A PERMUTATION REPRESENTATION OF THE GROUP OF THE BITANGENTS

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1. The group Γ of the bitangents has been studied in two recent papers ([3] and [4]). It was represented in [4] as a subgroup of index 2 of the group of symmetries of a regular polytope in Euclidean space of dimension 6, in [3] as the group of automorphisms of a non-singular quadric Q in the finite projective space [6] over F—the Galois Field $GF(2)$. The culmination of [4] is the compilation, for the first time, of the complete table of characters of Γ, and Frame uses this table to suggest possible degrees for permutation representations. Such representations, of degrees 28, 36, 63, 135, 288 are patent once the geometry of Q is known; but Frame, having observed that there is a combination of the characters that satisfies the several conditions known to be necessary, had proposed also 120 as a possible degree. As there is no guarantee that the set of necessary conditions is sufficient, and as no representation of Γ of degree 120 seems yet to have appeared in the literature, a description is here submitted of one that is incorporated with the geometry of Q.

Q consists, as explained in [3], of 63 points m; 315 lines g (all three points on a g being m) lie on Q, while through each g pass three planes d lying wholly on Q (in that all seven points in d are m, and all seven lines in d are g). These three d form the complete intersection of Q with E, the polar [4] of g.

There are, and it is intended to construct them, 120 figures $\mathscr{F}$; each $\mathscr{F}$ includes all 63 m together with 63 d, one d being associated with each m —having m for its *focus* as one may say. Those g in d that pass through its focus may be called *rays*; all three d containing a ray belong to $\mathscr{F}$, their foci being those three m that constitute the ray, so that, there being three rays in each of 63 d, there are 63 rays in $\mathscr{F}$. The plane of any two intersecting rays is on Q, and the third line therein through the intersection is a ray too. None of the 72 d extraneous to $\mathscr{F}$ includes a ray; of those d that pass through a g which is not a ray only one belongs to $\mathscr{F}$, the other two being extraneous to $\mathscr{F}$.

Although such a figure as $\mathscr{F}$ may not have been previously described it has been encountered, so to say, by implication, being obtainable when Q is regarded as a section of a ruled quadric S in [7]; one has then only to take, on S, those points that are autoconjugate (*i.e.* incident with their corresponding solids) in a certain triality. That such points make up a prime section of S is known (see 5.2.2 in [5]), and that there are 63 of

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them accords with putting $\kappa = \lambda = 2$ in 8.2.4 of [5]; 8.2.6 then says that, of 63 m, 32 lie outside the tangent prime T_0 to Q at a given point m_0 while 8.2.5 says that there are 63 rays, or "fixed lines" in Tits' phraseology.

2. Let δ, δ' be any two of the 135 planes on Q that are skew to one another; they span a [5] C and, being skew, belong to opposite systems on $\mathscr{K}$, the Klein section of Q by C.

Through any line g of δ passes another plane on $\mathscr{K}$ which, belonging to the opposite system to δ, is in the same system as δ' and so meets δ' at a point m'; moreover, the points m' so arising from g in δ concurrent at m lie on g', the line of intersection of δ' with the tangent space $[\delta g']$ of $\mathscr{K}$ at m The plane, other than δ', on $\mathscr{K}$ that contains g' is $[mg']$. So there is set up a correlation between δ and δ'; each point of either is correlative to a line of the other.

If m in δ and m' in δ' each lie on the line correlative to the other their join is on $\mathscr{K}$. There are 21 such joins; through each point m of δ there pass three, lying in the plane joining m to its correlative g', and likewise there pass three coplanar joins through each point m' of δ'. Since $\mathscr{K}$ consists of 35 m there are 21, which may be labelled temporarily as points μ, that lie neither in δ nor in δ'; through each μ passes one transversal to δ and δ'; these 21 lines, one through each μ, are the joins mm' of points each on the line correlative to the other.

Through each point on $\mathscr{K}$ pass nine lines lying on $\mathscr{K}$; if m is in δ three of them lie in δ while another three join m to the points on its correlative g'; there remain three others, so that 21 g on $\mathscr{K}$ meet δ in points and are skew to δ'. Another 21 meet δ' in points and are skew to δ. There are also among the 105 g on $\mathscr{K}$ seven in δ, seven in δ', 21 transversal to δ and δ'; there remain 28, which may be labelled g^*, skew to both δ and δ'. These 28 g^* may be identified as follows. Take any g in δ; the solid that joins it to any g' through its correlative m' in δ' meets $\mathscr{K}$ in two planes through mm', m being that point on g to which g' is correlative. But there are four lines g' in δ' that do not contain m'; then the solid $[gg']$ meets $\mathscr{K}$ in a hyperboloid whereon the regulus that includes g and g' is completed by g^*. As there are seven g in δ, and four g' in δ' not containing the correlative m, the 28 g^* are accounted for.

There being three μ on each g^*, but only 21 μ in all, one expects there to be four g^* through each μ; this is so. For let the transversal from μ to δ, δ' meet δ in m, δ' in m'; through m, and in δ, are lines g_1, g_2 other than the correlative g to m'; through m', and in δ', are lines g_1', g_2' other than the correlative g' to m; each solid

$$[g_1 g_1'], \quad [g_1 g_2'], \quad [g_2 g_1'], \quad [g_2 g_2']$$

meets $\mathscr{K}$ in a hyperboloid whereon a regulus is completed by a g^* through μ.

3. Take, now, one of these g^*: the transversals from its three μ to δ, δ' form a regulus whose complement includes g in δ and g' in δ', neither g nor g' being correlative to any point on the other. The correlative m in δ of g' is conjugate to every point of g and, by the defining property of the correlation, to every point of g'; so, likewise, is the correlative m' in δ' of g. Hence the polar plane j_0 ([3], §6) of $[gg']$ with respect to Q contains both m and m'; there is one remaining point m^* of Q in j_0, and it lies outside C—for to suppose that it belonged to C would put the whole of j_0 in C, whereas the kernel of Q, which is in j_0, is outside C. Now there are $63-35=28$ points m^* on Q that are not on $\mathscr{K}$; thus each m^* is linked to a g^*, and m^*g^* is a plane d on Q.

There are three planes on Q through any line thereon; if this line is a transversal $m\mu m'$ from one of the 21 μ to δ and δ' two of these planes are on $\mathscr{K}$, while the third contains a quadrangle $m_1^* m_2^* m_3^* m_4^*$ with its diagonal points at m, μ, m'. The tangent prime to Q at any vertex of this quadrangle contains $m\mu m'$ and meets δ, δ' in lines belonging to a regulus completed by g^* through μ. Thus four concurrent g^* are linked with coplanar m^* whose plane, containing the transversal to δ and δ' from the point of concurrence, lies on Q but not on $\mathscr{K}$.

4. Choose now, from among the 315 g on Q, the 21 transversals of δ, δ' and those, three through each m^*, that join m^* to those μ on the g^* that is linked with it. Each such join contains two m^*, the g^* that are linked therewith both passing through μ; hence, under this second heading, the number of g selected is $\frac{1}{2}(28\times 3)=42$. So 63 g are chosen: call them rays. Through each m on Q pass three rays, and they are coplanar. If m is m^* this is manifest from the prescription of choice, as it is too if m is in δ or δ'. If m is μ the rays are, say, $m_1^*\mu m_2^*$, $m_3^*\mu m_4^*$, $m\mu m'$ and lie in that d through $m\mu m'$ that is not on $\mathscr{K}$. So 63 d are chosen from among the 135 on Q; each contains three concurrent rays. Call the m wherein the rays concur the focus of d.

Through any g there pass three d; if g is a ray these d are those having the m on the ray for foci. The points of d other than its focus m are foci of those other d which belong to $\mathscr{F}$ and contain m; if d, d' in $\mathscr{F}$ are such that the focus of d' is in d then the focus of d is in d'. Whenever two rays meet the third line through their intersection and lying in their plane is a ray too. It is these 63 d, with the 63 rays and foci, that constitute the figure $\mathscr{F}$.

Each d in $\mathscr{F}$ contains, as well as three concurrent rays, a quadrilateral of g that are not rays; thus, by four in each of 63 d, the $315-63=252$ g that are not rays are accounted for. Through each such g pass two planes on Q in addition to d, but they are extraneous to $\mathscr{F}$. The $135-63=72$ extraneous planes may be labelled δ; the planes above denominated by δ and δ' are in this category. No g in δ is a ray and only one of the planes on Q that pass through it belongs to $\mathscr{F}$ whereas, were g a ray, all three would do so.

5. Label the m in any of the 72 δ by

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7: \qquad \text{I}$$

they lie on g that can be taken as

$$156,\ 246,\ 345,\ 147,\ 257,\ 367,\ 123. \qquad \text{II}$$

Through each such g there is a single d belonging to $\mathscr{F}$; label the foci of these d, none of which can lie in δ, respectively

$$1',\ 2',\ 3',\ 4',\ 5',\ 6',\ 7'. \qquad \text{I}'$$

Then those d whose foci are in δ join its points to the respective triads

$$1'4'7',\ 2'5'7',\ 3'6'7',\ 2'3'4',\ 1'3'5',\ 1'2'6',\ 4'5'6'. \qquad \text{II}'$$

Thus the join of every pair of points I′ is on Q and, there being no solid on Q, the points I′ lie in a plane δ' whose lines consist of the triads II′.

Each of the 72 δ has, it is now clear, a twin δ' coupled with it by $\mathscr{F}$. The correlation between δ and δ' is shown by I and II′ or, alternatively, by I′ and II. Those d that pass one through each line of δ' have for their foci the points of δ correlative to these lines; if d passes, say, through $1'3'5'$ its focus is the point 5 common to those d whose foci are $1'$, $3'$, $5'$.

Since, by the construction in §4, δ and δ' determine $\mathscr{F}$ uniquely there are $x/36$ figures $\mathscr{F}$ where x is the number of pairs of skew planes on Q. To calculate x note, in the first place (using d now to signify a plane on Q whether it be in $\mathscr{F}$ or extraneous thereto), that each d is met in lines by 14 others, two passing through each g in d. Note next, to ascertain how many d meet a given d_0 in points only, that the 15 d through a point m of d_0 project, from m, the figure of 15 g in [4] passing three by three through 15 points ([2], §§13-15). Since that one of these 15 g that lies in d_0 meets six others among these g it is skew to eight whose joins to m therefore meet d_0 at m only; hence there are, through any of the seven m in d_0, eight d that meet d_0 only at m. Wherefore the number of d skew to d_0 is

$$135-1-14-56=64$$

and there are $\frac{1}{2}(135\times 64)/36 = 120$ figures $\mathscr{F}$. They afford a permutation representation, of degree 120, of the group of the bitangents.

6. The 120 $\mathscr{F}$ are permuted transitively by Γ. For the C are certainly so permuted, each having a stabiliser isomorphic to the symmetric group $\mathscr{S}_8$, and the transitivity of Γ on the $\mathscr{F}$ will follow from that of this stabiliser on pairs of skew planes on the section $\mathscr{K}$ of Q by C. If $\mathscr{K}$ is regarded as mapping the lines of a solid Σ the stabiliser of C in Γ is put in isomorphism

with the group of collineations and correlations in Σ (*cf.* [**1**], §§18, 19; Γ is there used to denote the group of $\frac{1}{2}.8!$ collineations isomorphic to $\mathscr{A}_8$). The required transitivity is consequent upon that of the whole group in Σ on non-incident points and planes.

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