KLEIN'S ENCOUNTER WITH THE SIMPLE GROUP OF ORDER 660

By W. L. EDGE

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1. The simple groups g, of order 168, and G, of order 660 occur as linear fractional groups $\mathrm{LF}(2,p)$ with $p = 7, 11$. There is a close analogy between them, all the closer because 7 and 11 are congruent *modulo* 4. The smaller group, although encountered by Galois, is commonly called 'Klein's group' since Klein's famous paper [8] of 1878 was much involved with what later proved to be an irreducible representation of g of degree 3. This can be regarded as a group of collineations in a projective plane, and it is there found to have an invariant non-singular quartic curve k, Klein's quartic. The larger group has, analogously, an irreducible representation of degree 5; this can be regarded as a group of collineations in [4], a projective space of four dimensions. Indeed Klein himself so handled it, and discovered an invariant curve C of order 20 with some of whose properties we are to be concerned. But to encounter C after encountering k was to venture on *terra incognita* after journeying through thoroughly explored and meticulously mapped country. For the properties of the plane quartic, of which k is a specialization, were already known owing to the work of Steiner, Hesse, Cayley, Salmon, and others; Klein was well aware of this and ready to exploit his advantages. But there was nothing ready to his hand when he encountered C; organized projective geometry of hyperspace lay still in the future.

2. It is necessary now to give a short description of relevant properties of k since they indicate, by analogy, likely properties of C.

There are on k three special sets of 24, 56, 84 points, designated by Klein points a, b, c. They are invariant under cyclic subgroups $\mathscr{C}_7, \mathscr{C}_3, \mathscr{C}_2$ of g and are, respectively, inflections, contacts of bitangents and sextactic points of k. So there are on C three special sets of 60, 220, 330 points invariant under cyclic subgroups $\mathscr{C}_{11}, \mathscr{C}_3, \mathscr{C}_2$ of G.

g has eight subgroups $\mathscr{C}_7$; the collineations of any one of them share three fixed points which are all on k, all of them inflections; the tangent at each of them meets k again in another. Thus k has eight *inflectional triangles*; this is perhaps its best-known property and it will be seen in § 4 that, analogously, C has 12 *inflectional pentagons* associated one with each of 12 subgroups $\mathscr{C}_{11}$.

g has 28 subgroups $\mathscr{C}_3$; of the three points invariant for the collineations of a $\mathscr{C}_3$ two are on k and are contacts of one of its 28 bitangents. It will be seen in § 20 that, in association with 55 subgroups $\mathscr{C}_3$ of G, C has 55 quadritangent solids; the contacts of such a solid with C are among the points invariant for the collineations of the corresponding $\mathscr{C}_3$. But while the two contacts of a bitangent account for all its intersections with k the four contacts of a quadritangent solid with C leave 12 further intersections of C and the solid to be accounted for, as they will be in § 20.

g has 21 subgroups $\mathscr{C}_2$. Each involution in g is a harmonic homology the four intersections of whose axis with k are sextactic points. G has 55 subgroups $\mathscr{C}_2$. Each involution in G is harmonic inversion in a line λ and a plane π, π meeting C in six points on a conic while λ is skew to C.

Another property of k is explained by the occurrence in g of two sets of seven octahedral subgroups ω. Each ω contains four of the 28 $\mathscr{C}_3$; the four bitangents of k associated with them undergo all 4 ! permutations under the collineations of ω, and their eight contacts with k are on a conic. So there arise two sets of seven conics; as soon as he encountered them Klein recognized ([**8**] p. 106) that they were of great importance. Each set of seven conics cuts on k the whole set of 56 contacts of bitangents.

The four subgroups $\mathscr{C}_3$ of an octahedral subgroup ω of g belong to its intersections with four octahedral subgroups of the opposite set: indeed the actual intersections are dihedral, of order 6. The conics associated with the *other three* subgroups of this opposite set span a net to which the conic associated with ω belongs. This criss-crossing between the two sets of seven conics does not appear in Klein's original paper; but equations that imply it are in the Klein–Fricke treatise ([**10a**] p. 759).

3. Just as g has two conjugate sets of seven octahedral subgroups ω so G has two conjugate sets of 11 icosahedral subgroups j. Every icosahedral group has 10 subgroups $\mathscr{C}_3$; each $\mathscr{C}_3$ in j is associated with four points b, contacts of C with a quadritangent solid, and the 10 sets of four points compose the complete intersection of C with a quadric. Thus there appear two sets, each of 11 quadrics, and these, like Klein's conics, are of great importance. As there were seven conics in a set they were necessarily linearly dependent; the mere number 11 does not, of necessity, imply linear dependence among quadrics in [4], but the 11 of either set are linearly dependent nonetheless. Now an icosahedral group has five tetrahedral subgroups; if j is any of the 22 icosahedral subgroups of G its tetrahedral subgroups are its intersections with five icosahedral subgroups of the opposite set ([**10a**] p. 482). Thus each of the 22 quadrics is put in correspondence with five quadrics of the opposite set and is indeed linearly dependent on them.

Equations which imply this are on p. 429 of [**10b**] and were transcribed by Klein when his collected papers were published (p. 167). In his original 1879 paper he did mention the two sets of 11 quadrics but instantly ([**9**] p. 150) discarded one set as having the same properties as the other. So, indeed, it has; but its absence from the investigation there pursued postponed the recognition of any linear dependence between quadrics of different sets.

That the 22 quadrics in [4] are linearly dependent on only 10 calls for an explanation; it implies that there are $15-10 = 5$ linearly independent quadric envelopes inpolar to all 22 quadric loci. Could these five be polar quadrics of a cubic envelope? The mention of an envelope indicates outer automorphisms of G, and perhaps it is again advisable to summarize the corresponding circumstances for g, the more so as they are not commonly discussed in this geometrical setting.

Quadratic forms which, equated to zero, give Klein's 14 conics are labelled c_x on p. 108 of [**8**]. The matrix of each c_x is, in fact, a scalar multiple of a unitary matrix: this implies that the coefficients of the point equation of a conic of either set of seven are the same as those of the line equation of a conic of the other set which, in its turn, implies the existence of a conic reciprocating either set into the other. But this is only one of 28 such reciprocations all of which transform k into the same quartic envelope e (e is the equianharmonic envelope of k which, in its turn, is the equianharmonic locus of e). Indeed each conic $c_x = 0$ can, by these means, be reciprocated into four of the opposite set: namely into those other than the three spanning a net to which $c_x = 0$ itself belongs.

The analogy between g and G is now patent. Quadratic forms which, equated to zero, give Klein's 22 quadrics are labelled (3) and (4) on p. 428 of [**10b**]. The matrices of these quinary forms too are scalar multiples of unitary matrices, and the coefficients of the point equation of any quadric of either set are the same as those of the prime equation of a quadric of the other set. The reciprocation, hereby implied, of either set into the other is one of 66 such reciprocations which all transform the invariant cubic primal into the same cubic envelope $\mathscr{E}$: given any one of the 22 quadrics there are six of the opposite set into which it can be reciprocated, namely those other than the five on which it is linearly dependent. And it is $\mathscr{E}$ whose first polars are all inpolar to all 22 quadrics.

These 22 quadrics are therefore all outpolar to $\mathscr{E}$; it is this that explains their linear dependence. There is no analogous linear dependence among the 14 conics; e does not have any outpolar conic.

4. Klein discovered this cubic primal, therefore to be labelled K, in [4] that was invariant under the group G of 660 collineations. This was before Veronese and Corrado Segre had propelled the Italian geometers into hyperspace so that, in marked contrast to the circumstances attending his discovery of k, Klein was forced to construct any necessary geometrical apparatus for himself. But he remarked that, as the Hessian H is a covariant of K, it too was invariant under G as was also therefore its nodal curve C: and so he was provided with the curve and Riemann surface that he was seeking.

The order of C, and indeed of far more complicated loci, can now be found by mere substitution in a formula; put $\mu = 1$, $c = 3$ in the product on p. 111 of [**1**], and then $p = 4$; C has order 20. But Klein, in finding the order of C himself, acquired information that unveils the true nature of the 60 points a: information far beyond the scope of the mere formula. He gives the first terms of the expansions, in terms of a local parameter dt, of the coordinates of five points on C, namely the vertices of the simplex of reference for his homogeneous coordinates; these points, being invariant under a group $\mathscr{C}_{11}$, are points a. He deduces that the order of C is 20 by adding the exponents of the differential dt in a column of his table (14) ([**9**] p. 154).

But let us now note the exponents in a row of this table. They disclose that this linear branch of C is *inflectional* ([**14**] p. 183), the tangent having 3-point intersection: not only so, but the osculating plane has 6-point and the osculating solid 10-point intersection. There is an *inflectional pentagon* $a_1a_2a_3a_4a_5$ inscribed in C; a_1a_2 is the tangent, $a_1a_2a_3$ the osculating plane, and $a_1a_2a_3a_4$ the osculating solid at a_1; and likewise, cyclically permuting the suffixes, at the other vertices. And there are 12 such pentagons: they afford a permutation representation of G just as the eight inflectional triangles of k did of g. The 60 points a compose the whole set of intersections of C with K.

5. What of the genus of C? Here again Klein had to improvise, and he was sufficiently dissatisfied with the somewhat tentative nature of his reasoning to suggest its replacement when preparing his collected works for publication ([**9**] p. 155, footnote). But even his second thoughts should now yield place to an argument that takes advantage of the properties of linear series on algebraic curves.

The solids in [4] cut on C a linear series g^4_{20}; a standard formula ([**16**] p. 188; [**1**] p. 10; [**15**] p. 389) says that there are $(4+1)(20+4p-4)$ quintuple points of this series, p being the genus of C. Moreover, there are precise rules for counting points, whose multiplicity exceeds 5, the proper number of times. In the present situation each of the 60 points a is

simple for a triple infinity of sets,
triple for a double infinity of sets,
sextuple for a single infinity and
tenfold for one set of g^4_{20};

the rules ([**16**] p. 188) prescribe that it be counted

$$1+3+6+10-\tfrac{1}{2}.5.4 = 10$$

times. Since there are 60 such points a

$$20(4+p) \geqslant 600, \quad p \geqslant 26.$$

An opposite inequality is found by appealing to a theorem of Clifford ([**4**] p. 329; [**16**] p. 131; [**15**] p. 384), published, so it happens, in the same year 1878 as Klein's paper.

There are 35 linearly independent cubic primals in [4]; it will be seen in § 7 that none contains C, so that they cut on C a linear series g^{34}_{60}. Since its grade 60 is less than double its freedom 34 the series is, by Clifford's theorem, non-special; so the complete series to which it belongs has freedom $60-p$, not more. Hence $34 \leqslant 60-p$, or $p \leqslant 26$. The two inequalities leave no alternative; p is 26.

One may note the two consequences:

(i) C has *no* stationary osculating solids, i.e. solids with more than the statutory 4-point intersection, save those at the 60 points a.

(ii) The cubic primals in [4] cut a *complete* linear series on C.

It is, perhaps, appropriate here to apply the rule for the correct enumeration of multiple points also to the series g^{14}_{40} cut on C by the quadrics of [4]; the formula states that there are

$$(14+1)(40+350) = 5850$$

points on C where 15 of the 40 points of a set of g^{14}_{40} coincide. It appears, from the exponents in any of the rows of Klein's table (14), that the lowest powers of dt in the five squares and 10 products of the coordinates are, in non-descending order,

$$0, 1, 2, 3, 4, 6, 6, 7, 9, 10, 11, 12, 13, 16, 20.$$

The sum of these integers is 120, so that the rule prescribes

$$120-\tfrac{1}{2}.14.15 = 15$$

for the number of times each of the 60 a-points must be counted among the 5850. There remain $5850-900 = 4950$ further points to be accounted for; no doubt Klein's 220 b-points and 330 c-points make their due contributions.

6. The Hessian H of a general cubic primal F is the locus of points whose polar quadrics with respect to F are cones; if A is a point on H the vertex A' of its polar cone is on H too, and the vertex of the polar cone of A' is A: the points of H are thus linked in pairs; if A describes a curve or surface on H then A' describes a curve or surface linked thereto. Moreover, the polar solid of a point on H with respect to F is the tangent solid of H at the linked point—supposed non-singular.

But if A is on the nodal curve C of H its polar quadric (with respect to F) has for its vertex a line q on H, and H has the same tangent solid at all (non-singular) points of q ([**18**] p. 518). The polar cones of the points of q belong to a pencil with a common vertex, and therefore include four line-cones; q is quadrisecant to C, and the tangents to C at its intersections with q all lie in the solid touching H along q. Voss' paper is concerned with a cubic primal in projective space $[p-1]$; he certainly observed the lines on the Hessian meeting its nodal locus in $p-1$ points, and remarked too that $p-1$ of the lines pass through each point of the locus. When $p = 5$ these lines are quadrisecants of the nodal curve, but there was no investigation of their properties; indeed there seems, strangely, no other mention of them before Seifert's two notes of 1937; each is conjugate to a point of C and they generate a scroll S, like C of genus 26, having C for a quadruple curve. Among the properties found by Seifert perhaps two may be mentioned here.

(i) The solid Σ spanned by the three, other than any one q_0, of the four q concurring at a point of C contains also those q conjugate to the remaining three intersections of q_0 with C. Σ thus contains six q comprising two concurrent sets of three; its 20 intersections with C consist of the two points of concurrence and three further points on each of the six q ([**13b**] § 13).

(ii) The cubic curves, in which planes spanned by complementary pairs of concurrent q meet H, are linked.

The first of these two properties accords with the striking attribute of K. Let $XYZTU$ be an inflectional pentagon, XY, YZ, ZT, TU, UX being the tangents of K, at X, Y, Z, T, U respectively; the polar quadric of each point is a line-cone whose vertex is the opposite side of the pentagon.

For instance—to use Klein's coordinates for the moment—the polar of $(1, 0, 0, 0, 0)$ with respect to

$$x^2y+y^2z+z^2t+t^2u+u^2x = 0$$

is $u^2+2xy = 0$ with vertex ZT. On the other hand, the polar of $(0, 0, \zeta, \tau, 0)$ on ZT is

$$\zeta(y^2+2zt)+\tau(z^2+2tu) = 0,$$

and the discriminant of the quadratic form here is $\zeta\tau^3$: those points on ZT whose polar cones have line-vertices consist of Z, counted three times, and T.

The q through X therefore are XU, XU, XU, XY; take XY to be the q_0 of Seifert's first property. Its 'remaining' intersections with K are X, X, Y so that there is a solid whose 20 intersections with K consist of two sets of 10 points:

$$X+9U \quad \text{and} \quad T+6Z+U+2T.$$

The whole aggregate is $10U+6Z+3T+X$; the integer coefficients here are the same as the powers of dt in Klein's table ([**9**] p. 154). The solid is $y = 0$; it osculates K at U.

7. The condition for the quadric $x'Ax = 0$ to be outpolar to $x'Bx = 0$, or for the second quadric to be inpolar to the first, is that the trace of the matrix $A \operatorname{adj} B$ should be zero. Here A and B are symmetric matrices and x the column vector of n point coordinates; the condition is linear in the elements of A and of degree $n-1$ in those of B. For us, n is 5.

Those points of [4] whose polar quadrics with respect to F are inpolar to a given quadric Q lie on a quartic primal R. Since the condition for a point-cone to be inpolar to Q is that its vertex should lie on Q, R cuts H in the surface linked to the common surface of H and Q; R contains those, and only those, 40 q that are conjugate to the intersections of Q and C. And since every line-cone is inpolar to Q (the adjugate of the matrix of rank 3 of the point equation of such a cone being the zero matrix) R contains C. The 15 linearly independent quadrics of [4] are thus linked with 15 linearly independent quartics through C. That there are quartics through C is known because C is on all those primals given by equating to zero the 15 different first minors of the 5-rowed symmetric Hessian determinant Δ, and the elements of Δ are linear forms.

The intersection of R with the scroll S can only consist, in addition to C counted four times, of a certain number of generators. But it has just been seen that this number is 40 so that, if S has order n,

$$4n = 80+40, \quad n = 30.$$

This order was found otherwise by Seifert ([**13a**] p. 15), who used the Jacobian curve, of order 10, of the net of polar quadrics of points of a plane.

The existence of S proves there to be no cubic primal through C. For such a primal would, of necessity, contain every q, and so the whole of S; yet the surface in which it meets H cannot be of larger order than 15.

8. A certain number of q are tangents of C, and an application of Chasles' principle of correspondence discloses what this number is.

Take a plane α: two solids through α are to correspond whenever there is a q two of whose four intersections with C lie one in each solid. Any solid meets C in 20 points through each of which pass four q; given this solid through α, 240 others are in correspondence with it. The correspondence has, by Chasles' principle, 480 coincidences. But coincidences are of two kinds: the solid may join α to

(i) the contact of a q with C, or

(ii) one of the 30 q which meet α.

Each of the 30 solids in (ii) can be defined as the join of α to any one of four points on q; its coincident correspondent can then join α to any one of the other three; it contributes 12 to the number of coincidences. Hence there are

$$480-12.30 = 120$$

q that are tangents of C. The question of how many q have a prescribed cross-ratio for their four intersections with C appears to be still outstanding.

For Klein's special F the tangents to C are inflectional at its 60 a-points; they occur among the q because each meets C at another a-point in addition to that at which it has 3-point intersection. And since this is 3-point and not the 2-point intersection of an ordinary tangent this q must be counted twice among the 120. So there are no q tangent to this C other than the sides of the 12 inflectional pentagons.

9. The linkage between quartic primals R through C and quadrics Q yields relations between the orders n, n' of linked curves m, m' and the numbers i, i' of their intersections with C. For the points linked to the intersections of m and Q are those intersections of m' and R that do not lie on C; and likewise with the roles of m, m' transposed. Hence

$$2n = 4n'-i', \quad 2n' = 4n-i, \tag{9.1}$$

or, more symmetrically,

$$i+i' = 2(n+n'), \quad i-i' = 6(n-n').$$

For example: a plane section ϑ' of H has $n' = 5$, $i' = 0$; the curve ϑ linked to ϑ has therefore $n = 10$, $i = 30$; it is the Jacobian curve of the net of polar quadrics of all points in the plane of ϑ'.

10. The particular circumstances of Q becoming a pair of solids Σ_1, Σ_2 or, still more particularly, a repeated solid Σ, occur in Seifert's second note. If a quadric is inpolar to the pair Σ_1, Σ_2 then Σ_1, Σ_2, are conjugate for it; if a quadric is inpolar to a repeated solid Σ then Σ is one of its tangent solids. The quartic locus of the point, whose polar with respect to F the quadric is, touches H along a surface f, of order 10 and containing C; f is linked to the section of H by Σ and is the Jacobian surface of the web of polar quadrics of points of Σ. The surfaces f_1 and f_2 so arising from the solids Σ_1 and Σ_2 form the complete intersection of H with the quartic whose points are such that Σ_1 and Σ_2 are conjugate for their polar quadrics. Of course f contains those 20 q conjugate to the intersections of C with Σ. These quartic primals are those whose equations arise on bordering Δ with a row and column composed of the coefficients in the equations of Σ_1 and Σ_2.

11. This (1, 1) correspondence between quadrics of [4] and quartic primals through C is markedly reminiscent of Hesse's correspondence between quadric surfaces in [3] and plane cubics ([7] pp. 288–89); indeed algebra analogous to that by which Hesse sets up his correspondence shows that, if Δ_{ij} is the cofactor of $\partial^2 F/\partial x_i\, \partial x_j$ in the Hessian determinant Δ of F, the correspondence in [4] is just

$$x_i x_j \leftrightarrow \Delta_{ij}.$$

Δ_{ij} is quartic in the coefficients of F as well as in the x_i. The surface in which H is met by any quadric $\sum a_{ij} x_i x_j = 0$ is linked to that in which H is met by the quartic $\sum a_{ij} \Delta_{ij} = 0$.

The quartic primals are, admittedly, restricted to contain C whereas Hesse's cubic could be any of its plane. But corresponding quadrics and quartics are in the same projective space and the two surfaces in which they meet H are related mutually, each linked to the other. If one regards the quartic as a biquadratic, a quadric function of the quadrics, one can replace these quadrics by the appropriate linear combinations of the Δ_{ij} and so produce a polynomial of degree 8. This one would anticipate to be the product of the original quadric and a sextic polynomial $\mathfrak{S}$, the primal $\mathfrak{S} = 0$ meeting H in the scroll S of order 30. Of course when $\mathfrak{S} = 0$ does contain S so does every sextic primal whose equation has for its left-hand side any linear combination of $\mathfrak{S}$ and the product of Δ by any linear form; this accords with the fact that a quartic can be written as a biquadratic in various ways, e.g.

$$x_1 x_2 x_3 x_4 = \lambda x_1 x_2 . x_3 x_4 + \mu x_1 x_3 . x_2 x_4 + \nu x_2 x_3 . x_1 x_4$$

whenever $\lambda + \mu + \nu = 1$. Some rule is necessary if the procedure is to be unambiguous, and such a rule is provided by Sylvester's principle of unravelment ([17] p. 322; [5] pp. 251–52); not only so, but the adoption of this principle ensures that $\mathfrak{S}$ will be a covariant of F. Since it is of order 6 in the variables and of degree 12 in the coefficients of F its weight is w where $5w+6 = 36$, so that $w = 6$.

For the special case when F is K there is a generating function ([2] p. 301) giving the number of invariants, of different degrees, for the representation of G as an irreducible group of quinary substitutions. Its first terms prove to be $1+x^3+x^5+2x^6$; the two independent invariants of degree 6 are $\mathfrak{S}$ and the square of K. If σ denotes the 5-term sum generated by the cyclic permutation $(xyztu)$ from a single term Klein's quinary cubic and its Hessian are $\sigma.x^2y$ and $3xyztu+\sigma.x^2y(y^2-xt)$. The covariant $\mathfrak{S}$ turns out to be

$$\sigma.(xy^4z+x^3z^2t-xyz^2t^2).$$

There is, subordinate to the correspondence between quadrics in [4] and quartics through C, a less ample correspondence that should be mentioned. If P is common to H and the polar quadric of a point A with respect to F the polar solid of P with respect to F contains A. But this solid is ([18] p. 516) the tangent solid of H at the point P' linked to P, so that P' is on the first quartic polar of A with respect to H. In other words: the first polars of any point with respect to F and H cut H in linked surfaces.

12. The polar cones of points on a line of H belong to a pencil. There are, according to Segre ([12] p. 895) three kinds of pencils of quadric cones in [4].

(i) Cones with a common vertex A. They include four line-cones whose vertices q_1, q_2, q_3, q_4 concur at A. The polar solid of any point (other than A) on q_i with respect to every cone of the pencil is that spanned by the remaining q.

(ii) Cones with collinear vertices. The line l on which the vertices lie is on all the cones, which all have the same tangent solid Σ along l. There are two line-cones in this pencil; both their vertices meet l and lie in Σ, spanning this solid.

(iii) Cones whose vertices lie on a conic.

These facts suggest the occurrence of three kinds of lines on H:

(i) quadrisecants of C;

(ii) chords of C which will be linked in pairs, the polar cones of the points on either having their vertices on the other;

(iii) lines skew to C, linked to conics that meet C in six points.

Suppose that the chord PQ of C is on H; it is linked to a second chord RS of C, also lying on H. Let p, q, r, s be the quadrisecants conjugate to P, Q, R, S. Then, by Segre's results, the solid pq touches the polar cone of every point on PQ all along RS and the solid rs touches the polar cone of every point on RS all along PQ. So, if A is on PQ and B on RS, the polar solid of B with respect to the polar cone of A is pq and the polar solid of A with respect to the polar cone of B is rs. Hence pq and rs are the *same* solid, namely the 'mixed polar' of A and B with respect to F. The 20 intersections of this solid Σ with C consist of P, Q, R, S and four points on each of p, q, r, s.

Σ meets the polar cones of P and R in the repeated planes pRS and rPQ respectively; these repeated planes meet in a line and determine a pencil whose members are the pairs of planes of the involution whose double members are pRS and rPQ. But these are the intersections of Σ with the polar quadrics of points on PR, which quadrics must therefore be cones. So PR is on H, and the chord linked with it is common to the planes pRS and rPQ. Thus all six edges of the tetrahedron $PQRS$ are on H, each edge being linked with the opposite one. Each face of the tetrahedron contains the quadrisecant conjugate to the opposite vertex.

Since the solid which touches H along p does so in particular at its intersections with RS, SQ, QR it contains these three lines and their plane; the intersection of the plane QRS with H consists of p reckoned twice and of the three lines RS, SQ, QR. As the solid has 12 intersections with C off p there are four such planes through p, and 12 of its chords through a point P of C lie on H.

Alternatively: assume that there is a plane ϖ containing the quadrisecant p conjugate to P as well as three further points Q, R, S of C none of which is on p and which are not collinear. Every point of ϖ has the same polar solid T with respect to a line-cone with p for vertex; hence T contains the vertices q, r, s of the polar cones of Q, R, S—the polar solid of Q, say, with respect to the polar cone of P being the same as that of P with respect to the polar cone of Q and so containing q. Likewise Q has the same polar solid with respect to the polar cone of R as does R with respect to the polar cone of Q, and this solid is qr, i.e. T. Thus T is the polar solid of each of P, Q, R, S with respect to the polar cones of the other three. And so we find again a tetrahedron inscribed in C with a quadrisecant, conjugate to the opposite vertex, in each of its faces; for example: Q and R have the same polar with respect to the polar cone of S so that QR meets s, and so on. And so QR, as meeting both s and p and being a chord of the nodal curve, lies on H and is linked with SP.

13. The polar quadrics of the points of T constitute the web linearly dependent on the polar line-cones of P, Q, R, S. The whole system of polar quadrics of points of [4] is spanned by this web and the polar quadric Ψ of any point X outside T. Now the polar solid of P with respect to Ψ being also the polar solid of X with respect to the polar line-cone of P, contains p; p is the intersection of the plane QRS and the polar plane of P with respect to the quadric surface ψ which is the section of Ψ by T. So, likewise, are q, r, s the intersections of the faces of the tetrahedron $PQRS$ with polar planes of its opposite vertices with respect to this same quadric surface ψ.

So, by a classical theorem ([**3**] p. 402; [**11**] pp. 118–22) p, q, r, s belong to a regulus ρ. The quadric surface on which ρ lies meets the quintic section of H by T in p, q, r, s and a sextic curve through the intersections, 16 in all, of p, q, r, s with C. The curve linked to this is, on putting $n = 6$ and $i = 16$ in (9.1), seen to be a quartic through P, Q, R, S. Each line of the regulus complementary to ρ meets H, and so the sextic curve, only in one point in addition to points on p, q, r, s so that the sextic, and therefore the quartic linked with it, are rational.

14. The types of projectivities for which K is invariant are known, being deducible from Klein's original work. But they are more immediately identified by the characters of this representation of G since these are so familiar: for example, put $p = 11$ in the second table on p. 502 of [**2**] and use the second or third column. If $x = 3$ the entry in the table gives $+1$ for the character of every involution; the five latent roots of the matrix of the substitution are therefore $1, 1, 1, -1, -1$ and the projectivity is harmonic inversion in a line and plane. Likewise, with $x = 2$, the character of operations of period 3 is -1 so that the latent roots of the substitution must be $1, \omega, \omega, \omega^2, \omega^2$ with ω either complex cube root of unity. And so on.

The larger the number of projectivities for which a non-singular cubic primal F is invariant the more geometrically significant properties is it likely to have. So take F_2—the suffix indicating the order of the finite group of self-projectivities admitted by F—and then gradually raise the order of this group.

Suppose that F_2 admits a harmonic inversion J in a line λ and a plane π skew to λ: as F_2 is to be non-singular it does not contain π, and so meets π in a plane cubic f. Since every transversal t of λ and π is invariant so is the set of three intersections of t with F_2. But t need not meet π on f; it follows that λ is on F_2 and that, if t is not wholly on F_2, the remaining two intersections are transposed by J. In particular: the join of L on λ to P on f touches F_2 at P. The tangent solid of F_2 at P contains λ and, since this solid cuts F_2 in a cubic surface with a node at P, the plane

$P\lambda$ cuts F_2 in λ and two lines through P. Those lines on F_2 transversal to π and λ generate a scroll of order 9, the intersection of F_2 with the cubic cone of planes joining λ to the points of f. The scroll has f for a double curve and, since the tangent solid to F_2 at any point of λ cuts π in a line meeting f in three points, λ for a triple directrix line.

Since every point of λ lies in the tangent solids of F_2 at all the points of f every point of f, and so the whole plane π, lies on the polar quadrics of all the points of λ. These quadrics are therefore cones with vertices in π; λ is on the Hessian H_2 of F_2 and is linked to a curve λ' in π.

One naturally chooses a system of homogeneous coordinates for which λ is $x_1 = x_2 = x_3 = 0$ and π is $x_4 = x_5 = 0$. Then F_2 has an equation

$$\varphi + \alpha x_4^2 + 2\beta x_4 x_5 + \gamma x_5^2 = 0$$

where φ is cubic and α, β, γ are linear in x_1, x_2, x_3. H_2 is

$$\Delta_2 \equiv \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \alpha_1 x_4 + \beta_1 x_5 & \beta_1 x_4 + \gamma_1 x_5 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \alpha_2 x_4 + \beta_2 x_5 & \beta_2 x_4 + \gamma_2 x_5 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & \alpha_3 x_4 + \beta_3 x_5 & \beta_3 x_4 + \gamma_3 x_5 \\ \alpha_1 x_4 + \beta_1 x_5 & \alpha_2 x_4 + \beta_2 x_5 & \alpha_3 x_4 + \beta_3 x_5 & \frac{1}{2}\alpha & \frac{1}{2}\beta \\ \beta_1 x_4 + \gamma_1 x_5 & \beta_2 x_4 + \gamma_2 x_5 & \beta_3 x_4 + \gamma_3 x_5 & \frac{1}{2}\beta & \frac{1}{2}\gamma \end{vmatrix} = 0,$$

and, as a covariant of F_2, is also invariant under J. It is apparent that its intersection with π consists of the conic $\beta^2 = \gamma\alpha$ and the Hessian h of f, the plane cubic $\varphi = 0$; and, furthermore, that at the six intersections w of these two curves in π every first minor of Δ_2 vanishes: the points w are on the nodal curve C_2 of H_2.

The polar quadric of $(0, 0, 0, \rho, \sigma)$ on λ is

$$\rho(\alpha x_4 + \beta x_5) + \sigma(\beta x_4 + \gamma x_5) = 0,$$

a cone whose vertex, satisfying

$$\rho\alpha + \sigma\beta = \rho\beta + \sigma\gamma = x_4 = x_5 = 0, \tag{14.1}$$

is in π on the conic $\beta^2 = \gamma\alpha$; this conic is therefore λ'.

The polar quadric of $P(\xi_1, \xi_2, \xi_3, 0, 0)$ in π is

$$\sum_{i=1}^{3} \xi_i(\varphi_i + \alpha_i x_4^2 + 2\beta_i x_4 x_5 + \gamma_i x_5^2) = 0.$$

When P is on λ', so that

$$(\beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3)^2 = (\gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3)(\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3),$$

this quadric is

$$\xi_1\varphi_1 + \xi_2\varphi_2 + \xi_3\varphi_3 + (rx_4 + sx_5)^2 = 0$$

and so is a cone whose vertex $(0, 0, 0, s, -r)$ is on λ. For it to be a line-cone its discriminant must have rank 3; this symmetric 5-rowed matrix has zeros in the first three places of its last two columns and of its last two rows, the bottom right-hand corner being

$$\begin{bmatrix} r^2 & rs \\ rs & s^2 \end{bmatrix}.$$

On the other hand, the top left-hand corner is the discriminant of the ternary quadratic $\xi_1\varphi_1+\xi_2\varphi_2+\xi_3\varphi_3$, i.e. of the polar conic of P with respect to f. Since r, s are not both zero it is this polar conic that must be singular for the rank of the 5-rowed discriminant matrix to fall to 3; P is then on the Hessian h of f. It is thus the six intersections w of h and λ' that have polar line-cones: their vertices q join the points on λ linked to the w regarded as points of λ' to the points on h linked to these same w regarded as points of h.

15. The common surface of the polar cones of the points on λ consists of π and the cubic scroll Γ given by

$$\alpha/\beta = \beta/\gamma = -x_5/x_4;$$

Γ has λ for directrix and contains λ'. Indeed Γ is generated by the joins of pairs of linked points (cf. 14.1) on λ and λ'. It meets F_2 where it meets the cubic line-cone $\varphi(x_1, x_2, x_3) = 0$, i.e. in λ counted thrice and six generators, namely those through the intersections of λ' with f.

The points of F_2 whereat the tangent solid contains the whole of λ are on the polar cones of all points on λ, and so on Γ or in π; they consist of f, λ and those six generators of Γ just remarked. If g is any one of them any solid through the plane $g\lambda$ is bitangent to F_2, its contacts harmonic to the intersections of g with λ and λ'. The solid spanned by two such g is therefore quadritangent to F_2 meeting it in a four-nodal cubic surface. Thus 15 quadritangent solids of F_2 contain λ; these are invariant under J while any remaining ones are transposed in pairs. There are 480 of these, Fano having shown ([**6**] p. 282) that a cubic primal in [4] has, in general, 495 quadritangent solids.

Since C_2 is invariant under J the plane joining λ to a point of C_2 meets C_2 again; the projection of C_2 from λ onto π is a curve of order 10 covered twice. This curve in π is, by Zeuthen's formula ([**19**] p. 107; [**16**] p. 169), of genus 12 because the (1, 2) correspondence between it and C_2 has six branch points—the intersections w of C_2 with π. The quadrisecants q conjugate to these points w meet π at singular points of the projected curve which, because H_2 has the same tangent solid at every point of q, passes through the singular point with two branches both of which have there the same tangent as does h.

16. Since the 55 involutions in G belong to a single conjugate class the 55 harmonic inversions for which K is invariant are all of the same kind. When three such involutions belong to a 4-group—each pair commuting and having the third for their product—one may take lines $\lambda_1, \lambda_2, \lambda_3$ forming a triangle $X_1X_2X_3$ in a plane δ while the planes π_1, π_2, π_3 join X_1, X_2, X_3 to a line μ skew to δ.

Let X_4, X_5 be the Hessian duad of the three intersections of F_4 and μ, and use $X_1X_2X_3X_4X_5$ as simplex of reference. Since the simultaneous change of sign of any two of x_1, x_2, x_3 must not alter it the equation of F_4 has the form

$$6kx_1x_2x_3 + 3(ax_1^2 + bx_2^2 + cx_3^2)x_4 + 3(dx_1^2 + ex_2^2 + fx_3^2)x_5 + gx_4^3 + hx_5^3 = 0. \tag{16.1}$$

F_4 contains $\lambda_1, \lambda_2, \lambda_3$ and meets π_1, π_2, π_3 in cubic curves f_1, f_2, f_3; the equation of, say, f_2 is

$$3x_2^2(bx_4 + ex_5) + gx_4^3 + hx_5^3 = 0. \tag{16.2}$$

This curve is invariant under the harmonic homology in π_2 whose centre is X_2 and axis X_4X_5; X_2 is an inflection, X_4X_5 the corresponding harmonic polar.

H_4 meets π_i in the Hessian h_i of f_i and the conic λ_i' linked to λ_i; X_i and μ are pole and polar for λ_i', X_i an inflection on h_i with μ as the corresponding harmonic polar. H_4 meets δ in $\lambda_1, \lambda_2, \lambda_3$ and a conic χ for which $X_1X_2X_3$ is a self-polar triangle.

The point X_1' linked to the intersection X_1 of λ_2 and λ_3 is common to λ_2' and λ_3'; it thus lies in both π_2 and π_3 and so on μ. So X_1', X_2', X_3' are among the five intersections of H_4 and μ; the other two are X_4 and X_5 whose polar quadrics are seen, from the equation of F_4, each to be a cone whose vertex is the other.

Each of $\lambda_1, \lambda_2, \lambda_3$ is met by six of the 30 q which meet δ; there remain 12 to meet χ, and the points in which they do meet χ form three quadrangles each with X_1, X_2, X_3 for diagonal points. The curve linked to χ is a rational quartic χ' meeting C_4 12 times and each of $\lambda_1', \lambda_2', \lambda_3'$ twice.

17. The joins of linked points on λ_i and λ_i' generate a cubic scroll Γ_i; those six which pass through the intersections of λ_i' with f_i are on F_4. These intersections are collinear in pairs with X_i; if the join of a pair meets μ at L the solid δL, containing two of the six generators of Γ_i that are on F_4, is quadritangent to F_4. There are three of these solids, there being three points L on μ. But the same points L occur for $i = 1, 2, 3$. If, say, i is 2 then λ_2' is $\beta^2 = \gamma\alpha$ where, as seen in the discussion of F_2, α, β, γ are the multipliers of $x_1^2, 2x_3x_1, x_3^2$ in (16.1), i.e. λ_2' is

$$k^2x_2^2 = (cx_4+fx_5)(ax_4+dx_5).$$

The three joins, of pairs of points common to f_2 and λ_2', concurrent at X_2 are, by (16.2),

$$3(ax_4+dx_5)(bx_4+ex_5)(cx_4+fx_5)+k^2(gx_4^3+hx_5^3) = 0. \tag{17.1}$$

This same equation appears for each i.

That the solids δL meet F_4 in four-nodal cubic surfaces also appears as follows. The solid $x_5 = \rho x_4$ gives, for its section of F_4, a cubic surface

$$6kx_1x_2x_3+3x_4(ax_1^2+bx_2^2+cx_3^2)+3\rho x_4(dx_1^2+ex_2^2+fx_3^2)+(g+h\rho^3)x_4^3 = 0$$

and this has the standard form for the equation of a four-nodal cubic surface, namely

$$\begin{vmatrix} k^{-1}(a+\rho d)x_4 & -x_3 & -x_2 \\ -x_3 & k^{-1}(b+\rho e)x_4 & -x_1 \\ -x_2 & -x_1 & k^{-1}(c+\rho f)x_4 \end{vmatrix} = 0,$$

provided only that

$$3k^{-2}(a+\rho d)(b+\rho e)(c+\rho f)+g+h\rho^3 = 0;$$

in other words provided that $x_5 = \rho x_4$ joins δ to one of the three points L. This solid contains pairs of generators of each of $\Gamma_1, \Gamma_2, \Gamma_3$; these pairs of generators join the same four nodes, i.e. are pairs of opposite edges of the tetrahedron of nodes.

Each scroll Γ_i affords, spanned by pairs among six of its generators, 15 quadritangent solids of F_4; the same three solids δL occur for $i = 1, 2, 3$. Each Γ_i affords 12 more, so that $3+36 = 39$ of the 495 quadritangent solids of F_4 are accounted for.

18. One embeds the above 4-group in a tetrahedral group $\mathscr{T}$ by adjoining a projectivity of period 3 that permutes X_1, X_2, X_3, cyclically and leaves X_4, X_5 invariant; for example, that whose matrix is

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega & \cdot \\ \cdot & \cdot & \cdot & \cdot & \omega^2 \end{bmatrix},$$

with ω either complex cube root of unity, will serve. F_4 will not admit this new projectivity unless the coefficients in (16.1) satisfy

$$b = \omega a, \quad c = \omega^2 a, \quad e = \omega^2 d, \quad f = \omega d;$$

a primal F_{12} is given by

$$6kx_1x_2x_3 + 3ax_4(x_1^2 + \omega x_2^2 + \omega^2 x_3^2) + 3dx_5(x_1^2 + \omega^2 x_2^2 + \omega x_3^2) + gx_4^3 + hx_5^3 = 0. \tag{18.1}$$

It has the properties already found for F_4 as well as others consequent upon its invariance under the four subgroups $\mathscr{C}_3$ of $\mathscr{T}$.

The 4-group leaves X_1, X_2, X_3 all invariant and permutes the points of δ not on any side of the triangle in tetrads: each tetrad forms a quadrangle having X_1, X_2, X_3 for its diagonal points. $\mathscr{T}$ now permutes these quadrangles in threes save for three special ones: those, namely, any of whose vertices is invariant under a $\mathscr{C}_3$. Such a special quadrangle has one vertex invariant for each of the four $\mathscr{C}_3$, and this $\mathscr{C}_3$ permutes the remaining vertices cyclically. So far as the restriction of $\mathscr{T}$ to δ goes the matrices of period 3 consist of

$$\begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \\ -1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & -1 & \cdot \\ \cdot & \cdot & 1 \\ -1 & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \\ 1 & \cdot & \cdot \end{bmatrix}$$

and their squares. Each matrix has latent roots $1, \omega, \omega^2$; the associated latent vectors of the first are

$$(1\ 1,\ 1), \quad (1,\ \omega,\ \omega^2), \quad (1,\ \omega^2,\ \omega);$$

those of the others are the transforms of these by the 4-group.

What is significant here is the occurrence of ω and ω^2 as latent roots; it implies, in [4], that the joins u of $(1, \omega, \omega^2, 0, 0)$ to X_4 and u' of $(1, \omega^2, \omega, 0, 0)$ to X_5 are both pointwise invariant under $\mathscr{C}_3$.

19. Consider, therefore, a cubic primal F_3 that admits a $\mathscr{C}_3$ of self-projectivities for which a point O and the points of two lines u, u' are all invariant; u, u' are skew and O is outside the solid Σ spanned by them.

Σ meets F_3 in a surface E. The three intersections of E with the join of P on u to Q on u' are cyclically permuted by a projectivity on PQ whose fixed points are P and Q themselves; P, Q are the Hessian duad of the triad of intersections of PQ with E. Should one of P, Q be on E, PQ has three-point intersection with E there; should P, Q both be on E so is the whole line PQ. Thus all nine joins of the intersections P_1, P_2, P_3 of E with u to its intersections Q_1, Q_2, Q_3 with u' are on E.

A transversal PQ of u and u' meets the Hessian H_3 of F_3 in five points, the whole set invariant under $\mathscr{C}_3$; it must consist of a cyclically permuted triad and of two invariant points which can only be P and Q themselves. So u and u' both lie on H_3.

Since the points of u are all invariant so are the vertices of their polar cones, i.e. the points of the curve linked to u on H_3; this must be either u itself or u'. Now the polar cone of a point P, on u, other than P_1, P_2, P_3 cuts u at the polar pair of P for this triad—a pair generally distinct—so that its vertex is not on u. The conclusion is that u and u' are linked on H_3 and so are both chords of its nodal curve C_3. Had P been, exceptionally, one of the Hessian pair of P_1, P_2, P_3 its polar pair would not have been distinct, but would have been the other member, taken twice, of this Hessian pair.

20. Let A be an intersection of u and C_3. Its conjugate quadrisecant meets u' and, being invariant, meets u too, say at B. Then the vertex of the polar cone of B is A so that A, B is the Hessian duad of P_1, P_2, P_3: the intersections of C_3 with u and u' are the Hessian duads of the triads of their intersections with F_3; the q conjugate to any of these four points passes through the complementary point of the duad and is transversal to u and u'. Moreover, the tangents to C_3 at these four points, since they are invariant, are also transversals of u and u'; Σ is quadritangent to C_3, and its 12 intersections, other than its four contacts, with C_3 are the remaining triads on the four q. These triads are invariant too, so that C_3 cuts all four q equianharmonically.

Each point of C_3 on neither u nor u' belongs to a cycle of three; four of these cycles are collinear on q, the other cycles all span planes through O. The only coincidences in the $(3, 1)$ correspondence between C_3 and the aggregate of triads are the four points on u and u'; each of these counts, when reckoned three times, as a triad and so contributes the equivalent of two coincidences; the osculating planes of C_3 there contain O. Zeuthen's formula shows the aggregate to have genus 8.

If the intersection of the tangent to C_3 at A with u' is, say, T the osculating plane of C_3 at A is OAT; the osculating solid contains this plane and so, being invariant, joins it either to u or to u'. But the latter alternative cannot occur: for the solid would then meet C_3 at $20-4-2 = 14$ points not on u or u'; none of these could be invariant, nor do they admit permutation as cycles of three. So the osculating solid joins OAT to u and has $20-4-1 = 15$ intersections with C_3 not on u or u'. Similar considerations apply at each contact of Σ with C_3.

$\mathscr{T}$ contains four $\mathscr{C}_3$, each with an isolated invariant point O_i and axes u_i, u'_i $(i = 1,2,3,4)$. The 4-group V in $\mathscr{T}$ is a normal subgroup, but each $\mathscr{C}_3$ permutes the involutions of V cyclically, as it therefore does X_1, X_2, X_3. Thus δ contains each O_i, meets u_i at, say, U_i and u'_i at U'_i; $X_1X_2X_3$ is the diagonal point triangle of all these three quadrangles in δ. Moreover, each $\mathscr{C}_3$ induces a projectivity on μ of which X_4, X_5 are the fixed points; each u_i passes through, say, X_4, each u'_i through X_5. The quadrangles of U_i and U'_i are both inscribed in χ; each of them is equianharmonic on χ and is the Hessian tetrad of the other. This last remark is validated by the fact that each $\mathscr{C}_3$ in $\mathscr{T}$ permutes the others cyclically: the one which has, say, u_1 and u'_1 for axes induces permutations $(u_2u_3u_4)$ and $(u'_2u'_3u'_4)$.

21. The tetrahedral group $\mathscr{T}$ can be extended to an icosahedral group $\mathscr{I}$ by adjoining a suitable operation of period 5. One way of achieving this is to take a standard irreducible representation of $\mathscr{I}$ as a group of monomial quinary substitutions ([**2**] p. 353). If a tetrahedral subgroup in this representation is transformed into $\mathscr{T}$ the same transformation, of a monomial substitution of period 5, will produce an appropriate operation to adjoin to $\mathscr{T}$. So one finds the substitution

$$\left.\begin{array}{rrrrrr} 2\xi_1 = & & -\ x_2 & +\ x_3 & -\ x_4 & -\ x_5 \\ 2\xi_2 = & -x_1 & -\ x_2 & & -\omega^2x_4 & -\ \omega x_5 \\ 2\xi_3 = & -x_1 & & +\ x_3 & -\ \omega x_4 & -\omega^2x_5 \\ 2\xi_4 = & x_1 & +\ \omega x_2 & -\omega^2x_3 & & +\ x_5 \\ 2\xi_5 = & x_1 & +\omega^2x_2 & -\ \omega x_3 & +\ x_4, & \end{array}\right\} \qquad (21.1)$$

the multipliers at the left ensuring that the substitution has determinant $+1$.

In order that the quinary cubic in (18.1) be invariant under (21.1) it is necessary that k, a, d, g, h be proportional to the corresponding coefficients in the cubic produced by the substitution; this is found to happen (the verification is routine) provided that

$$k+a+d = 0, \quad g = a+2d, \quad h = 2a+d. \qquad (21.2)$$

A quinary cubic contains 35 terms, only nine of which appear in (18.1); it is necessary for the invariance of the primal that none of the 26 missing terms intrudes in consequence of the substitution; happily the conditions (21.2) ensure that none does.

So, invariant under $\mathscr{I}$, there is F_{60} given by

$$a\{3x_4(x_1{}^2+\omega x_2{}^2+\omega^2x_3{}^2)-6x_1x_2x_3+x_4{}^3+2x_5{}^3\}$$
$$+d\{3x_5(x_1{}^2+\omega^2x_2{}^2+\omega x_3{}^2)-6x_1x_2x_3+2x_4{}^3+x_5{}^3\} = 0.$$

It is important to note that there is a single quadric Q invariant under $\mathscr{I}$; in the present coordinate system Q is

$$x_1^2+x_2^2+x_3^2+2x_4x_5 = 0.$$

Each of the ten subgroups $\mathscr{C}_3$ of $\mathscr{I}$ provides a solid Σ spanned by two axes u, u' both pointwise invariant under $\mathscr{C}_3$; any transversal t of u, u' meets Q in two points which, invariant as a pair under $\mathscr{C}_3$, are individually invariant so that one is on u, the other on u'. Thus u, u' both lie on Q. They are chords of C_{60}, the nodal curve of the Hessian H_{60} of F_{60}; as there are 10 such pairs of axes u, u' they together account for all 40 intersections of Q with C_{60}. Of course each of the ten Σ is quadritangent to C_{60}.

The group G for which Klein's curve C is invariant has 55 subgroups $\mathscr{C}_3$; thus C has 55 pairs of chords, the chords of a pair being linked on H, the solid spanned by each pair quadritangent to C. It is the contacts of these solids with C that make up the set of 220 points b. Each icosahedral subgroup $\mathscr{I}$ of G is one of a conjugate set of 11; the 11 quadrics, one invariant for each subgroup $\mathscr{I}$ of the set, together cut all 220 points b on C.

22. The latent roots of

$$\frac{1}{2}\begin{bmatrix} \cdot & -1 & 1 & -1 & -1 \\ -1 & -1 & \cdot & -\omega^2 & -\omega \\ -1 & \cdot & 1 & -\omega & -\omega^2 \\ 1 & \omega & -\omega^2 & \cdot & 1 \\ 1 & \omega^2 & -\omega & 1 & \cdot \end{bmatrix}$$

are the five fifth roots of unity. The latent column vector associated with unity itself is

$$(0,\ 0,\ \omega-\omega^2,\ \omega,\ -\omega^2)' \qquad (22.1)$$

while that associated with any primitive root η is

$$(\eta-\eta^4,\ \eta^3-\eta^2,\ -1,\ 1-\omega\eta-\omega\eta^4,\ 1-\omega^2\eta-\omega^2\eta^4)'. \qquad (22.2)$$

The four points O_1, O_2, O_3, O_4 of which these latter, i.e. the vectors occurring on writing $\eta, \eta^2, \eta^3, \eta^4$ for η in (22.2), are coordinate vectors are all on Q; the solid spanned by them is the polar solid

$$\omega^2(x_3+x_4) = \omega(x_3+x_5)$$

of the other point O whose coordinate vector is (22.1).

This subgroup $\mathscr{C}_5$ of $\mathscr{I}$ has thus four isolated fixed points O, O_1, O_2, O_3, O_4; the join of any two of them is invariant, its points, other than the two, being permuted by $\mathscr{C}_5$ in cycles of five. Any primal invariant under $\mathscr{C}_5$, and all primals invariant under $\mathscr{I}$ in particular, can only meet the join in such cycles apart form the two fixed points themselves. Q has been seen to touch OO_i at O_i; it cannot meet it elsewhere. It is also found to contain two pairs of opposite edges of the tetrahedron $O_1O_2O_3O_4$; the two edges not on Q are O_1O_4 and O_2O_3.

F_{60} does not pass through O. But it does pass through O_i; not only so but, as can be verified by substituting the coordinates of O_i in the equations of the polar quadric and polar solid of O, OO_i has 3-point intersection with F_{60} at O_i.

Each point of C_{60} belongs to a pentad whose members are subjected to cyclic permutation by $\mathscr{C}_5$. There are no coincidences in the (5, 1) correspondence between C_{60} and this aggregate of pentads, which aggregate therefore has genus 6 by Zeuthen's formula. Since there are 66 subgroups $\mathscr{C}_5$ in G Klein's curve is in (5, 1) correspondence with a curve of genus 6 in 66 different ways.

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Mathematical Institute
20 *Chambers Street*
*Edinburgh EH*1 1*HZ*