THE FLECNODAL CURVE OF A RULED SURFACE

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1. In 1849 Salmon [**8**; p. 260] proved that the points on a non-singular algebraic surface F, of order n in [3], at which one of the two inflectional tangents has 4-point intersection with F, are those of the common curve $\mathscr{F}$ of F and a covariant surface of order $11n-24$. The tangent plane at any ordinary point cuts F in a curve with a node at the contact, and the two nodal tangents to this plane curve are inflectional tangents to F; should either of these have 4-point intersection with F, and so with the curve, the node is what Cayley [**4**; p. 28] called a flecnode, there being an inflection on one of the two branches there. So Cayley [**4**; p. 29] naturally called $\mathscr{F}$, of order $n(11n-24)$, the flecnode-curve of F. But if F is a scroll R, one of the two inflectional tangents at a point X is the generator g through X, and the question arises whether the other inflectional tangent can have 4-point intersection with R at X. The locus of such X is the flecnodal curve $\mathscr{F}$ of R.

In 1874 Voss investigated in the course of a lengthy and, to one reader at least, somewhat difficult paper, the two special cases of R being

(a) the complete set of lines common to three complexes [**14**; p. 90];

(b) rational [**14**; p. 107];

a year later he obtained [**15**; p. 485] the order of $\mathscr{F}$ for a general scroll of order n and genus p. He also gave, under both (a) and (b), an order for the scroll ϕ generated by these tangents having 4-point intersection—correctly in (b), but the result he prints for (a) is not correct. There seems to be no flaw in Voss's geometry; but when he alters the last row and column of his 7-rowed determinant he seems unconscious of thereby subtracting 2 from its degree in his coordinates x_i; consequently what is printed as 19 in the *last* equation on p. 90 ought to have been 20. There is no manuscript evidence available; but if one may judge from the printed page this may have been an unlucky accident of dittography. It is regrettable that some devil's advocate was not at hand to forestall the canonization of this incorrect statement in standard works of reference ([**17**; p. 1214], [**3**; p. 726]).

2. Voss's declared purpose was to exploit the then new theory of line geometry, developed by Plücker and Klein, and so investigate the geometry of R by using only line coordinates, never point or plane coordinates. Here it is proposed to use Klein's mapping of the lines of [3] by the points of a non-singular quadric Ω in [5]; R is then mapped by a curve C, of the same order n and genus p as R, on Ω; nor is the mapping in any way restricted to special cases, whether (a) or (b) or others. The map C of R in (a) is the complete intersection of Ω and three primals of orders, say, M, N, P; the genus p of C is then determined by the fact (not available in 1874, nor for some years later) that the canonical series is cut on C by primals of order

$$2+M+N+P-6 = M+N+P-4,$$

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so that $2p-2 = 2MNP(M+N+P-4)$. Since the order n of C is here $2MNP$, Voss's calculation makes the order of $\mathscr{F}$ $5n+12(p-1)$. The order he gives for ϕ would, after the correction, be $8(n+3p-3)$. Both the orders are found by him, correctly, when $p = 0$ [**14**; p. 107 and p. 106] and both are found, for any n and p, below.

3. It was Veronese, profiting by Cayley's handling of curves in [3] both by projecting them into plane curves and taking plane sections of their scrolls of tangents, who inaugurated the study of algebraic curves in higher space. He immediately introduced [**13**; p. 198] what he called their ranks—orders of manifolds generated by osculating spaces. He expresses [**13**; p. 201] each rank in terms of lower ranks; from these equations three-term recurrence relations between the ranks can be obtained and were used by Segre [**11**; p. 245] when he encountered these ranks in another context. Thus all the ranks can be calculated if the first two are known. But the first rank is merely the order n of C and the second the order $2(n+p-1)$ of the scroll of tangents of C—or the number of points in the Jacobian set of a g_n^1 on a curve of genus p. Hence it is found that the order of the manifold generated by (*cf.* [**12**; p. 389]) the osculating $[k]$'s of C is $(k+1)(n+pk-k)$. It is this result that we wish to use. It is relevant to note that, as these manifolds are generated by linear spaces, the order of each is the same as that of its polar reciprocal in a non-singular quadric.

4. The essential main features of Klein's map can be briefly summarised. Lines in [3] meet or are skew according as points mapping them on Ω are conjugate or not; in the former alternative the join of the two points is on Ω, in the latter it is not. There are also planes lying wholly on Ω; they are of two systems. Points of a plane α map a star of concurrent lines; points of a plane β map a field of coplanar lines. Any two planes of the same system have a single common point. A plane α and a plane β have a common line or are skew according as the vertex of the star and the plane of the field are, or are not, incident ([**2a**; p. 40], [**12**; p. 238]).

5. Three generators g_1, g_2, g_3 of R are mapped by points P_1, P_2, P_3 on C; their plane meets Ω in a conic whose points map the regulus to which g_1, g_2, g_3 belong (other lines of this regulus are not on R). The complementary regulus is mapped by the conic in which the polar plane of $P_1 P_2 P_3$ meets Ω. Consider, in particular, the case in which g_1, g_2, g_3 all coincide with a generator g of R. Let π be the polar plane of the osculating plane ω of C at the point P mapping g. Then the conic in which Ω meets π maps the regulus of inflectional tangents of R at the points of g. Since ω generates a threefold of order $3(n+2p-2)$, so does π; this latter threefold cuts Ω in a surface J of order $6(n+2p-2)$ whose points map the inflectional tangents of R. The surface J contains a singly-infinite set of conics.

The osculating solids of C generate a fourfold of order $4(n+3p-3)$, so that their polar lines generate a scroll Σ of this same order. The osculating solid of C at P contains ω, so that its polar line lies in π, its two intersections with Ω mapping those two members of the regulus of inflectional tangents along g that have 4-point intersection with R. Since Σ meets Ω in a curve Γ of order $8(n+3p-3)$, this is the order of ϕ, the scroll of tangents of R having 4-point intersection. This seems as direct a way as any of finding this order.

Equally direct is the determination of the number of lines having 5-point intersection with R. Each such line is mapped by a point at which the tangent [4] of

Ω is an osculating [4] of C. But the locus of poles of these osculating [4]'s is a curve of order $5(n+4p-4)$ which meets Ω at $10(n+4p-4)$ points; this is, then, the number of 5-point tangents of R. For $p = 0$ this was found by Voss [**14**; p. 108]. He also [**14**; p. 99] offers the correct number when R is the set of lines common to three complexes; but he guardedly makes the offer conditional by disclaiming success in pushing his work to a conclusion through a highly complicated manipulation of determinants. The contrast of the two methods has some interest.

6. The tangent lines to R at the points of a generator g are mapped by the intersection of Ω with the polar solid T of the tangent t to C at that point P which maps g. This intersection is an ordinary quadric cone, with vertex P, each of whose generators maps the pencil of lines through a point of g and lying in the tangent plane to R there. The situation is specialised if t lies on Ω, for T then meets Ω in the two planes α and β which contain t. The points of β map the lines in a plane, and all of these touch R on g; the plane touches R at every point of g which is what Cayley [**5**; p. 334] called a torsal generator; it is known ([**1**; p. 20], [**2b**; p. 26], [**12**; p. 206], [**16**; p. 220]) that there are $2(n+2p-2)$ such g. At any point P of C other than these special ones the tangent [4] to Ω, touching C at P, meets C in $n-2$ further points; $n-2$ chords of C through P lie on Ω. But if P maps a torsal generator these $n-2$ chords include t and the osculating plane of C at P is in the tangent [4] of Ω. There are, however, in general no points of C at which its osculating solid is in the tangent [4] of Ω.

7. Reconsider, now, the 4-point tangents of R. The polar line of the solid osculating C at P meets Ω at A, B which, as P traces C, trace on Ω the curve Γ of order $8(n+3p-3)$. Were either A or B to coincide with P, the osculating solid of C there, and its polar line, both containing P, would also lie in the tangent prime of Ω at P—a circumstance which, it has just been said, will not occur. Hence there is a (1, 2) correspondence between C and Γ, without united points. It follows that the lines PA generate a scroll Ψ of order

$$8(n+3p-3)+2n = 10n+24(p-1);$$

this is clear on setting up a correspondence among the [4]'s through a solid S of general position, [4]'s being in correspondence when they join S to corresponding points of C and Γ. For this correspondence has indices $8(n+3p-3)$ and $2n$ and so, by the elementary Chasles principle [**7**; p. 1175], $10n+24(p-1)$ coincidences. And such a coincidence involves a [4] containing a pair of corresponding points, whose join thus meets S. Conversely: if such a join meets S it affords a coincidence of the correspondence. The scroll Ψ is the complete intersection of Ω with the threefold M generated by the planes PAB, so that the order M is $5n+12(p-1)$.

An arbitrary plane $\tilde{\omega}$ therefore meets $5n+12(p-1)$ planes of M; should $\tilde{\omega}$ be on Ω, so are such intersections, which are then on Ψ; thus Ψ meets a plane on Ω, be this plane α or β, in $5n+12(p-1)$ points—one here presumes the plane to be " of general position " on Ω, and so it will not contain a generator of Ψ. If, say, β meets AP then it is incident with the plane α^* that contains AP so that the plane in [3] whose lines are mapped on Ω by the points of β contains the intersection of the generator of R mapped by P and the 4-point tangent of R mapped by A: i.e. the plane in [3] meets $\mathscr{F}$ whose order is therefore $5n+12(p-1)$.

The dual reasoning, with α instead of β, shows that the tangent planes to R at the points of $\mathscr{F}$ form a developable of class $5n+12(p-1)$.

8. Γ is bisecant to the generators of Σ. The correspondence between the primes through S, wherein primes correspond that join S to points of Γ on the same generator of Σ, is symmetric, with both indices equal to the order of Γ; hence these are, by Chasles' principle, double this number of coincidences in the correspondence. But these include the primes joining S to those generators of Σ which meet it, and indeed twice over, once for each intersection of Γ with such a generator. Hence there are

$$2.8(n+3p-3)-2.4(n+3p-3) = 8(n+3p-3)$$

remaining coincidences; these can only occur when Γ touches a generator of Σ. This result does not depend on the genus π of Γ, but π, as Segre showed [**10**; p. 127], can be found by applying Zeuthen's formula ([**16**; p. 107], [**12**; p. 82]) to the (2, 1) correspondence between Γ and a prime section, of genus p, of Σ. The branch points are the contacts just noted, so that the formula gives

$$8(n+3p-3) = 2(\pi-1)-4(p-1),$$

$$\pi = 4n+12p-13.$$

The planes α through PA and PB map the two points of $\mathscr{F}$ on that generator of R whose map is P. If a generator of Γ touches Σ, A and B coincide and $\mathscr{F}$ touches a generator of R. Voss found the numbers of contacts of $\mathscr{F}$ with generators of his two special scrolls [**14**; p. 91 and p. 106]; both numbers are in accord with $8(n+3p-3)$. In passing he drops a hint [**14**; p. 91] that there may be particular scrolls on which $\mathscr{F}$ touches every generator. He also obtains [**14**; p. 107] the value of π when $p = 0$.

9. The points of $\mathscr{F}$ are linked to the osculating solids of C. Were C itself to lie in a solid the geometry would collapse: for this solid would osculate C at all its points and the only 4-point tangents of R would be its two, possibly coincident, directrices. The simplest scroll having a proper flecnodal curve is the rational quartic scroll R^4, with $\mathscr{F}$ an octavic, of genus 3, touching eight generators. The 4-point tangents generate an octavic scroll ϕ^8; such a tangent cannot meet R^4 elsewhere unless it is itself a generator, and this, as remarked in §6, does not happen. The common curve of R^4 and ϕ^8 is $\mathscr{F}$, reckoned four times

The quartic C^4 lies on the section of Ω by a prime Π, and any curve so situated maps a scroll R whose generators belong to a linear complex Λ. Every osculating solid of such a curve C is in Π, so that its polar line contains the pole O of Π; Σ is a cone with vertex O, and two 4-point tangents of R whose contacts are on the same generator are polars of each other in Λ. Those generators of Σ that touch Ω do so at the intersections of Γ and Π and this makes it, one might say, visually clear that, when R belongs to Λ, the number of generators of R which touch $\mathscr{F}$ is equal to the order of Γ, and so of ϕ.

10. The quadric containing the generators g_1, g_2, g_3 of R intersects that containing g_0, g_1, g_2 in g_1, g_2 and the two transversals of g_0, g_1, g_2, g_3. When g_1 and g_2 are coincident with a generator g these quadrics touch each other along g. If, subsequent to this, one allows g_0 and g_3 to approach g, it would seem that the 4-point tangents of R belong to the envelope of its osculating quadrics—the osculating hyperboloids of Salmon [**9**; p. 425] and Voss [**14**; p. 99]. The envelope will also include R, indeed multiply. This could afford an approach towards finding an equation for ϕ.

Take, by way of example, a scroll R generated by chords of a twisted cubic γ.

Let the parametric form of γ be, as customary,

$$x:y:z:t=\theta^3:\theta^2:\theta:1.$$

One first requires the equation of the quadric containing three chords of γ; a search through standard texts has not disclosed it, but it is, if α_1, β_1; α_2, β_2; α_3, β_3 are the parameters of the pairs joined by the chords,

$$\begin{vmatrix} xz-y^2 & xt-yz & yt-z^2 & -\mathscr{A}\mathscr{B} \\ \alpha_1\beta_1 & \alpha_1+\beta_1 & 1 & (\beta_1-\alpha_2)(\beta_1-\alpha_3)(\alpha_1-\beta_2)(\alpha_1-\beta_3) \\ \alpha_2\beta_2 & \alpha_2+\beta_2 & 1 & (\beta_2-\alpha_3)(\beta_2-\alpha_1)(\alpha_2-\beta_3)(\alpha_2-\beta_1) \\ \alpha_3\beta_3 & \alpha_3+\beta_3 & 1 & (\beta_3-\alpha_1)(\beta_3-\alpha_2)(\alpha_3-\beta_1)(\alpha_3-\beta_2) \end{vmatrix}=0,$$

where

$$\mathscr{A}\equiv x-(\alpha_1+\alpha_2+\alpha_3)y+(\alpha_2\alpha_3+\alpha_3\alpha_1+\alpha_1\alpha_2)z-\alpha_1\alpha_2\alpha_3 t$$

and $\mathscr{B}$ is got by replacing in $\mathscr{A}$ each α_i by the corresponding β_i. This follows because, when x, y, z, t are replaced in the top row of the determinant by

$$\lambda\alpha_i{}^3+\mu\beta_i{}^3,\ \lambda\alpha_i{}^2+\mu\beta_i{}^2,\ \lambda\alpha_i+\mu\beta_i,\ \lambda+\mu,$$

this top row becomes the product of $R_{i+1}(\text{row}_{i+1})$ by $\lambda\mu(\alpha_i-\beta_i)^2$.

The restriction placed on the chord to generate R makes β_i a function of α_i. To bring the chords to coincidence let θ, $\theta+\delta$, $\theta+\varepsilon$ be the parameters α_1, α_2, α_3 and drop the suffix of α_1;

$$\alpha_2=\alpha+\delta\alpha'+\tfrac{1}{2}\delta^2\alpha''+\ldots,\quad \alpha_3=\alpha+\varepsilon\alpha'+\tfrac{1}{2}\varepsilon^2\alpha''+\ldots$$

with exactly corresponding equations for β_2 and β_3 in terms of $\beta(=\beta_1)$ which is also a function of $\theta(=\alpha)$—not one-valued, but a definite chord among generators of R through α has been chosen.

Substitute these values for the six parameters in the determinant, and break off all expansions after terms of the second order in δ and ε. The details are tiresome, but the bottom right-hand entry becomes

$$(\alpha-\beta)^4+(\alpha-\beta)^3\{(\delta+2\varepsilon)(\alpha'-\beta')+(\tfrac{1}{2}\delta^2+\varepsilon^2)(\alpha''-\beta'')\}$$
$$+(\alpha-\beta)^2\{-\delta^2\alpha'\beta'+2\varepsilon\delta(\alpha'^2-\alpha'\beta'+\beta'^2)+\varepsilon^2(\alpha'^2-4\alpha'\beta'+\beta'^2)\};$$

the entry above this is got by merely transposing δ and ε. The entry in the second row will be symmetric in δ and ε, and of course all entries are symmetric in α and β.

The avenue to the limiting form opens after making appropriate linear combinations of the last three rows. For it transpires that the infinitesimals of lowest order in

$$\delta R_4-\varepsilon R_3+(\varepsilon-\delta)R_2$$

are of the third order (this applies to every column); these terms are all multiples of $\varepsilon\delta(\varepsilon-\delta)$, which can then be cancelled along this new fourth row, and δ can likewise be cancelled from the terms of lowest order along a new third row R_3-R_2. So, in the limit, the osculating quadric of R along its generator $\alpha\beta$ is

$$\begin{vmatrix} xz-y^2 & xt-yz & yt-z^2 & -\mathscr{A}^*\mathscr{B}^* \\ p & s & 1 & (\alpha-\beta)^4 \\ p' & s' & 0 & (\alpha-\beta)^3(\alpha'-\beta') \\ \tfrac{1}{2}p'' & \tfrac{1}{2}s'' & 0 & \tfrac{1}{2}(\alpha-\beta)^3(\alpha''-\beta'')-3(\alpha-\beta)^2\alpha'\beta' \end{vmatrix}=0,$$

where
$$\mathscr{A}^* \equiv x-3\alpha y+3\alpha^2 z-\alpha^3 t,$$
$$\mathscr{B}^* \equiv x-3\beta y+3\beta^2 z-\beta^3 t,$$

and s, p are the sum and product of α and β. Since everything is symmetric in α and β, so is the whole determinant, which is, after doubling its bottom row,

$$\begin{vmatrix} xz-y^2 & xt-yz & yt-z^2 & Q \\ p & s & 1 & (s^2-4p)^2 \\ p' & s' & 0 & (s^2-4p)(ss'-2p') \\ p'' & s'' & 0 & (s^2-4p)(ss''-2p'')+2ps'^2-2ss'p'+2p'^2 \end{vmatrix}$$

where

$$Q \equiv -x^2+3sxy-9py^2-3(s^2-2p)zx+s(s^2-3p)xt$$
$$+9spyz-9p^2z^2-3(s^2-2p)pyt+3p^2szt-p^3t^2.$$

Here it would seem profitable to take, as a new fourth column

$$C_4+(s^2-4p)(2C_1-sC_2+2pC_3)$$

and so arrive at the equation for the osculating quadric in the form

$$\begin{vmatrix} xz-y^2 & xt-yz & yt-z^2 & * \\ p & s & 1 & 0 \\ p' & s' & 0 & 0 \\ p'' & s'' & 0 & 2ps'^2-2ss'p'+2p'^2 \end{vmatrix} = 0$$

where

$$* \equiv Q+(s^2-4p)\{2(xz-y^2)-s(xt-yz)+2p(yt-z^2)\}.$$

11. Not every ruled surface contains twisted cubics bisecant to its generators, but the general quartic scroll R^4 has such a cubic γ for its nodal curve [6; p. 1100] and is generated by a symmetric (2, 2) correspondence between the points of γ. Such a correspondence has an equation

$$(p \quad s \quad 1)\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} p \\ s \\ 1 \end{pmatrix} = 0$$

and so permits p and s to be expressed as quadratics in a parameter. The pinch points on γ are those through which the two generators are coincident: these are the torsal generators. Since the condition for the quadratic

$$a\alpha^2\beta^2+b(\alpha+\beta)^2+c+2f(\alpha+\beta)+2g\alpha\beta+2h\alpha\beta(\alpha+\beta) = 0$$

in β to have equal roots is

$$\{h\alpha^2+(b+g)\alpha+f\}^2 = (a\alpha^2+2h\alpha+b)(b\alpha^2+2f\alpha+c),$$

the parameters on γ of the pinch points are the roots of this biquadratic. Choose the parameter on γ so that it takes the values 0, ∞ at two of these four pinch points; then

$$h^2 = ab \text{ and } f^2 = bc,$$

so that one may take

$$a:b:c:h:f = \rho^2 : 1 : \tau^2 : \rho : \tau$$

when the (2, 2) correspondence is

$$\rho^2\alpha^2\beta^2+(\alpha+\beta)^2+\tau^2+2\tau(\alpha+\beta)+2g\alpha\beta+2\rho\alpha\beta(\alpha+\beta) = 0,$$

$$(\rho p+s+\tau)^2 = 2(\rho\tau-g)\,p = \sigma^2 p, \text{ say,}$$

and one may use the parametric forms

$$p = \psi^2, \qquad s = -\rho\psi^2-\sigma\psi-\tau.$$

When these forms of p and s are substituted in the determinant the outcome is a sextic polynomial in ψ with quaternary quadratics for its coefficients; that there are six osculating quadrics of R^4 through an arbitrary point is of course known—a general scroll of order n and genus p has $3(n+2p-2)$ of its inflectional tangents through an arbitrary point. But the explicit equation here yields, by using the discriminant of the sextic polynomial, an equation of order 20 in x, y, z, t. This envelope of the osculating quadrics of R^4 presumably consists of R^4 itself, reckoned thrice, and the octavic scroll ϕ^8 of 4-point tangents.

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