31-POINT GEOMETRY

By W. L. Edge.

The following paragraphs have been assembled in consequence of my reading Dr. Cundy's note on 25-point geometry.* Towards the end of it, apparently mindful of the adjunction of a " line at infinity " to the Euclidean plane, he adjoins a line to the 25-point plane and so obtains a geometry of 31 points. Here I reverse this procedure : I start with the 31-point geometry and thereafter assign to one of its 31 lines the rôle of the " line at infinity ". This seems more in the spirit of Cayley's dictum at the end of his Sixth Memoir on Quantics, that " descriptive geometry is *all* geometry " and metrical geometry only a part thereof.

Finite geometries arise whenever coordinates are confined to a finite field. Any such field consists of p^n " marks ", p being a prime number, and the resulting plane geometry consists of $p^{2n}+p^n+1$ points. If $p^n=5$ we obtain the 31-point geometry and in it, as we shall see, harmonic properties are prominent.

1. The integers, positive, negative and zero, may be distributed among five residue classes

$$0, 1, -1, 2, -2$$

to modulus 5. Each integer belongs to one and only one of the classes, which are added and multiplied commutatively according to the modulus 5 so that, for example

$$1+2=-2, \qquad (-2)(-2)=-1.$$

Moreover each non-zero class has a unique inverse :

$$1 \cdot 1 = -1 \cdot -1 = 2 \cdot -2 = -2 \cdot 2 = 1.$$

The classes indeed form a *field* F, and we call them the *marks* of F. In F 1 and -1 each have two square roots, those of 1 being 1 and -1 and those of -1 being 2 and -2. But 2 and -2 have no square roots in F.

If x_1 and x_2 are two marks we can, save when $x_1=x_2=0$, regard them as homogeneous coordinates of a point of a line L. The point is unaltered if both x_1 and x_2 are multiplied by the same non-zero mark, so that L consists of $24 \div 4 = 6$ points. Whenever $x_2=0$ we label the point by the mark $x=x_1/x_2$; when $x_2=0$ we use the sixth mark ∞.

There are projectivities

$$axx'+bx+cx'+d=0 \qquad \text{(1.1)}$$

among the points of L, such a projectivity being uniquely determined when three corresponding pairs of points x, x' are assigned. The number of projectivities is therefore ${}^6P_3=120$; they form a group, triply transitive on the points of L and subjecting them to 120 permutations. It is not, in general, possible to fit a projectivity on to four pairs, but we do have, for any four distinct points A, B, C, D,

$$ABCD \barwedge DCBA \barwedge CDAB \barwedge BADC. \qquad \text{(1.2)}$$

It is of course understood that the coefficients a, b, c, d are all marks of F.

2. Any four points of L have cross ratios, and cross ratio is invariant under projective transformation. But since every cross ratio of four among

$$0, 1, -1, 2, -2, \infty$$

is itself among these six marks the cross ratios of four distinct points on L can only be -1, 2, -2 ; the other marks 0, 1, ∞ occur as cross ratios only when two of the four points coincide.

* *Math. Gazette* 36 (1952) 158–166.

The transposition of any pair of the four points cannot, without the simultaneous transposition of the complementary pair, be brought about by a projectivity; it permutes the three cross ratios and, being of period 2, must transpose two of them and leave the third unchanged. The transposition of the complementary pair of points imposes the same permutation of the cross ratios. There are three ways of separating the four points into complementary pairs and each of the three cross ratios is unchanged by the transposition of either pair for one of the three separations; the separation for which -1 is the invariant cross ratio is *harmonic separation*, and *every set of four distinct points on L admits a harmonic separation.* This is surely the most significant feature of the geometry. We say, as usual, that either pair is harmonically conjugate to the other, and the condition for A, B and C, D to be harmonically conjugate is

$$(x_A + x_B)(x_C + x_D) = 2(x_A x_B + x_C x_D). \quad\text{(2.1)}$$

We may also say that A and B are harmonic inverses of one another in C and D, or that A and B are a pair of the involution whose foci are C and D. If harmonically conjugate pairs on L are given by quadratics

$$a_1x^2 + 2h_1x + b_1 = 0, \qquad a_2x^2 + 2h_2x + b_2 = 0$$

then

$$a_1b_2 + a_2b_1 = 2h_1h_2, \quad\text{(2.2)}$$

it being understood that ∞ is a root of a " quadratic " wherein x^2 has coefficient zero. Any quadratic whose roots are both marks of F has its coefficients in F too although, as $x^2 = 2$ exemplifies, the converse is not true. We cannot, for instance, as we could in the field of complex numbers, assert that two pairs determine a unique pair harmonic to them both: an involution need not have real foci.

The projectivity 1.1 is an involution when $b = c$, and the foci of

$$axx' + b(x + x') + d = 0$$

are the roots of

$$ax^2 + 2bx + d = 0 \quad (f)$$

These belong with a, b, d to F whenever $b^2 - ad$ is a square in F; but should $b^2 - ad$ be either of the non-squares ± 2 the foci are " conjugate imaginaries ". In this event neither a nor d can be zero, for this would involve the contradiction that the square of b was a non-square; we may therefore suppose, without affecting the roots of (f), that $a = 1$ and

$$d = b^2 \pm 2.$$

The mark b can be any one of the five; once b has been chosen there are two values for d, and so there are ten involutions whose foci are not " real ". On the other hand any two points of L determine an involution of which they are the foci, so that there are fifteen involutions whose foci are " real ". These are the harmonic inversions, each point of L having, by 2.1, a unique harmonic conjugate in regard to any two given points.

3. Take any pair A, B of points on L; the remaining four points are thereby separated into pairs C, D and E, F each of which is harmonic to A, B. Not only so: C, D and E, F are harmonic to each other. For the harmonic conjugate of C in regard to E and F is distinct from C, E, F and cannot be A or B each of which has the other for its harmonic conjugate; it must therefore be D. Each of the fifteen pairs of points induces such a separation of the six into three mutually harmonic pairs so that there are $15 \div 3 = 5$ such *sextuples*, the term sextuple implying the pairing and harmonic separations. A sextuple is indeed what Sylvester called a syntheme, but only five of the fifteen synthemes are sextuples. Since every pair determines one and only one sextuple

no two sextuples have a pair in common; the five sextuples together constitute what Sylvester called a synthematic total. It is one, T call it, of the six such totals that arise from the six points. Any projectivity on L leaves invariant cross ratio and the harmonic relation, and therefore T; and the fact that no projectivity can change T into any other of the six totals accords with the fact that only one sixth of the 6! permutations of the points of L can be achieved by projectivities. We now display the sextuples of T, with the quadratics whose roots are the mutually harmonic pairs; any two quadratics in the same row satisfy 2.2. Once a sextuple is given the others are obtainable from it by projectivities; indeed by relations $x'=x+m$ where m belongs to F. Thus the remaining sextuples of T follow at once from the one in the top row.

$\infty, \ 0$	$1, -1$	$2, -2$	x	x^2-1	x^2+1
$\infty, \ 1$	$2, \ 0$	$-2, -1$	$x-1$	x^2-2x	x^2-2x+2
$\infty, \ 2$	$-2, \ 1$	$-1, \ 0$	$x-2$	x^2+x-2	x^2+x
$\infty, -2$	$-1, \ 2$	$0, \ 1$	$x+2$	x^2-x-2	x^2-x
$\infty, -1$	$0, -2$	$1, \ 2$	$x+1$	x^2+2x	x^2+2x+2

4. If A, B; C, D; E, F is a sextuple then

$$ABED \barwedge ABFC \barwedge BACF, \quad \text{.............................}(4.1).$$

the first projectivity being harmonic inversion in A and B and the second the simultaneous transposition of complementary pairs in accordance with 1.2 This shows that AB, CE, DF are in involution, and we prove in this manner that the three pairs of any syntheme extraneous to T are in involution. These are the ten involutions with " conjugate imaginary " foci.

These ten involutions may also be obtained as follows. Take any three distinct points X, Y, Z on L and let

X' be the harmonic inverse of X in Y and Z,
Y' be the harmonic inverse of Y in Z and X,
Z' be the harmonic inverse of Z in X and Y.

All the points of L are hereby accounted for. Now the sextuple determined by the pair YZ must be

$$Y, Z \qquad X, X' \qquad Y', Z'$$

so that X is the harmonic inverse of X' in Y' and Z'.
Likewise Y is the harmonic inverse of Y' in Z' and X'
and Z is the harmonic inverse of Z' in X' and Y'.

The complementary triads XYZ and $X'Y'Z'$ are similarly related, and every pair of complementary triads bear this relation to each other. Moreover, just as in 4.1,

$$XX'Y'Z \barwedge XX'Z'Y \barwedge X'XYZ'$$

so that XX', YY', ZZ' are in involution. The foci of this involution cannot be among the points of L since all six are accounted for by the three pairs, and the number of such involutions is ten, the number of pairs of complementary triads. In this notation the synthematic total of sextuples is

YZ	XX'	$Y'Z'$
ZX	YY'	$Z'X'$
XY	ZZ'	$X'Y'$
XY'	YZ'	ZX'
XZ'	YX'	ZY'

and the ten synthemes extraneous to it are each composed of three pairs in involution.

This geometry, wherein a line consists of six points that can be separated in five ways into mutually harmonic pairs while each of the other ten synthemes yields pairs in involution, is very briefly, though quite explicitly, alluded to by Fano in his paper " Sui postulati fondamentali della geometria proiettiva " in volume 30 of *Giornale di Matematiche* (1892) ; see therein page 123.

5. Geometry can be based on F for a space of any number of dimensions. In a plane the number of points is the number, 5^3-1, of triplets (x_1, x_2, x_3) of marks not all of which are zero, divided by the number, $5-1$, of non-zero marks. This yields a 31-point geometry ; there are 31 points lying 6 on each of 31 lines, each point lying on 6 lines. Similarly we can set up a three dimensional geometry of 156 points, and so on. In all these spaces every set of four distinct collinear points admits a harmonic separation. These geometries are all self-dual. In the plane for example a line answers to a triplet (u_1, u_2, u_3) of marks which are not all zero, and line and point are incident whenever $u_1x_1+u_2x_2+u_3x_3=0$.

6. We now consider the geometry in such a plane Π, and first obtain the numbers of polygons. In this context the use of such a word as triangle, — —, hexagon implies automatically that no three vertices are collinear.

There are 25 points not on the join of two given points and 16 not on any side of a given triangle ; hence the number of quadrangles is

$$31 \,.\, 30 \,.\, 25 \,.\, 16 \div 4! = 15500.$$

We can commence the construction of a pentagon by taking one of the quadrangles, whose four vertices and three diagonal points are then not eligible for the remaining vertex of the pentagon. Neither are any of the other three points on any of the six sides of the quadrangle (each side contains two vertices and one diagonal point). The number of points which are eligible is therefore

$$31-4-3-6 \,.\, 3=6$$

and so the number of pentagons is $15500 \times 6 \div 5 = 18600$.

If a pentagon is to be amplified, by choosing a further vertex, to a hexagon we are debarred from choosing any of its five vertices, fifteen diagonal points or any of the other points one on each of the ten sides (each side containing two vertices and three diagonal points). The number of eligible vertices is therefore

$$31-5-15-10=1$$

so that the number of hexagons is $18600 \times 1 \div 6 = 3100$.

7. Let α, β be any two vertices of a hexagon $h \equiv \alpha\beta\gamma\delta\epsilon\zeta$. There are four other points on $\alpha\beta$ and the six sides of the quadrangle $\gamma\delta\epsilon\zeta$ must each pass through one of them. Hence $\alpha\beta$ contains two diagonal points of $\gamma\delta\epsilon\zeta$, say Q, the intersection of $\delta\epsilon$ and $\gamma\zeta$, and R, the intersection of $\gamma\epsilon$ and $\delta\zeta$. The intersections Y, Z of QR with $\epsilon\zeta$ and $\gamma\delta$ are harmonic to Q, R and α, β complete the sextuple. Thus on each *side* s of h, and constituting a sextuple thereon, are two *vertices* V, two *diagonal points* D and two *Brianchon points* B, these last being points of concurrence of three s that join the six V in pairs. Since there are fifteen s, two through each D and three through each B, h has in all fifteen D and ten B which, together with its six V, account for all the points of Π.

Of the six lines through any vertex V_0 five are sides s ; they account for all ten B, two of which are on each s, and all the other V, but only for ten of the fifteen D. The remaining five D therefore lie on the one remaining line through V_0. Such a line, consisting of one V and five D, we call a *tangent* t

all these names are of course related to h and a point or line is differently named in relation to different hexagons. There are six t, one associated with each V, forming a hexagram H. Any two of them meet in a D and the fifteen D are the intersections of the pairs of t just as the fifteen s are the joins of the pairs of V.

We have now seen that when two V α, β have been chosen two of the three separations of the other four into complementary pairs yield pairs of s whose intersection is a B on $\alpha\beta$. The third pairing however does not: $\alpha\beta$, $\gamma\delta$, $\epsilon\zeta$ are not concurrent but form a triangle XYZ each of whose vertices is a D. Each side of XYZ determines the other two, just as $\alpha\beta$ above determined $\gamma\delta$ and $\epsilon\zeta$, and each s determines one and only one such triangle. Hence there are five of these triangles: they answer to the five synthemes of V in one synthematic total. The vertices of the triangles account for all the D, their sides for all the s.

8. Take any D, for example the intersection X of $\gamma\delta$, $\epsilon\zeta$. Through it pass the s XY, XZ each containing two V, two B and one D other than X. There are also two t, $X\alpha$ and $X\beta$, each containing four D other than X and one V. These two pairs of lines are harmonic. The third pair XQ, XR of the sextuple has to account for the remaining four D and six B. Neither line contains any V and we call this third type of line a Pascal line p; each p contains three D (hence its name) and three B, and since two p pass through each D there are ten p altogether. The three B, one on each of $\alpha\beta$, $\gamma\zeta$, $\delta\epsilon$ apart from their point of concurrence Q, are collinear for they are, respectively, R, the harmonic conjugate of Q in regard to γ and ζ, and the harmonic conjugate of Q in regard to δ and ϵ; the two latter points lie, by the harmonic property of $\gamma\delta\epsilon\zeta$, on RX, which is a p. The ten p are thus associated one with each B.

Since there are ten p with three of the ten B on each, each B lies on three p. Indeed the p could have been thus obtained. For through Q there pass three s which together account for six V, six D and three B other than Q. None of the three remaining lines through Q can contain a V and they have together to account for six B (other than Q) and nine D.

If we delete the three p through Q and the one, q call it for the moment, that is associated with Q there remain six p passing two through each of the three B on q. Thus the ten p and ten B form a Desargues figure.

9. Since each of the 3100 hexagons distributes the 31 lines of π as six t, ten p and fifteen s it follows that any given line L of Π is a tangent of 600, a Pascal line of 1000 and a side of 1500 hexagons.

10. We may define a conic Σ in Π as the set of intersections of corresponding rays of two related pencils, with the proviso that the join $\alpha\beta$ of their vertices is not self-corresponding. The pencils can be related in ${}^6P_3 = 120$ ways, the relation being determined when to three fixed lines through α correspond three distinct lines through β. But in ${}^5P_2 = 20$ of these relations $\alpha\beta$ and $\beta\alpha$ correspond, so that there are only 100 conics through α and β. Likewise there are ${}^5P_2 - {}^4P_1 = 16$ conics circumscribing a triangle and ${}^4P_1 - 1 = 3$ circumscribing a quadrangle. There is a unique conic circumscribing a pentagon. Indeed a conic consists, as a locus, simply of the V of a hexagon h; the t are *tangents* of Σ and the s are *chords* of Σ. For of the six lines through α the one which, in the pencil with vertex α, corresponds to $\beta\alpha$ does not meet Σ elsewhere while each of the other five meets Σ once elsewhere, namely at its intersection with the corresponding line through β. The fifteen D, each the intersection of two t, are *external* to Σ and are the *poles* of the fifteen s; the D and s are the vertices and sides of five *self-polar triangles*. None of the ten B can lie on any t; they are *internal* to Σ. A second kind of self-polar triangle, of which there are fifteen, has one vertex external and the other two both internal to Σ.

11. There are on Σ involutions and harmonic pairs, sections of involutions

and harmonic pairs of lines of any pencil whose vertex is on Σ. Thus any four points α, β, γ, δ of Σ admit one separation, say $\alpha\beta$, $\gamma\delta$ into harmonic pairs. Each of $\alpha\beta$, $\gamma\delta$ passes through the pole of the other since

$$\alpha(\alpha\beta\gamma\delta) \barwedge \beta(\alpha\beta\gamma\delta) \barwedge \beta(\beta\alpha\gamma\delta),$$

a projectivity between the pencils with vertices α, β in which $\alpha\beta$ is self-corresponding, so that γ, δ are collinear with the pole of $\alpha\beta$. The chords $\alpha\beta$, $\gamma\delta$ are *conjugate*, and $\epsilon\zeta$ is conjugate to both of them; the three chords form a self-polar triangle whose vertices are all D. We see again how the five triangles of this kind answer to the five synthemes of V in one synthematic total.

The involution whose foci are α and β has γ, δ and ϵ, ζ as two pairs; it is cut on Σ by the lines through the pole X of $\alpha\beta$, although the two p through X do not meet Σ. There are also involutions centred at the points B. For example, in the figure already used for h,

$$\alpha(\alpha\beta\gamma\delta) \barwedge XZ\gamma\delta \barwedge XY\epsilon\zeta \barwedge \alpha(\alpha\beta\epsilon\zeta),$$

the projection from XZ on to XY being from R, so that, on Σ,

$$\alpha\beta\gamma\delta \barwedge \beta\alpha\zeta\epsilon$$

and the concurrent chords $\alpha\beta$, $\gamma\zeta$, $\delta\epsilon$ do give the pairs of an involution. There are ten involutions of this kind on Σ, one centred at each B.

12. Pascal's Theorem has to hold for Σ since it is a consequence of the projective generation. But while six points of general position on a conic give rise, in projective geometry over the field of complex numbers, to sixty Pascal lines, here only the ten p are eligible. It was Veronese who discovered that the general figure of sixty Pascal lines is composed of six Desargues figures; here a single Desargues figure serves six times over. That each p arises for six different orderings of the V is clear. For let the V be separated into two complementary triads in any one of the ten possible ways, and use the notation of § 4; the six orderings

$$\begin{array}{lll} XX'YY'ZZ' & XY'YZ'ZX' & XZ'YX'ZY' \\ XYZX'Y'Z' & YZXY'Z'X' & ZXYZ'X'Y' \end{array}$$

are found all to yield the same p, opposite sides of any one of these hexagons always intersecting on this p.

13. It scarcely needs saying that the statements dual to all those in §§ 6–12 also hold, the geometry in Π being self-dual. Σ may be regarded not only as the assemblage of its six points but also as the assemblage of its six tangents. It can be generated by a projective relation between any two of its tangents in which their intersection does not correspond to itself.

14. Choose now any one, L say, of the 31 lines in Π as absolute line. Any two lines which meet on L we call parallel, and the remaining 30 lines are distributed in six parallel sets of five. We have a geometry, when L is omitted, of 25 points and 30 lines, six lines passing through each point and five points being on each line, and the completely symmetrical duality so evident in Π is now destroyed. The mid-point of $\alpha\beta$ is the harmonic conjugate, in regard to α and β, of the intersection of $\alpha\beta$ with L; given any three collinear points (none of them on L) some one of them bisects the join of the other two.

Those 600 conics of which L is a tangent are parabolas; they fall into six families of 100, all of any one family having parallel axes.

Those 1500 conics of which L is a chord are hyperbolas; they fall into fifteen families of 100, all of any one family having parallel asymptotes. The four points on any hyperbola are vertices of a parallelogram whose diagonals are diameters, meeting at the centre of the curve. Of the 100 members of any family four have their centre at any one of the 25 points not on L; this is merely equivalent to saying that, given any two points α, β on L and a point C not on L there are four projectivities between the pencils whose vertices are α, β in which $\alpha\beta$, αC correspond respectively to βC, $\beta\alpha$.

Those 1000 conics of which L is a Pascal line are ellipses; they may be separated into ten families of 100 by using the ten involutions on L (cf. § 4) with conjugate imaginary foci. The B which is the pole of L is the centre of the ellipse, whose six points lie two on each of three diameters.

15. Let us now take any one of these ten involutions and decree that its foci, I and J, be the absolute points. Their coordinates are not marks of F so that they do not belong to Π any more than Poncelet's points belong to the real Euclidean plane; but they may serve as absolute points just as Poncelet's do. Let the pairing of points on L in the involution, be

$$X, X'; \quad Y, Y'; \quad Z, Z';$$

each pair is harmonic to I and J. And now there are six directions distributed as three perpendicular pairs as Dr Cundy so perspicuously describes. The hyperbolas of three of the fifteen families are now rectangular and the ellipses of one of the ten families are now circles.

16. We add an instance or two of the use of coordinates. Recall that, as laid down in § 1,

(i) All coordinates x, y, z are marks of F and any marks serve so long as all three coordinates are not zero simultaneously,

(ii) (x, y, z) and (mx, my, mz) are the same point for any non-zero mark m.

If a conic circumscribes the triangle of reference its equation is

$$fyz + gzx + hxy = 0$$

where f, g, h all belong to F and, if the conic is presumed non-degenerate, $fgh \neq 0$. We may therefore take $f = 1$ without affecting the conic; there remain four choices for each of g and h so that, as the projective generation made clear in § 10, there are sixteen conics circumscribing a triangle.

The six points on $yz + zx + xy = 0$ are

$$\begin{array}{ccc} 1, 0, 0 & 0, 1, 0 & 0, 0, 1 \\ 2, 1, 1 & 1, 2, 1 & 1, 1, 2. \end{array}$$

When arranged in this manner in three pairs their joins are concurrent at (1, 1, 1), which is therefore a Brianchon point. The fifteen D are the intersections of pairs of the six tangents

$$\begin{array}{ccc} y+z=0 & z+x=0 & x+y=0 \\ y+z=x & z+x=y & x+y=z. \end{array}$$

Three of these D lie on $x+y+z=0$, which is a Pascal line.

A similar discussion shows that there are sixteen non-degenerate conics for which a given triangle is self-polar.

The six points on $x^2 + y^2 + z^2 = 0$ are those whose coordinates are permutations of (0, 1, 2). The fifteen D, arranged as vertices of five self-polar triangles, are

$$\begin{array}{ccccc} 1, 0, 0 & 2, 1, 1 & 1, 2, 2 & 2, -1, 1 & 2, 1, -1 \\ 0, 1, 0 & 1, 2, 1 & -1, 2, 1 & 2, \ \ 1, 2 & 1, 2, -1 \\ 0, 0, 1 & 1, 1, 2 & -1, 1, 2 & 1, -1, 2 & 2, 2, \ \ 1 \end{array}$$

and the ten B are

0, 1, 1	1, 0, 1	1, 1, 0	−1, 1, 1	1, −1, 1
0, 1, −1	−1, 0. 1	1, −1, 0	1, 1, −1	1, 1, 1.

$x+y+z=0$ is a Pascal line for this conic too, containing

three B:	0, 1, −1	−1, 0, 1	1, −1, 0
and three D:	1, 2, 2	2, 1, 2	2, 2, 1

but for the former conic $yz+zx+xy=0$ the roles of these six points are reversed, the three former then being D and the three latter B. But if we choose $x+y+z=0$ as absolute line both conics are ellipses with centre (1, 1, 1).

17. We are able to retain symmetry in the three coordinates by selecting I and J to satisfy

$$x^2+y^2+z^2=x+y+z=0,$$

whereupon the above two ellipses become concentric circles. Indeed all the conics

$$x^2+y^2+z^2=k(x+y+z)^2$$

with k a mark of F are circles with centre (1, 1, 1) except when $k=2$; the discriminant then vanishes and

$$x^2+y^2+z^2-yz-zx-xy=0$$

is a point-circle. This equation may, among other forms, be written

$$(x+2y+2z)^2+2(y-z)^2=0,$$

and since 2 is not a square both linear forms appearing here must be zero simultaneously for the equation to hold.

More generally: the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

cannot be a circle unless

$$a+2f=b+2g=c+2h,$$

when its equation is

$$a(x^2-yz)+b(y^2-zx)+c(z^2-xy)+d(yz+zx+xy)=0$$

and represents a circle provided that its discriminant is not zero. The coordinates of the centre are

$$2a+b+c+2d, \qquad a+2b+c+2d, \qquad a+b+2c+2d.$$

Thus the circles with centre (A, B, C) have equations

$$(2A+B+C)(x^2-yz)+(A+2B+C)(y^2-zx)+(A+B+2C)(z^2-xy)$$
$$+d(2x^2+2y^2+2z^2-yz-zx-xy)=0.$$

The discriminant of this quadratic form is

$$\Delta \equiv (A+B+C)\{(A+B+C)d+A^2+B^2+C^2-BC-CA-AB\}.$$

It is implied that $A+B+C$ is not zero, and the four genuine circles occur for those four marks d for which $\Delta \neq 0$.

Notes and References.

1. Sylvester introduces synthemes of six objects in his paper " Elementary Researches in the Analysis of Combinatorial Aggregation "; this is on p. 91 of Vol. I of his *Mathematical Papers* and he gives an etymology for his neologism in a footnote on this page. On p. 92 he displays a synthematic total: i.e. a set of five synthemes of which the pairs, three in each syntheme, together

exhaust all ${}^6C_2 = 15$ pairs. Given any one syntheme two totals can be built to include it, wherefore the number of distinct totals is $15 \times 2 \div 5 = 6$.

2. When we adjoin to F either root of any one of those ten quadratics (f) whose discriminant is a non-square we thereby generate a larger field Φ of twenty-five marks. If the roots of $x^2 = 2$ are $\pm j$ each of the ten quadratics has a pair of roots $A \pm Bj$ where A is a mark and B a non-zero mark of F. These twenty roots, with the five marks of F, constitute Φ. Just as there are primitive marks, namely ± 2, in F of which all non-zero marks are powers, so in Φ; the first twenty-four powers of, for example, $2 - j$ yield all the non-zero marks of Φ. The one-dimensional geometry based on Φ consists of the six points of L and the pairs of foci of the ten involutions, 26 points in all.

3. Veronese's long account of the Hexagrammum Mysticum of Pascal is in vol. 1 of the third series of Memoirs of the *Atti della Reale Accademia dei Lincei* (1877); pp. 649–702. The separation into six Desargues figures is on p. 661. It was on reading the manuscript of this memoir before its publication that Cremona realised how to obtain the whole Pascal figure by projection from a node of a cubic surface. Concerning these matters see H. W. Richmond: *Mathematische Annalen* 53 (1899), 161–176.

4. Cayley's Sixth Memoir on Quantics was published in vol. 149 of the *Philosophical Transactions* in 1859 and occupies pp. 561–592 of vol. II of his *Collected Mathematical Papers*. It does not presume a knowledge of its five predecessors and should be read by every mathematician. It is a pleasure, as well as an education, to read it and Forsyth says, in his obituary notice of Cayley, that it could not be presented in more attractive form.

The choice as absolute of a pair of points was made in § 15 in order to derive the geometry described by Dr. Cundy. But it is clear from the Memoir that we might equally well have chosen as absolute any of the 3100 conics and so obtained a geometry that is "non-euclidean" but in which the duality between points and lines is still symmetrical.

5. Given a quadrangle in Π there is a unique projectivity transforming its vertices into those of the same or any other quadrangle taken in any prescribed order. The group G of projectivities in Π is therefore of order 372000, 4! times the number of quadrangles. G is transitive on the lines and on the conics of Π.

Any non-singular three-rowed matrix M imposes, whenever its nine elements all belong to F, one of these projectivities, but the same projectivity is imposed by all four matrices $\pm M$, $\pm 2M$. The number of non-singular matrices is therefore 1488000, four times the number of projectivities; these matrices form the general linear homogeneous group (on three variables, over F) whose order is given in the classical treatises, for instance on p. 97 of C. Jordan's *Traité des substitutions* (Paris 1870) and on p. 77 of L.E. Dickson's *Linear Groups* (Leipzig 1901). From these sources we derive the order 1488000 in the form $(5^3-1)(5^3-5)(5^3-5^2)$. Since the multiplication of M by a mark m of F causes the determinant $|M|$ to be multiplied by m^3 we can set up a(1, 1) correspondence between matrices and the projectivities which they impose by stipulating that $|M|$ is always 1, and we thus obtain G as the special linear homogeneous group of order $(5^3-1)(5^3-5)5^2$.

Any conic Σ is invariant for $372000 \div 3100 = 120$ projectivities of G. Since, by choice of the triangle of reference, Σ has the equation $x^2+y^2+z^2=0$, the 120 unimodular matrices constitute the orthogonal group R (in three variables and over F). This is of the same order as the group, encountered in § 1, of projectivities on L; indeed the two groups are not merely of the same order but are isomorphic, both of them being symmetric groups of degree 5. The projectivities on L permute five of six synthematic totals while R permutes those five self-polar triangles of Σ whose vertices are all D.

W. L. E.