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GEOMETRY RELEVANT TO THE BINARY QUINTIC

by W. L. EDGE

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Introduction

The aim in this paper is to indicate one way of interpreting covariants of a binary quintic F geometrically, interpretations having been found recently [7] for its quadratic covariant ${}_2C_2$, called Γ in [7], and its invariant I_4. The symbol ${}_dC_n$ will denote a covariant of order n in the binary variables x, y and degree d in the coefficients of F, I_d being used in preference to ${}_dI_o$ for *invariants*. The sum $d+n$ is 4 for both ${}_2C_2$ and I_4, and no other covariant affords as small a sum; so it is natural to have begun by interpreting these two and to use them as auxiliaries in interpreting others.

Similar procedures apply to the covariant ${}_2C_2$ and invariant I_4 when F has any odd order $2m+1$, but it will be enough first to concentrate on $m=2$ and perhaps allude to the generalisation thereafter.

The rational normal curve and its osculating spaces

1. The whole discussion hinges on the rational normal quintic curve

$$C: \quad x_i=(-t)^i \qquad i=0,1,2,3,4,5$$

in projective space S of five dimensions. The point $P(a,b,c,d,e,f)$ maps the binary quintic

$$F: ax^5+5bx^4y+10cx^3y^2+10dx^2y^3+5exy^4+fy^5$$

whose zeros x/y are consequently the parameters of contact with C of those five of its osculating primes ω_4 that contain P [6, p. 312]. One uses ω_k to denote any osculating $[k]$ of C, including its points ω_0. When P is of general position it does not lie in any ω_k with $k<4$; if, however, it does happen to lie in such an $\omega_k(t)$ then t is a $(5-k)$-fold zero of F. As t, and so the point on C, varies $\omega_k(t)$ generates a locus Ω_{k+1}; these loci are [11, p. 95]

$$\Omega_1^5(\equiv C),\ \Omega_2^8,\ \Omega_3^9,\ \Omega_4^8.$$

The points of the octavic primal Ω_4^8 map those F having a repeated zero.

2. Any two ω_3 meet in a line, and this doubly infinite set of lines generates a threefold V double on Ω_4^8; the points of V map those F having a pair of repeated zeros. In $\omega_3(t)$ these lines are the tangents of the locus $T(t)$ of its intersections with the ω_2: $T(t)$ is a twisted cubic sharing the osculating spaces $\omega_0(t)$, $\omega_1(t)$, $\omega_2(t)$ with C. This is rapidly substantiated by taking $t=0$, $\omega_3(0)$ being $x_4=x_5=0$. Since $\omega_2(\phi)$ is

$$\phi^3x_i+3\phi^2x_{i+1}+3\phi x_{i+2}+x_{i+3}=0 \qquad i=0,1,2$$

its intersection with $\omega_3(0)$ satisfies

$$\phi^3x_0+3\phi^2x_1+3\phi x_2+x_3=\phi^2x_1+3\phi x_2+3x_3=\phi x_2+3x_3=0,$$

$$x_0:x_1:x_2:x_3=10:-6\phi:3\phi^2:-\phi^3,$$

the parametric form of $T(0)$, whose tangent joins this point to

$$x_0:x_1:x_2:x_3=0:-2:2\phi:-\phi^2$$

and both these points satisfy both the equations

$$\phi^4x_i+4\phi^3x_{i+1}+6\phi^2x_{i+2}+4\phi x_{i+3}+x_{i+4}=0 \qquad i=0,1 \tag{2.1}$$

of $\omega_3(\phi)$.

The tangents of $T(t)$ generate a quartic scroll, and V is generated by this scroll as t varies.

The equation of Ω_4^8 is the outcome of eliminating ϕ between the pair of equations (2.1), and Sylvester's dialytic process provides an 8-rowed determinant immediately. Only the single term $x_0^4x_5^4$ in its expansion involves x_0 to a power as high as the fourth; this, therefore, is the *leading term* or *leader* in the sense of the dictionary or lexicon. Replacing the x_i in order by $a, b, \ldots, f$ produces the discriminant of F, an invariant I_8 led by a^4f^4. It is labelled Q' by Cayley; its 59 terms are written out by him in full [4, p. 288; see, too, **12**, p. 209].

When handling a sole covariant any non-zero constant multiplier need not be relevant. This is no longer so if more than one covariant is involved; for example: a linear combination of I_4^2 and I_8 has to be identified precisely, as have the leaders of the two invariants. The natural procedure is to decide that the lexicographically leading terms of invariants, and of the leading seminvariants of covariants, all have coefficient $+1$.

The first linear covariant

3. Baker [**1**, p. 137] proved that Ω_2^8 is the base surface of a net N of quadrics, and the way to the sought interpretation called in aid [7, p. 136] those quadrics of N that are singular; these are all point-cones with vertices on C, the Jacobian curve of N consisting of C reckoned thrice. There are two of these cones through P; it is the parameters ϕ, ψ of their vertices A, B on C that are [7, p. 137] the zeros of

$${}_2C_2: \eta_0x^2-\eta_1xy+\eta_2y^2$$

where

$$\eta_0 = ae - 4bd + 3c^2, \quad \eta_1 = -af + 3be - 2cd, \quad \eta_2 = bf - 4ce + 3d^2$$

and so

$$I_4 \equiv \eta_1^2 - 4\eta_0\eta_2$$

led by a^2f^2.

Were this geometrical setting to determine a point on C *uniquely* its parameter would be the zero of a *linear* covariant. One such point J is the residual intersection of C with the [4] spanned by P, $\omega_1(A)$, $\omega_1(B)$. As $\omega_1(A)$ is given by the four equations

$$t_i \equiv \phi^2 x_i + 2\phi x_{i+1} + x_{i+2} = 0 \qquad i = 0, 1, 2, 3$$

and as $\psi^2 t_i + 2\psi t_{i+1} + t_{i+2}$ is symmetric in ϕ, ψ the bitangent solid of C spanned by $\omega_1(A)$ and $\omega_1(B)$ is given by the two equations

$$\psi^2 t_i + 2\psi t_{i+1} + t_{i+2} = 0 \qquad i = 0, 1$$

or, since $\phi + \psi = \eta_1/\eta_0$ and $\phi\psi = \eta_2/\eta_0$,

$$E_i \equiv \eta_2^2 x_i + 2\eta_1\eta_2 x_{i+1} + (\eta_1^2 + 2\eta_0\eta_2)x_{i+2} + 2\eta_0\eta_1 x_{i+3} + \eta_0^2 x_{i+4} = 0 \tag{3.1}$$

$$[\eta_2 \quad \eta_1 \quad \eta_0] \begin{bmatrix} x_i & x_{i+1} & x_{i+2} \\ x_{i+1} & x_{i+2} & x_{i+3} \\ x_{i+2} & x_{i+3} & x_{i+4} \end{bmatrix} \begin{bmatrix} \eta_2 \\ \eta_1 \\ \eta_0 \end{bmatrix} = 0$$

$$\eta' M_i^x \eta = 0. \tag{3.2}$$

The [4] $\lambda E_0 + E_1 = 0$ meets C where

$$(\lambda - t)(\phi - t)^2(\psi - t)^2 = 0$$

so that the parameter of J is

$$t = -(\eta' M_1 \eta)/(\eta' M_0 \eta)$$

where

$$M_0 = \begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}, \qquad M_1 = \begin{bmatrix} b & c & d \\ c & d & e \\ d & e & f \end{bmatrix}$$

and J maps the linear covariant

$$_5C_1 : (\eta' M_0 \eta)x + (\eta' M_1 \eta)y. \tag{3.3}$$

The leader of $\eta' M_0 \eta$ (it will be remembered that a is absent from η_2) comes from $\eta_1^2 c$ and is $a^2 cf^2$.

4. The above discussion is, as the null polarity set up [6, p. 312] by C in S shows, tantamount to determining $_5C_1$ as that linear form which, when multiplied by the square of $_2C_2$, produces a quintic apolar to F. The condition [9, p. 213] for apolarity of F and

$$(\alpha x + \beta y)\{\eta_0^2 x^4 - 2\eta_0\eta_1 x^3 y + (\eta_1^2 + 2\eta_0\eta_2)x^2 y^2 - 2\eta_1\eta_2 x y^3 + \eta_2^2 y^4\}$$

gives (3.2) instantly.

5. This exhibition of the coefficients of $_5C_1$ as ternary quadratic forms in η_0, η_1, η_2 places no burden on the memory and enables one just to "write down" the covariant. It does rely on having prior knowledge of $_2C_2$, but this we do have, there being an automatic procedure for "writing down" η_0, η_1, η_2 [7, p. 139]. It is perhaps a welcome alternative to producing a string of 20 quintic monomials as Cayley [2, p. 274] and Salmon [12, p. 216] were constrained to do. There are of course two strings, one for each coefficient, but either comes from the other on imposing the triple transposition (af) (be) (cd), this being the special instance for $n=5$ of the "symmetry" of covariants of a binary $n-ic$. [8, p. 124].

Cayley displayed *in extenso* 23 covariants of F. His resource in constructing generating functions and his skill in manipulating them enabled him to forge a tool that ascertained the number of linearly independent covariants of given order and degree. He could then employ differential annihilators to calculate their actual form; these matters are described in Elliott's book [8, Chs. VI, VII, VIII]. For the recorded *existence* of $_5C_1$ see [2, p. 264], for its actual expression [2, p. 274]. One can read the facts duly recorded by his successors [5, p. 277; 12, p. 216; 9, p. 131; 8, p. 299]. One must also record, as a tiny by-product, one might almost say a throw-away, of a work of Hilbert [10, pp. 115–116] the determinantal form

$$\begin{bmatrix} 4c & 8b & a & . & . \\ 3d & 5c & b & b & a \\ 2e & 2d & c & 2c & 2b \\ f & -e & d & 3d & 3c \\ . & -4f & e & 4e & 4d \end{bmatrix},$$

led by $8a^2cf^2$, of the coefficient of x.

6. The occurrence of M_0 and M_1 suggests that the matrix

$$\begin{bmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{bmatrix} \tag{6.1}$$

should be used, the more especially when it is readily checked that

$$[\eta_2 \quad \eta_1 \quad \eta_0]\begin{bmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{bmatrix} = -(3\Delta_4, \Delta_3, \Delta_2, 3\Delta_1)$$

Δ_i being the determinant of the residue when the ith column is dropped; in particular $\Delta_1 = |M_1|$, $\Delta_4 = |M_0|$. Then

$$\eta' M_0 \eta = -(3\Delta_4\eta_2 + \Delta_3\eta_1 + \Delta_2\eta_0),$$

$$\eta' M_1 \eta = -(\Delta_3\eta_2 + \Delta_2\eta_1 + 3\Delta_1\eta_0)$$

and one has another form for ${}_5C_1$:

$$\begin{bmatrix} -3\eta_0 y & a & b & c \\ \eta_0 x + \eta_1 y & b & c & d \\ -\eta_1 x - \eta_2 y & c & d & e \\ 3\eta_2 x & d & e & f \end{bmatrix}$$

in which, too, the leader of the coefficient of x proves to be a^2cf^2.

Geometrical interpretations of other concomitants

7. The harmonic conjugate J' of J on C with respect to A, B maps a second linear covariant. The chords of C joining pairs of that involution of which J is a focus and AB a pair generate [**11**, p. 97] a quartic scroll; two of the generators are tangents of C; the tangent at J is one, that at J' is the other. The harmonic property is equivalent to saying that this second linear covariant is the Jacobian [**9**, p. 133; **8**, p. 300] of ${}_2C_2$ and ${}_5C_1$; its degree is therefore $2+5=7$; it is a ${}_7C_1$ and duly registered by Cayley [**3**, p. 286], each coefficient involving 49 terms. The Jacobian of $\eta_0 x^2 - \eta_1 xy + \eta_2 y^2$ and $\alpha x + \beta y$ is

$$(2\beta\eta_0 + \alpha\eta_1)x - (\beta\eta_1 + 2\alpha\eta_2)y$$

and the substitution of $\eta' M_0 \eta$ for α and $\eta' M_1 \eta$ for β produces ${}_7C_1$.

Since the harmonic pairing involves J and J' symmetrically not only is ${}_7C_1$ the Jacobian of ${}_5C_1$ and ${}_2C_2$: ${}_5C_1$ is the Jacobian of ${}_7C_1$ and ${}_2C_2$; that this latter Jacobian is a ${}_9C_1$ can only mean that it is the product of ${}_5C_1$ and I_4. This is indeed a special instance of the reducibility of the Jacobian of a Jacobian [**9**, p. 78]. In that text take $f\equiv{}_2C_2\equiv\psi$, $\phi\equiv{}_5C_1$; the first and third terms of equation XIX are zero because ϕ is linear, leaving only the middle term, the product of ${}_5C_1$ and the discriminant of ${}_2C_2$.

That J, J' are harmonic to A, B is among the facts announced by Todd [15, p. 5]; in his notation the statement is "β is the harmonic conjugate of α with respect to i".

8. The matrix (6.1) is central to the geometry of the trisecant planes of C, there being [**11**] a *unique* trisecant plane through any point P of general position. For the plane spanned by the points on C having $t=\alpha,\beta,\gamma$ is, if

$$\alpha+\beta+\gamma=e_1 \qquad \beta\gamma+\gamma\alpha+\alpha\beta=e_2 \qquad \alpha\beta\gamma=e_3$$

determined by the linear equations

$$e_3x_i+e_2x_{i+1}+e_1x_{i+2}+x_{i+3}=0 \qquad i=0,1,2$$

so that *one and only one* such plane contains P, namely that for which α,β,γ are the zeros of

$$\begin{vmatrix} 1 & -\theta & \theta^2 & -\theta^3 \\ a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{vmatrix} \tag{8.1}$$

or

$$\Delta_1+\Delta_2\theta+\Delta_3\theta^2+\Delta_4\theta^3$$

so that

$$j\equiv\Delta_4x^3+\Delta_3x^2y+\Delta_2xy^2+\Delta_1y^3$$

is the covariant ${}_3C_3$ that plays an important part [**12**, p. 148; **5**, p. 277; **9**, p. 132] in the theory of the binary quintic. The qualifying phrase "of general position" restricting P means that (6.1) has rank 3. As P is in the plane spanned by α,β,γ on C, F is linearly dependent on $(x-\alpha y)^5$, $(x-\beta y)^5$, $(x-\gamma y)^5$ permitting the canonical form so profitably exploited by Salmon [**12**, pp. 206–215]. Note, to gain full access to this treatment, that

$$(\beta-\gamma)(x-\alpha y)+(\gamma-\alpha)(x-\beta y)+(\alpha-\beta)(x-\gamma y)\equiv 0$$

so that F is a linear combination of the fifth powers of three linear forms summing identically to zero. The shapes that all 23 concomitants take for this canonical form are listed [**12**] by Sylvester. See also [**15**].

Note, too, the special case of P lying on Ω_3^9, and so in an $\omega_2(t)$. This implies not only that F has (above, p. 311) a triple factor but that j is a perfect cube. Since $\omega_4(t)$ now accounts for three of the five ω_4 through P j is a factor of F [**9**, p. 230].

9. The discriminant of j is an invariant I_{12} whose full expression was elaborated in the earliest stage of the theory [**3**, p. 294; **12**, p 210]. As it is the eliminant of

$$3\Delta_4 x^2+2\Delta_3 xy+\Delta_2 y^2 \quad \text{and} \quad \Delta_3 x^2+2\Delta_2 xy+3\Delta_1 y^2 \tag{9.1}$$

one merely solves, equating these two polars of j to zero, for x^2, xy, y^2 and obtains

$$(9\Delta_1\Delta_4-\Delta_2\Delta_3)^2=4(\Delta_2^2-3\Delta_1\Delta_3)(\Delta_3^2-3\Delta_2\Delta_4). \tag{9.2}$$

This is more concise than the 252 terms of the original expression; not only so, but it affords some insight into the geometry of Π, the duodecimic primal in S obtained on substituting the x_i for $a, b, \ldots$ in I_{12}. Denote the outcome of this substitution in Δ_i by $\Delta_i(x)$. Then Π is generated, as x/y varies, by ∞^1 threefolds M_3^9 given by equating both quadratics (9.1) to zero. It may well not have escaped notice that (9.2) is a precise analogue of the quartic equation of the developable of tangents of a twisted cubic; the M_3^9 correspond one to each of these tangents while the threefold

$$3\Delta_1(x)/\Delta_2(x)=\Delta_2(x)/\Delta_3(x)=\Delta_3(x)/3\Delta_4(x),$$

of order $6^2-3^2=27$, corresponds to the cubic itself and so is cuspidal on Π.

Π is the locus of points P such that the unique trisecant plane through P touches C, and so contains every plane joining an ω_0 to an ω_1. So it is generated by

(a) the cubic line-cones projecting C from its own tangents,

(b) the sextic point-cones projecting Ω_2^8 from the points of C.

When each of $a, b, \ldots$ is replaced by the appropriate $(-t)^i$ the rank of (6.1) sinks to 1; C is therefore a double curve on each $\Delta_i(x)=0$ and so, at least, octuple on Π. This implies that every chord of C, and so [**7**, p. 137] the threefold M_3^6 that the chords generate, lies wholly on Π. Indeed, as will be seen in a moment, M_3^6 lies on each $\Delta_i(x)=0$ and so is, at least, a quadruple threefold on Π.

To vindicate this last statement substitute $\lambda(-u)^i+\mu(-v)^i$ for $a, b, \ldots$, in (6.1). Each Δ_j then becomes the sum of $2^3=8$ determinants each of which involves multiples of at least two columns of either $\Delta_j(-u)^i$ or $\Delta_j(-v)^i$, and so is zero.

If one looks upon the pair of equations (3.2) as equations of conics it is clear that, since two conics intersect in four points, there are four bitangent solids of C through an arbitrary point P. The condition for two of these to coalesce is that the two conics touch: their tact-invariant must be zero. But it is readily seen that the invariants generally designated in treatises on conic sections by

$$\Delta, \quad \Theta, \quad \Theta', \quad \Delta'$$

are here (each of Θ, Θ' is at first sight the sum of three determinants, but two of these are zero)

$$\Delta_4(x), \quad \Delta_3(x), \quad \Delta_2(x), \quad \Delta_1(x)$$

and so the condition in question is (9.2). One therefore has a double interpretation of I_{12}.

If P maps a quintic having $I_{12}=0$ then two of the three intersections of C with its trisecant plane through P coalesce, as also do two of the four bitangent solids of C through P.

If one follows Hammond [8, p. 301] and takes an F with $c=d=0$, so lacking its two central terms, its map P is in $x_2=x_3=0$. So one achieves this by using one of the four bitangent solids through P and assigning 0 and ∞ to be the parameters of its two contacts.

10. The geometry proffers many sets of points on C as representatives of covariants to be identified. A quadratic suggests itself at once: the two points harmonic on C to both pairs AB, JJ'. Now the quadratic harmonic to both of two given quadratics is their Jacobian, and here the two are ${}_2C_2$ and the product ${}_5C_1 . {}_7C_1$. Since

$$\frac{\partial(U, VW)}{\partial(x, y)} = V\frac{\partial(U, W)}{\partial(x, y)} + W\frac{\partial(U, V)}{\partial(x, y)}$$

the new covariant is seen, on taking $U={}_2C_2$, $V={}_5C_1$, $W={}_7C_1$ to be a linear combination of $I_4({}_5C_1)^2$ and $({}_7C_1)^2$.

A second proposal is to pair the residual intersection of C with the [4] spanned by P, A, $\omega_2(B)$ with the corresponding point in the [4] spanned by P, $\omega_2(A), B$. The parameters ϕ, ψ are involved symmetrically and the pair maps a quadratic covariant.

A different procedure is to take, on the unique transversal line from P to $\omega_2(A)$ and $\omega_2(B)$, the harmonic conjugate P' of P with respect to the intersections of the transversal with the planes; P', P are harmonic inverses in $\omega_2(A)$, $\omega_2(B)$. Then P' maps a quintic covariant F' of F. But the symmetry of the harmonic relation does not imply symmetry between F and F': the pair A', B' on C derived from P' as A, B were from P will, in general, be another pair and so map another quadratic covariant.

The geometric interpretation of the skew invariant I_{18} is immediately apparent from the opening sentence [12, p. 212] of Salmon's Section 229. If $I_{18}=0$ *P lies in the ω_4 at one of the three intersections of C with its trisecant plane through P.* Dialytic elimination, using (8.1) and F, produces I_{18} as an 8-rowed determinant; the Δ_i provide the elements of five, the a_i of the other three, rows.

The binary form of odd order $2m+1$

11. Similar proceedings to those used above apply to the mapping of a binary form F of order $2m+1$ by a point P of the projective space $[2m+1]$, the cardinal feature being the rational normal curve C of order $2m+1$. All the ω_{m-1} of C lie on the quadrics of Baker's net N [1, p. 137]; the singular members of N are all point-cones with vertices on C which, reckoned $m+1$ times, is the Jacobian curve of N. There are two of these cones through a point P of general position, the parameters ϕ, ψ on C of their vertices A, B are the zeros of

$$ {}_2C_2\colon \quad \eta_0 x^2 - \eta_1 xy + \eta_2 y^2, $$

the quadratic covariant $(ab)^{2m}a_x b_x$ of F, where η_j can be written down by rule of thumb [7, p. 139]. Then

$$ I_4' \equiv \eta_1^2 - 4\eta_0\eta_2. $$

The residual intersection J of C with the $[2m]$ spanned by P, $\omega_{m-1}(A), \omega_{m-1}(B)$ maps a linear covariant ${}_{2m+1}C_1$; its harmonic conjugate J' on C with respect to A, B mapping a second linear covariant ${}_{2m+3}C_1$. The covariant ${}_{2m+1}C_1$ is that linear form which, when multiplied by $({}_2C_2)^m$, provides a $(2m+1)ic$ apolar to F.

Take the residual intersection of C with the $[2m]$ spanned by $P, \omega_r(A), \omega_s(B)$ where $r+s=2m-2$ and pair it with its analogue in the prime spanned by $P, \omega_s(A), \omega_r(B)$; here, as equality has already been considered, r, s are unequal but each can take any value other than $m-1$ between 0 and $2m-2$ inclusive. The pair so obtained on C involves ϕ and ψ symmetrically and provides a quadratic covariant; a string of such covariants occurs on taking the different values of r, s.

There is, just as was shown for $m=2$, a unique $(m+1)$-secant $[m]$ of C through a point P of general position; the parameters of its $m+1$ intersections are zeros of a determinant analogous to (8.1) and supply the covariant ${}_{m+1}C_{m+1}$, or canonizant, of F. Its coefficients are $(m+1)$-rowed determinants Δ_i whose elements are coefficients in F and its discriminant, of degree $2m$ in its coefficients, provides an invariant $I_{2m(m+1)}$. Replacing the a_i by the coordinates x_i in this invariant produces a primal of order $2m(m+1)$, the assemblage of all those secant $[m]$'s that touch C, meeting C in $m-1$ further points.

Since the ω_{2m-1} generate a primal Ω_{2m}^{4m} the discriminant of F itself is an invariant I_{4m}.

12. As it will provide an intriguing problem for solution a final word about the nonic is in order. Its coefficients, shorn of their binomial multipliers, are

$$ a, \quad b, \quad c, \quad d, \quad e, \quad f, \quad g, \quad h, \quad i, \quad j \tag{12.1} $$

to which the respective weights

$$ 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9 $$

are assigned, and the rule gives

$$ \eta_0 = ai - 8by + 28cg - 56df + 35e^2, $$

$$ \eta_1 = -aj + 7bi - 20ch + 28dg - 14ef, $$

$$ \eta_2 = bj - 8ci + 28dh - 56eg + 35f^2, $$

so that

$$I_4 = \eta_1^2 - 4\eta_0\eta_2$$

led by a^2j^2. This tallies with Cayley's expression [**4**, p. 318] led by $-a^2j^2$, but Cayley would have had to perform some calculations to obtain it. What is surprising at first sight is that Cayley produces a second quartic invariant J_4. But any surprise is misplaced if one recalls Hermite's law of reciprocity [**12**, p. 135; **8**, p. 155]: the number of invariants of degree n of a binary $p-ic$ is equal to the number of degree p of binary $n-ic$, "number" here implying "functionally independent". The binary quartic has, in the customary notation, two invariants I, of degree 2, and J, of degree 3; so J^3 and I^3J are both of degree 9. The binary nonic has therefore two quartic invariants.

Cayley would start from knowing that the number of linearly independent invariants of degree 2σ is the coefficient of $r^{9\sigma}$ in

$$\frac{(1-r^{2\sigma+1})(1-r^{2\sigma+2})----(1-r^{2\sigma+9})}{(1-r^2)(1-r^3)-----(1-r^9)}$$

which, for $\sigma=2$, is the coefficient of r^{18} in

$$(1-r^{10})(1-r^{11})(1-r^{12})(1-r^{13})/[(1-r^2)(1-r^3)(1-r^4)],$$

a polynomial of degree 37. He would then catalogue those quartic monomials in (12.1) that are of weight 18 and seek those linear combinations of them that are annihilated by the two differential operators or, alternatively, are annihilated by either operator and are either invariant or multiplied by -1 under the fivefold transposition $(aj)(bi)(ch)(dg)(ef)$. His two eligible expressions appear juxtaposed in two columns [**4**, p. 318]; the column led by $-a^2j^2$ is $-I_4$, that led by $2aci^2$ is J_4. There is one clear misprint (ce^3i for ce^2i) in this column, and it is suggested that the multiplier of d^2g^2 should be -47, not $+47$ as printed. The numerical multipliers in J_4 are noticeably lower than those in I_4.

The occurrence of two quartic invariants of the nonic was later noted by Sylvester [**14**, p. 281].

13. This is, surely a challenging situation. The interpretation, given a point P of general position in regard to a rational normal curve in $[2m+1]$, of I_4 was found in (7); it now appears—or did away back in 1856—that for $m=4$ there is a *second quartic primal* inherent in the figure about which nothing is yet known save that the left-hand side of its equation is Cayley's string of isobaric quartic monomials of weight 18. But how is it identified in the geometry? Indeed there is now a whole pencil of quartic primals to be interpreted and their common intersection, of order 16 and dimension 7, may be significant for the algebraic theory of the nonic. Which members of this pencil have higher singularities and do they acquire greater significance thereby? Does the fact that the numerical multipliers in J_4 are comparatively small have any import, or is it a mere accident?

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Montague House
67 Mill Hill
Musselburgh, EH21 7RL
Scotland