$\begin{aligned} ax_0 &= -a^2b^3x - ab^3y - b^3z + a^2t + pa + q, \\ \omega a x_2 &= -\omega^2 a^2 b^3 x - \omega a b^3 y - b^3 z + \omega^2 a^2 t + \omega p a + q, \\ \omega^2 a x_4 &= -\omega a^2 b^3 x - \omega^2 a b^3 y - b^3 z + \omega a^2 t + \omega^2 p a + q, \end{aligned}$$

it follows that

$$a(x_0 + \omega^2 x_2 + \omega x_4) = 3(a^2 t - a^2 b^3 x).$$

Similarly

$$ib(x_1 + \omega^2 x_3 + \omega x_5) = 3(b^2 t - a^3 b^2 x)$$

so that (15·1) is

$$a^{\frac{3}{2}}(t - b^3 x) + b^{\frac{3}{2}}(t - a^3 x) = 0,$$

$$(a^{\frac{3}{2}} + b^{\frac{3}{2}})\{t - (ab)^{\frac{3}{2}} x\} = 0,$$

showing μ_0, μ_2, μ_4 to be in the plane $t - (ab)^{\frac{3}{2}} x = 0$. Similarly μ_0', μ_2', μ_4' are in

$$t + (ab)^{\frac{3}{2}} x = 0.$$

Since, now, $e_1 = e_2 = e_4 = e_5 = 0$ the equation of W_3 is

$$\begin{vmatrix} x & y & e_3(y^2 + 2zx) - 2zt \\ y & z & e_6 x^2 + 2e_3 yz - t^2 \\ z & t & 2e_6 xy + e_3 z^2 \end{vmatrix} = 0$$

$$e_6(3x^2yz - 2xy^3 - x^3t) + e_3(y^3t - xz^3) + xt^3 + 2z^3t - 3yzt^2 = 0, \tag{17·1}$$

where $e_6 = a^3b^3$ and $e_3 = a^3 + b^3$. The six lines μ_j, μ_j' lie three in each of two planes through $x = t = 0$, the intersection of the planes $t = a^3x$, $t = b^3x$ spanned by the triads of points on $\gamma(1, \theta, \theta^2, \theta^3)$ whose parameters are

$$a, \omega a, \omega^2 a \quad \text{and} \quad b, \omega b, \omega^2 b.$$

The lines become conspicuous on writing (17·1) in the form (abbreviating a square root of a^3b^3 by the letter ρ)

$$(t^2-a^3b^3x^2)(xt-3yz)+\frac{a^3+b^3}{2\rho}\{(\rho y^3-z^3)(\rho x+t)-(\rho y^3+z^3)(\rho x-t)\}$$
$$-\{(\rho y^3-z^3)(\rho x+t)+(\rho y^3+z^3)(\rho x-t)\}=0.$$

The μ_j are the intersections of $\rho x=t$ with the planes $\rho y^3=z^3$ and concur at $(1,0,0,\rho)$; the μ_j' occur on writing $-\rho$ for ρ.

W_3 is invariant under biaxial inversions $\mathscr{H}_0, \mathscr{H}_2, \mathscr{H}_4$, the axes of $\mathscr{H}_j$ being μ_j and μ_j'. These, like the H_j which have given rise to them on projection from λ, generate a symmetric group of degree 3 under which W_3 is invariant. Indeed a glance at (17·1) wherein y and z only occur cubed and in the product yz, shows that W_3 admits the self-projectivity which replaces x, y, z, t by $x, \omega y, \omega^2 z, t$ and this does in fact happen to be $\mathscr{H}_2\mathscr{H}_4=\mathscr{H}_4\mathscr{H}_0=\mathscr{H}_0\mathscr{H}_2$. Since the axes of $\mathscr{H}_{2p}$ are $\rho x/t=\omega^p\rho^{\frac{1}{3}}y/z=\pm 1$ the effect of the inversion is clear from (6·3), or rather from the analogue in [3] of this rule in [5]; it is to replace

$$x, y, z, t \quad \text{by} \quad t/\rho,\ z\omega^{2p}/\rho^{\frac{1}{3}},\ \omega^p\rho^{\frac{1}{3}}y,\ \rho x.$$

This, as $e_6=\rho^2$, multiplies the left-hand side of (17·1) by -1.

F_4

(18) While the vertices of any two equilateral triangles with the same circumcircle can be paired in three involutions I_0, I_2, I_4 it is noticeable that a fourth such partitioning, in an involution J, is available when the two triangles supply the vertices of a regular hexagon, for they then lie two on each of three diameters. This, in the notation of section 14, happens when $a+b=0$; the pairing in J is

$$a,\ -a;\quad \omega a,\ -\omega a;\quad \omega^2 a,\quad -\omega^2 a.$$

The quadrics are then precisely as in section 17 of (6); F_4 admits, among its self-projectivities, not only H_0, H_2, H_4 but also H of section 7 here. Since H imposes the triple transposition

$$(X_0X_1)(X_2X_3)(X_4X_5)$$

on the vertices of S it transposes the planes $X_0X_2X_4$ and $X_1X_3X_5$. These vertices now undergo a dihedral group of twelve permutations and F_4 admits 384 self-projectivities. The product H_2H_0H, for example, permutes the six X_i in a single cycle.

$h_0h_2h_4$ is still central in this larger group.

When $b=-a$ (17·1) becomes

$$a^6(x^3t+2xy^3-3x^2yz)+xt^3+2z^3t-3yzt^2=0$$

in agreement, when $a^6=1$, with (5·1). The six lines μ_j, μ_j' are apparent from

$$(t^2+a^6x^2)(xt-3yz)=(ia^3y^3-z^3)(ia^3x+t)+(ia^3y^3+z^3)(ia^3x-t).$$

This equation may also be written

$$3(t^2+a^6x^2)(xt-yz)+2t(z^3-a^6x^3)-2x(t^3-a^6y^3)=0,$$

showing that $xt = yz$ cuts W_4, apart from γ, in the five lines

$$z = a^2x, \quad t = a^2y; \quad z = \omega a^2x, \quad t = \omega a^2y; \quad z = \omega^2a^2x, \quad t = \omega^2a^2y;$$
$$x = z = 0, \quad y = t = 0.$$

The former three join nodes of W_4 paired in J; the latter two, in the opposite regulus, pass one through each focus of J on λ and are the axes ν, ν' of a biaxial inversion $\mathscr{H}$ leaving W_4 invariant. W_4 admits a dihedral group of 12 self-projectivities: the symmetric group, already available for W_3, extended by $\mathscr{H}$.

The lines μ_j, μ_j' on W_4 are

$$t = \pm ia^3x, \quad z = \mp i\omega^j ay$$

and lie in a regulus with ν, ν' on Q_j, the quadric $zt = \omega^j a^4 xy$ which touches W_4 all along the join of the two nodes that are paired both in J and in I_j; i.e. those whose parameters on γ are $\pm\omega^j a$.

F_6

(19) There are on conics hexads admitting six partitionings as three pairs in involution; such a hexad, as Corrado Segre first remarked ((10), p. 133) affords a sextuple tetrahedroid K_6. In birational correspondence with K_6 is a specialized Weddle surface W_6 which is the projection of a non-singular model F_6.

One way to obtain such a hexad is to use a regular sextuple, i.e. three pairs each harmonic to both the others. Each pair belongs to two of the six involutions: one supplements it by crossings of the other two. For example: a standard regular sextuple is

$$0, \infty; \quad 1, -1; \quad i, -i;$$

the pair $0, \infty$ is to be supplemented either by $1, i$ and $-1, -i$ or by $1, -i$ and $-1, i$.

Take, then, the three pairs

$$a, b; \quad \omega a, \omega b; \quad \omega^2 a, \omega^2 b.$$

Since they undergo cyclic permutation on multiplication by ω any two of them are harmonic if one pair of pairs is; this occurs when

$$(\omega a + \omega b)(\omega^2 a + \omega^2 b) = 2(\omega^2 ab + \omega ab),$$
$$a^2 + 4ab + b^2 = 0. \tag{19·1}$$

There are, in these circumstances, six involutions each containing three pairs of the hexad, namely

I_0	I_1	I_2	I_3	I_4	I_5
a, b	a, b	$a, \omega b$	$a, \omega a$	$a, \omega^2 b$	$a, \omega^2 a$
$\omega a, \omega^2 b$	$\omega a, \omega^2 a$	$\omega a, b$	$b, \omega b$	$\omega a, \omega b$	$\omega a, \omega b$
$\omega^2 a, \omega b$	$\omega b, \omega^2 b$	$\omega^2 a, \omega^2 b$	$\omega^2 a, \omega^2 b$	$\omega^2 a, b$	$b, \omega^2 b$

The quadratics whose roots are the foci of these involutions are

$$\left.\begin{aligned}
&I_0\colon \theta^2 = ab. \quad I_1\colon \theta^2 + 2(a+b)\theta + ab = 0.\\
&I_2\colon \theta^2 = \omega ab. \quad I_3\colon \theta^2 + 2\omega^2(a+b)\theta + \omega ab = 0.\\
&I_4\colon \theta^2 = \omega^2 ab. \quad I_5\colon \theta^2 + 2\omega(a+b)\theta + \omega^2 ab = 0.
\end{aligned}\right\} \tag{19·2}$$

The I_j admit coupling as

$$I_0, I_1; \quad I_2, I_3; \quad I_4, I_5;$$

the pair common to any couple belonging to the hexad; F_6 has the properties of F_2 in triplicate and so, after projection, W_6 will have the properties of W_2 in triplicate.

Three involutions do not, in general, share a common pair; they do so when the quadratics whose roots are their foci are linearly dependent; this occurs for

$$I_0, I_2, I_4; \quad I_0, I_3, I_5; \quad I_2, I_5, I_1; \quad I_4, I_1, I_3.$$

So F_6 has the properties of F_3 in quadruplicate, as, after projection, W_6 will have those of W_3. One expects four sets of three double-fours of planes all meeting F_6 in conics and, as each I_j belongs to two sets of three with a common pair, each double-four will belong to two of the four sets of three double-fours.

One set is $\mathscr{D}_0, \mathscr{D}_2, \mathscr{D}_4$, already encountered in section 14; the others will be labelled

$$\mathscr{D}_0, \mathscr{D}_3, \mathscr{D}_5; \quad \mathscr{D}_2, \mathscr{D}_5, \mathscr{D}_1; \quad \mathscr{D}_4, \mathscr{D}_1, \mathscr{D}_3; \tag{19·3}$$

they can, of course, only occur when (19·1) is satisfied and we now proceed to identify them.

(20) By (14·1) the sum $a^2\Omega_0 + a\Omega_1 + \Omega_2$ lacks the terms in x_2^2 and x_4^2, while

$$b^2\Omega_0 + b\Omega_1 + \Omega_2$$

lacks those in x_3^2 and x_5^2. Is there, in the pencil of quadratic forms spanned by these two, one lacking x_0^2 and x_1^2? The only possibility is $ab\Omega_0 - (a+b)\,\Omega_1 + \Omega_2$ which is not in the pencil unless

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ ab & -a-b & 1 \end{vmatrix} = 0,$$

$$(a-b)\,(a^2+4ab+b^2) = 0,$$

and this is so by (19·1). Now

$$\begin{aligned} a^2\Omega_0 + a\Omega_1 + \Omega_2 &\equiv 3a^2x_0^2 + (a^2+ab+b^2)\,x_1^2 \\ &\qquad + (a^2+\omega ab+\omega^2b^2)\,x_3^2 + (a^2+\omega^2ab+\omega b^2)\,x_5^2 \\ &\equiv 3a(ax_0^2 - bx_1^2) + (a-b)\,\{(a-\omega^2b)\,x_3^2 + (a-\omega b)\,x_5^2\} \end{aligned}$$

by a further appeal to (19·1). But (19·1) is

$$(\omega a - \omega^2 b)^2 + (\omega^2 a - \omega b)^2 = 0$$

and to take

$$\omega a - \omega^2 b = i(\omega^2 a - \omega b) \tag{20·1}$$

is merely to select one of two sets of quadrics possessing the properties now demanded. Then

$$a^2\Omega_0 + a\Omega_1 + \Omega_2 \equiv 3a(ax_0^2 - bx_1^2) + (a-b)\,(a-\omega^2b)\,(x_3^2 + i\omega x_5^2).$$

Likewise

$$b^2\Omega_0 + b\Omega_1 + \Omega_2 = 3b(bx_1^2 - ax_0^2) + (b-a)\,(b-\omega^2a)\,(x_2^2 - i\omega x_4^2)$$

so that the double-four of planes

$$ax_0^2-bx_1^2 = x_2^2-i\omega x_4^2 = x_3^2+i\omega x_5^2 = 0 \qquad (\mathscr{D}_1)$$

is on every quadric of the pencil. The remaining two double-fours are, by using $\wp$,

$$ax_4^2-bx_5^2 = x_0^2-i\omega x_2^2 = x_1^2+i\omega x_3^2 = 0, \qquad (\mathscr{D}_3)$$

$$ax_2^2-bx_3^2 = x_4^2-i\omega x_0^2 = x_5^2+i\omega x_1^2 = 0. \qquad (\mathscr{D}_5)$$

Every plane of any of these six double-fours cuts eight lines on F_6 at the foci of the appropriate involutions. This may be verified by substituting the parametric forms (14·2) in the equations for the $\mathscr{D}_j$; the resulting quadratic for θ must be the same as that in (19·2) with θ changed in sign. One instance must suffice. The two primes $x_4^2 = i\omega x_0^2$ cut λ, where

$$\omega^2\theta^2+2\omega a\theta+a^2 = i\omega(\theta^2+2a\theta+a^2)$$

which accords with I_5 in (19·2), with $-\theta$ instead of θ, provided that

$$\omega^2-i\omega = \frac{\omega a(1-i)}{-\omega(a+b)} = \frac{(1-i\omega)a}{\omega^2 b}.$$

These conditions are indeed satisfied, by (20·1).

(21) It was noted in section 14 that $\mathscr{D}_0$, $\mathscr{D}_2$, $\mathscr{D}_4$ are all on $\Omega_1 = 0$; it is therefore to be expected that each of the sets of three double-fours in (19·3) is on some quadric containing F_6. Since $\mathscr{D}_1$ is on

$$a^2\Omega_0+a\Omega_1+\Omega_2 = 0 \quad \text{and} \quad b^2\Omega_0+b\Omega_1+\Omega_2 = 0$$

it follows, on applying $\wp$, that $\mathscr{D}_5$ is on

$$\omega a^2\Omega_0+a\Omega_1+\omega^2\Omega_2 = 0 \quad \text{and} \quad \omega b^2\Omega_0+b\Omega_1+\omega^2\Omega_2 = 0$$

and that $\mathscr{D}_3$ is on

$$\omega^2a^2\Omega_0+a\Omega_1+\omega\Omega_2 = 0 \quad \text{and} \quad \omega^2b^2\Omega_0+b\Omega_1+\omega\Omega_2 = 0.$$

The quadric that is common to the pencil containing $\mathscr{D}_5$ and the pencil containing $\mathscr{D}_3$ contains both $\mathscr{D}_5$ and $\mathscr{D}_3$; which quadric this is appears from the identity

$$\begin{aligned} b(\omega^2a-\omega b)(\omega a^2\Omega_0+a\Omega_1+\omega^2\Omega_2)+a(\omega a-\omega^2b)(\omega b^2\Omega_0+b\Omega_1+\omega^2\Omega_2) \\ \equiv (a-b)\{ab(a+b)\,\Omega_0-ab\Omega_1+(a+b)\,\Omega_2\}, \end{aligned}$$

and since the quadratic form in brackets is $(a+b)(ab\Omega_0+\Omega_2)-ab\Omega_1$ the corresponding quadric, to be called Q_{035}, contains $\mathscr{D}_0$. Moreover, by (19·1), the equation of Q_{035} is

$$ab\Omega_0+\tfrac{1}{2}(a+b)\,\Omega_1+\Omega_2 = 0.$$

From this, using $\wp^2$ and $\wp$, one finds that $\mathscr{D}_2$, $\mathscr{D}_5$, $\mathscr{D}_1$ are all on

$$Q_{251}\colon \omega^2ab\Omega_0+\tfrac{1}{2}(a+b)\,\Omega_1+\omega\Omega_2 = 0$$

and that $\mathscr{D}_4$, $\mathscr{D}_1$, $\mathscr{D}_3$ are all on

$$Q_{413}\colon \omega ab\Omega_0+\tfrac{1}{2}(a+b)\,\Omega_1+\omega^2\Omega_2 = 0.$$

The quadric $\Omega_1 = 0$ is Q_{024}.

The two planes of $\mathscr{D}_j$ that meet λ are fundamental spaces of a harmonic inversion H_j leaving F_6 invariant; H_j induces in Σ a biaxial harmonic inversion $\mathscr{H}_j$ for which W_6 is invariant. The effects of H_0, H_2, H_4 are indicated in (16·1); they show that these three inversions act on Ω_0, Ω_1, Ω_2 and the above four quadrics as follows.

	Ω_0	Ω_1	Ω_2	Q_{024}	Q_{035}	Q_{251}	Q_{413}
H_0	$-\Omega_2/ab$	$-\Omega_1$	$-ab\Omega_0$	Q_{024}	Q_{035}	Q_{413}	Q_{251}
H_2	$-\omega^2\Omega_2/ab$	$-\Omega_1$	$-\omega ab\Omega_0$	Q_{024}	Q_{413}	Q_{251}	Q_{035}
H_4	$-\omega\Omega_2/ab$	$-\Omega_1$	$-\omega^2 ab\Omega_0$	Q_{024}	Q_{251}	Q_{035}	Q_{413}

Thus each H_j leaves invariant those two of the four quadrics on which the planes of $\mathscr{D}_j$ lie, while transposing the other two quadrics. So, enlisting H_1, H_3, H_5, the six H_j impose the six transpositions of pairs and the group generated by the H_j imposes the symmetric group of 4! permutations on the quadrics.

If every one of Q_{024}, Q_{035}, Q_{251}, Q_{413} is invariant under a projectivity P so is every quadric of the net to which they belong and so, therefore, is their common self-polar simplex S whose bounding primes are thus permuted among themselves. But no permutation, other than the identity, of these primes $x_j = 0$ can leave every quadric of the net invariant; so P must be the identity. Thus the group of projectivities generated by the H_j is of order 24, a symmetric group of degree 4; F_6 has a group of 768 self-projectivities.

(22) F_6 has, as already remarked, the properties of F_3 in quadruplicate; for example, each set of three double-fours provides 32 prime sections of F_6 consisting of two lines and three concurrent conics. Of the 128 such sections of F_6 a single example must suffice: the planes (σ being a square root of i)

$$\begin{aligned}
x_0\sqrt{a}+ix_1\sqrt{b} &= \omega^2x_2\sqrt{a}+i\omega x_5\sqrt{b} = \omega x_4\sqrt{a}+i\omega^2x_3\sqrt{b} = 0,\\
x_4\sqrt{a}+x_5\sqrt{b} &= \quad x_0+\sigma\omega^2x_2 \quad = \quad x_1+i\sigma\omega^2x_3 = 0,\\
x_2\sqrt{a}-x_3\sqrt{b} &= \quad x_4-\sigma\omega^2x_0 \quad = \quad x_5-i\sigma\omega^2x_1 = 0
\end{aligned}$$

belonging respectively to $\mathscr{D}_0$, $\mathscr{D}_3$, $\mathscr{D}_5$ are all in the prime

$$\sigma(x_0\sqrt{a}+ix_1\sqrt{b})+i\omega^2(x_2\sqrt{a}-x_3\sqrt{b})-\omega(x_4\sqrt{a}+x_5\sqrt{b}) = 0$$

and all pass through the point

$$(\sigma\omega^2\sqrt{b},\ i\sigma\omega^2\sqrt{a},\ -\sqrt{b},\ -\sqrt{a},\ i\omega\sqrt{b},\ -i\omega\sqrt{a}).$$

Since λ meets an opposite pair of planes of each of the six double-fours there are, in addition to the lines on an unspecialized Weddle surface, 12 lines μ_j, μ'_j ((11), p. 360) on W_6; μ_j and μ'_j meet γ at the foci of I_j and are the transversals from these points to the chords joining points paired by I_j on γ. The 12 lines fall into four pairs of coplanar and concurrent sets of three, such a set being the projection of three concurrent conics on F_6 that all meet λ. The intersection of such a pair of planes is on W_6 and is also the intersection of planes spanned by complementary triads of nodes. The six biaxial inversions in pairs of lines μ_j, μ'_j belong to a group of 24 projectivities under which W_6 is invariant.

These, and other properties, are analogous to those of W_3; but W_6 has also properties that are analogous to those of W_2. For example: μ_0, μ_0', μ_1, μ_1' – the suffices as in one of the three couplings of the I_j – belong to a regulus on a quadric q_{01} which meets W_6 further in two lines and touches it along another line.

Of the ten lines of intersection of planes, spanned by complementary triads of nodes of W_6, four are also intersections of planes such as μ_0, μ_2, μ_4 and μ_0', μ_2', μ_4'; these four lines are

$$l_{024} \equiv l_{135}, \quad l_{035} \equiv l_{124}, \quad l_{251} \equiv l_{340}, \quad l_{413} \equiv l_{502}$$

and indicate the possession by W_6 of a property of W_3 in quadruplicate. The remaining six of the ten lines lie two on each of three quadrics that touch W_6 along the join of two nodes, and indicate the possession by W_6 of a property of W_2 in triplicate. Indeed there are quadrics:

q_{01}, touching W_6 along $n_0 n_1$ and containing $l_{023} \equiv l_{145}$ with $l_{123} \equiv l_{045}$,

q_{23}, touching W_6 along $n_2 n_3$ and containing $l_{245} \equiv l_{301}$ with $l_{345} \equiv l_{201}$,

q_{45}, touching W_6 along $n_4 n_5$ and containing $l_{401} \equiv l_{523}$ with $l_{501} \equiv l_{423}$.

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