AN OPERAND FOR A GROUP OF ORDER 1512

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1. The tableau on p. 105 built with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 is one of a possible 1920 and was compiled in the year 1963 at about the time when [5] was published. Whether it is new or not, its construction, described below, with the help of the geometry of Study's quadric $\mathcal{S}$, in projective space of 7 dimensions over the finite field $GF(2)$, may be. One can observe the visual effects on the tableau of 1512 even permutations of the group $H$ for which it is, as a whole, invariant. $H$ could, in this context, be identified as the intersection of three alternating groups of degree 9 derivable from each other by using the triality on $\mathcal{S}$. But $H$ and its simple subgroup $h$ of order 504, isomorphic to the linear fractional group $LF(2, 2^3)$, occur also in a less elaborate geometry. $LF(2, 2^3)$ is the group of projectivities on a 9-point line $\lambda$ and is extended to a group isomorphic to $H$ by the automorphism of period 3 of $GF(2^3)$ which, replacing each mark by its square, leaves three points (those with parameters 0, 1, $\infty$) of $\lambda$ unmoved while permuting the others in two cycles of three—a permutation unattainable by projectivities because any projectivity which leaves three points all unmoved can only be the identity.

The most recent appearance of $H$ in the literature may be Dye's recognition [3] of it as a maximal subgroup, of index 960, of the group of the bitangents. Among its other appearances one might mention that as leader [8; p. 452] of a series of groups—all, save itself, simple—of order $q^3(q^3 + 1)(q - 1)$ where $q = 3^{2n+1}$. But, as permutation groups of degree 9, both $H$ and $h$ were identified by Cole [1] and one may grant his claim [1; p. 250] to be the first to have asserted, and proved, that $h$ is simple. His claim, however, to be the original discoverer of this group must be disallowed because it had been found some 30 years earlier by Kirkman. Certainly Kirkman lumps together all permutations of the same cycle type, so distinguishing conjugate classes only in the symmetric group $S_9$; he does not refine the distinction in any of its subgroups. Yet $H$ and $h$, along with other groups, are plainly recorded [6; pp. 147 and 146]. Kirkman's essay appears to have escaped observation even in the telescopic sights of Coxeter and Moser.

2. The notation used is that of [5]. $\mathcal{S}$ is Study's non-singular quadric consisting of 135 points $m$ in projective space [7] over $GF(2)$; at each $m$ there is a prime $M$ tangent to $\mathcal{S}$. The $m$ are vertices of 960 enneads $\mathcal{E}$ [5; pp. 6, 7], no two vertices of the same $\mathcal{E}$ are ever conjugate for $\mathcal{S}$. $\mathcal{S}$ is ruled by two systems of solids $\omega$, $\omega'$ each with 135 members, and the existence of the enneads $\mathcal{E}$ implies, by triality, that of enneagrams $\eta$, $\eta'$ each consisting of nine solids of the same system all skew to one another. It is with such $\eta$ that we shall be concerned, and they will be obtained by elementary arguments which do not invoke triality.

The equation of $\mathcal{S}$ is $\Sigma_{i<j} x_i x_j = 0$, with its left-hand side the sum of the 28 products of pairs of eight homogeneous coordinates $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$. The simplex of reference consists of all but one of the vertices of an ennead $\mathcal{E}_0$, the remaining vertex of $\mathcal{E}_0$ being the unit point $U$.

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The intersection of any two of the nine primes $M$ tangent to $\mathcal{S}$ at the vertices of an ennead is a $[5]$ $C$, the polar of the chord $c$ that joins the contacts. The section of $\mathcal{S}$ by $C$ is ruled by planes, but no solid lies wholly on it. Thus all 270 solids on $\mathcal{S}$ are accounted for by 30 in each of nine $M$: that $M$ does contain 30 follows because it meets $\mathcal{S}$ in a cone projecting a Klein quadric from a point $m$ outside the $[5]$ containing it, and on a Klein quadric there are, over $\text{GF}(2)$, 30 planes.

3. Consider now those $\omega$ in $M_8$, the tangent $M$

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0 \quad (3.1)$$

to $\mathcal{S}$ at $U$. Any such $\omega$ is determined as the intersection of four linearly independent primes; among these one may or may not include (3.1). It will best serve our purpose to denote a prime by the symbol composed of the suffixes of such coordinates as are present on the left-hand side of its equation; the order of these suffixes does not matter. For instance, the unit prime is $01234567$ or any permutation of these digits, while $0357$ denotes $x_0 + x_3 + x_5 + x_7 = 0$, and so forth. Symbols may, just as left-hand sides of equations may, be added; any digit that occurs an even number of times in an added set of symbols disappears, any that occurs an odd number of times survives.

One set of four linearly independent primes whose intersection is on $\mathcal{S}$ consists of

$$0357, \quad 2571, \quad 4713, \quad 6135.$$ 

This is seen, for instance, by remarking that each prime contains the four linearly independent points

$$(1.1.1.1.), \quad (1..1.1.), \quad (1.1..1.), \quad (1.1.1.1),$$

all of which are on $\mathcal{S}$ and every pair of them conjugate. Alternatively, one may simply refer to 10.3 in [5]. The sums of pairs of the four symbols are

$$0123, \quad 4567, \quad 0145, \quad 2367, \quad 0167, \quad 2345;$$

their sums in threes

$$1246, \quad 0346, \quad 0256, \quad 0247,$$

while the sum $01234567$ of all four completes the set of fifteen primes which contain the solid. The same solid $\omega$ is determined by any four linearly independent symbols among those fifteen. If the octad is omitted the other symbols consist of seven complementary pairs of tetrads, and tetrads that are not complementary always share two digits.

4. It is helpful momentarily to regard these tetrads as labelling sets of four coplanar points in [3]. For, over $\text{GF}(2)$, there are eight points of [3] not in a given plane $\pi$; through each of the seven lines in $\pi$ pass two planes other than $\pi$ itself, so that the eight points fall into complementary tetrads in seven ways. Non-complementary tetrads share two digits because the line common to the corresponding planes is not in $\pi$ and so consists of one point in $\pi$ and two points outside $\pi$.

Call the points outside $\pi$ 0, 1, 2, 3, 4, 5, 6, 7 and let the planes 0167, 2345 meet $\pi$ in the line $ABC$; A, B, C are the collinear diagonal points of both quadrangles and these quadrangles, after the manner of desmic tetrahedra, are in perspective from each point of $\pi$ not on $ABC$. We may suppose that
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The seven separations into complementary coplanar tetrads are

\[
\begin{align*}
0124 & \quad 0135 & \quad 0167 & \quad 0236 & \quad 0257 & \quad 0347 & \quad 0456 \\
3567 & \quad 2467 & \quad 2345 & \quad 1457 & \quad 1346 & \quad 1256 & \quad 1237
\end{align*}
\] (4.1)

No permutation of any tetrad alters the meaning of this scheme. But the perspectives just mentioned serve to build a less arbitrary scheme, for one can prescribe that every pair of complementary tetrads has the digits so ordered that the joins of points in the same column are four concurrent lines; once the digits of either tetrad are placed this permits four orderings of the other. And the arbitrariness can be further restricted by demanding that the seven centres of perspective, one for each pair of tetrads, are all the points of \(\pi\); none serves for more than one pair of tetrads. The columns of the scheme then account for all twenty-eight pairs of the eight digits.

Any two of 03, 15, 26, 47 meet since they are in a plane containing one of A, B, C; but such a plane does not contain either of the other two joins, so that all four are concurrent. Other sets of four concurrent joins are

\[
05, 13, 27, 46; \quad 04, 12, 56, 37; \quad 02, 14, 36, 57.
\]

So one rebuilds (4.1) thus:

\[
\begin{align*}
0124 & \quad 0135 & \quad 0167 & \quad 0236 & \quad 0257 & \quad 0347 & \quad 0456 \\
6735 & \quad 2467 & \quad 5342 & \quad 7541 & \quad 4163 & \quad 1526 & \quad 3712
\end{align*}
\] (\(\Omega_8\))

5. The primes whose symbols are the fourteen tetrads in \(\Omega_8\) intersect in a solid, which we now label \(\omega_8\), on \(\mathcal{S}\); \(\omega_8\) lies in \(M_8\) and contains \(U\). Both \(\mathcal{S}\) and \(M_8\) have equations symmetric in the coordinates, so that any scheme derived from \(\Omega_8\) by permuting the digits represents a solid \(\omega\) or \(\omega'\), on \(\mathcal{S}\), in \(M_8\), containing \(U\). That there are thirty such solids is corroborated by noting how many permutations leave \(\Omega_8\) unchanged, i.e. permute its tetrads among themselves; the group \(\Gamma\) of projectivities in [3] for which \(\pi\) is invariant has [4; p. 335] order 1344, and \(8!/1344 = 30\). Alternatively
one can show that there are 30 schemes such as \( \Omega_8 \) determining solids on \( \mathfrak{S} \) in \( M_8 \) without alluding to \( \Gamma \). For \( \Omega_8 \) evolves from any of its seven blocks by linking the three bisections of either tetrad one with each bisection of its complement. Take, say, 0167 and 5342. Then \( \Omega_8 \) evolves by linking

\[ 01.67 \text{ with } 24.35, \quad 06.17 \text{ with } 23.45, \quad 07.16 \text{ with } 25.34 \]

so that another block consists of 0124 and 3567, yet another of 0135 and 2467, and so on. Since there are six ways of linking the bisections of a tetrad with those of its complement, and since a scheme can so evolve from any of its seven blocks, the number of schemes is \( \frac{1}{6} \cdot 6 \cdot 7 = 30 \).

When two of the eight digits are transposed, six tetrads in \( \Omega_8 \) are unchanged: the solids designated by \( \Omega_8 \) and the altered scheme lie together in seven primes—the unit prime and those six symbolised by the unchanged tetrads—linearly dependent on three among them. Thus the two solids lie together in a \([4]\) and are in opposite systems on \( \mathfrak{S} \). A scheme got from \( \Omega_8 \) by permuting the digits designates a solid belonging to the same system as \( \omega_0 \) or to the opposite system according as the permutation is even or odd.

6. The discussion has so far been confined to the solids on \( \mathfrak{S} \) which lie in \( M_8 \). But consider now solids on \( \mathfrak{S} \) which lie in the tangent prime at some vertex of \( \mathfrak{S}_0 \) other than \( U \), say in the tangent prime \( M_0 \) at \( X_0 \); they all contain \( X_0 \). The equation of \( M_0 \) being

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0, \]

its symbol, according to the convention adopted in §3, is 1234567 and, since this is the sum of 347, 567 and 127, \( M_0 \) contains the solid \( \omega_0' \) common to the primes

\[ 136, \quad 347, \quad 567, \quad 127. \quad (6.1) \]

This solid \( \omega_0' \) is seen to be on \( \mathfrak{S} \) by arguments like those of §3. The symbols of the ten primes, other than \( M_0 \) and the four in (6.1), that contain \( \omega_0' \) are the sums

\[ \begin{array}{ccc} 3456 & 1357 & 1467 \\ 2367 & 1234 & 1256 \end{array} \]

of pairs of triads (6.1), together with the sums

\[ \begin{array}{cc} 235 & 246 \\ 145 \end{array} \]

of three triads that include 136, and the sum 2457 of all four. So, of the fourteen primes other than \( M_0 \) through \( \omega_0' \), seven have tetrads and seven have triads for their symbols. But let the triads all be augmented by a dummy digit 8. Then \( \omega_0' \) is determined by the scheme

\[ \begin{array}{cccccccc} 1234 & 1256 & 1278 & 1357 & 1368 & 1548 & 1467 \\ 7856 & 3478 & 6543 & 8642 & 4275 & 2637 & 5823 \end{array} \quad (6.2) \]

and the thirty solids on \( \mathfrak{S} \) through \( X_0 \) appear on permuting the eight digits. In building this scheme it is again helpful to regard the digits as labels of points outside a plane in [3]; but this is a mere accessory and not the same as the [3] used in §4.

So, by using nine digits, one obtains the same type of scheme for every solid on \( \mathfrak{S} \). Should \( \omega \) be in \( M_0 \), and so contain \( X_0 \), the scheme lacks the digit 0; when \( \omega \) lay in \( M_8 \), and so contained \( U \), the digit 8 was absent. For \( \omega \) in \( M_1 \) and containing \( X_1 \),
the digit \( i \) will be missing. In any scheme arranged to accord with the prescriptions of §4, the pairs of the eight digits all occur, one pair in each column, once and only once.

7. Take \( \Omega_8 \) and apply to it repeatedly the cyclic permutation \( \Psi = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8) \). The resulting schemes are in Table 1; they determine nine \( \omega \), one through each of the vertices of \( \sigma_0 \). No tetrad occurs more than once in the table; all \( 9C_4 = 126 \) appear, each once and only once. All primes, save one, through any \( \omega \) are symbolised by the tetrads of its scheme, and there is no prime containing two of the nine \( \omega \). So these nine \( \omega \) are skew and belong to the same system on \( \mathcal{S} \). They compose an enneagram \( \eta \) and, with 15 \( m \) in each, account for all 135 \( m \) on \( \mathcal{S} \). The scheme \( \Omega_0 \) that leads the table is (6.2) with 4 and 5 transposed.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1235 1246 1278 1347 1368 1458 1567 (( \Omega_0 ))</td>
</tr>
<tr>
<td>7846 3578 6453 8652 5274 2637 4823</td>
</tr>
<tr>
<td>2346 2357 2380 2458 2470 2560 2678 (( \Omega_1 ))</td>
</tr>
<tr>
<td>8057 4680 7564 0763 6385 3748 5034</td>
</tr>
<tr>
<td>3457 3468 3401 3560 3581 3671 3780 (( \Omega_2 ))</td>
</tr>
<tr>
<td>0168 5701 8675 1874 7406 4850 6145</td>
</tr>
<tr>
<td>4568 4570 4512 4671 4602 4782 4801 (( \Omega_3 ))</td>
</tr>
<tr>
<td>1270 6812 0786 2085 8517 5061 7256</td>
</tr>
<tr>
<td>5670 5681 5623 5782 5713 5803 5012 (( \Omega_4 ))</td>
</tr>
<tr>
<td>2381 7023 1807 3106 0628 6172 8367</td>
</tr>
<tr>
<td>6781 6702 6734 6803 6824 6014 6123 (( \Omega_5 ))</td>
</tr>
<tr>
<td>3402 8134 2018 4217 1730 7283 0478</td>
</tr>
<tr>
<td>7802 7813 7845 7014 7035 7125 7234 (( \Omega_6 ))</td>
</tr>
<tr>
<td>4513 0245 3120 5328 2841 8304 1580</td>
</tr>
<tr>
<td>8013 8024 8056 8125 8146 8236 8345 (( \Omega_7 ))</td>
</tr>
<tr>
<td>5624 1356 4231 6430 3052 0415 2601</td>
</tr>
<tr>
<td>0124 0135 0167 0236 0257 0347 0456 (( \Omega_8 ))</td>
</tr>
<tr>
<td>6735 2467 5342 7541 4163 1526 3712</td>
</tr>
</tbody>
</table>

8. Any permutation of the nine digits that leaves \( \eta \) as a whole unchanged must, since the \( \omega \) stay in the same system, be even. How many of them leave unchanged not only \( \eta \) as a whole but also three of its individual members—say \( \Omega_6, \Omega_7, \Omega_8 \)? Since the triad 678 is accompanied in \( \Omega_0 \) by 4, in \( \Omega_4 \) by 3, in \( \Omega_3 \) by 0: in \( \Omega_1 \) by 2, in \( \Omega_2 \) by 5, in \( \Omega_5 \) by 1, the permutation (043)(125) is suggested, and the suggestion is reinforced on observing that the duads which accompany

\[
\begin{align*}
78 & \text{ in } \Omega_6 \text{ are 02, 45, 31;} \\
86 & \text{ in } \Omega_7 \text{ are 05, 41, 32;} \\
67 & \text{ in } \Omega_8 \text{ are 01, 42, 35.}
\end{align*}
\]

Moreover, the triad 043 is accompanied in

\[
\begin{array}{cccccc}
\Omega_1 & \Omega_5 & \Omega_2 & \Omega_6 & \Omega_7 & \Omega_8 \\
5 & 2 & 1 & 8 & 6 & 7
\end{array}
\]

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while 125 is accompanied in

\[
\begin{array}{cccccccc}
\Omega_0 & \Omega_3 & \Omega_4 & \Omega_6 & \Omega_7 & \Omega_8 \\
3 & 4 & 0 & 7 & 8 & 6
\end{array}
\]

by

Indeed, \((043)(125)\) does leave each of \(\Omega_6, \Omega_7, \Omega_8\) unchanged and permutes the remaining members of \(\eta\), with their suffixes, in two cycles of three. There is a corresponding statement when any three members of \(\eta\) are selected to be invariant; the three-fold stabilisers are of order three. Since any two of them share only the identity permutation, 168 operations of period 3 are accounted for in \(H\).

When any three members, say \(\Omega_i, \Omega_j, \Omega_k\), of \(\eta\) are chosen the residual six congregate automatically into complementary trios, each having its three schemes permuted cyclically by the threefold stabiliser \(H_{ijk}\). The 84 trios thus fall into 28 companion sets of three. Take, to be specific, \(i = 0, j = 3, k = 6\); Table 2 tells which digit accompanies the triad to the left of each row in the scheme heading each column.

<table>
<thead>
<tr>
<th>(\Omega_0)</th>
<th>(\Omega_3)</th>
<th>(\Omega_6)</th>
<th>(\Omega_1)</th>
<th>(\Omega_4)</th>
<th>(\Omega_7)</th>
<th>(\Omega_2)</th>
<th>(\Omega_5)</th>
<th>(\Omega_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>036</td>
<td>.</td>
<td>.</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>147</td>
<td>3</td>
<td>6</td>
<td>0</td>
<td>.</td>
<td>.</td>
<td>8</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>258</td>
<td>6</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

and implies that \(H_{036}, H_{147}, H_{258}\) are subgroups of \(x\), the direct product of any two of them; the two operations of \(x\) that do not belong to any threefold stabiliser permute the members of each of the trios cyclically. This accounts for 56 more operations of period 3, two for each of the 28 sets of complementary trios. Nor are there any further \([6; p. 147]\) operations of period 3 in \(H\).

9. \(H_{ijk}\) is a normal subgroup of \(G_{ijk}\), the subgroup of \(H\) that permutes \(\Omega_i, \Omega_j, \Omega_k\) among themselves; \(x\) subjects them to even permutations. It will be seen in a moment that \(G_{ijk}\) is the direct product \(\mathfrak{S}_3 \times \mathfrak{S}_3\) of cyclic and dihedral groups.

Again, to be specific, let \(i = 0, j = 3, k = 6\). In order to identify a permutation, in the stabiliser \(H_0\) of \(\Omega_0\), which transposes \(\Omega_3\) and \(\Omega_6\), note that the column in \(\Omega_0\) composed of the pair 36 is in the central block; the involution \(R = (18)(27)(36)(45)\) is thereby indicated. It does belong to \(G_{036}\); it transposes \(\Omega_3\) with \(\Omega_6\) and the other two trios of the companion set with one another. The three involutions

\[
(18)(27)(36)(45), \quad (06)(15)(24)(78), \quad (03)(12)(48)(57)
\]

suggested by the central blocks in \(\Omega_0, \Omega_3, \Omega_6\) all belong to \(G_{036}\), which is generated by \(R\) and \(S = (036)(285)\). For \(RS\) and \(SR\), both of period 6, have the same square and the generating relations for \(\mathfrak{S}_3 \times \mathfrak{S}_3\) are satisfied \([2; p. 134]\). This group includes six operations of period 6 which, with the three involutions, compose the coset of \(x\). As \(H\) has 84 subgroups \(G_{ijk}\), it includes 504 operations of period 6 \([6; p. 147]\).

10. The elementary Abelian group of order 9 which, while permuting members of each trio cyclically, leaves invariant each companion trio of a set of three is normal in a group of order 54 which, while leaving the whole set invariant, includes every operation that permutes the three companion trios. Those 27 operations that impose odd permutations on the trios

\[
\Omega_0, \Omega_3, \Omega_6; \quad \Omega_1, \Omega_4, \Omega_7; \quad \Omega_2, \Omega_5, \Omega_8;
\]
are the cosets of \( \alpha \) in \( G_{036}, G_{147}, G_{258} \); there remain 18 operations that permute the three trios cyclically. Each of these is expressible, in different ways, as a product of operations from any two of \( G_{036}, G_{147}, G_{258} \) just as a cyclic permutation of three objects is expressible, in different ways, as a product of two transpositions. All these 18 operations are cyclic, of period 9; \( \Psi \), used in constructing \( \eta \), must be among them; indeed

\[
\Psi = (012345678) = (18)(27)(36)(45).(08)(17)(26)(35),
\]

the product of involutions in \( G_{036} \) and \( G_{147} \). Such a group of order 54 is duly registered by Kirkman [6; p. 145].

The companion trios of the set chosen to illustrate the above discussion had the virtue of visual convenience, proferring the central columnar blocks of Table 1 for scrutiny. But the other 27 companion sets are of equal standing and have the same properties.

There are 63 involutions in \( H \), one for each of the 63 pairs of complementary tetrads of digits. Each of the 84 \( G_{ijk} \) has been seen to include three involutions; on the other hand, each involution belongs to four \( G_{ijk} \). For instance,

\[
(17)(28)(34)(56),
\]

corresponding to a complementary pair of tetrads in \( \Omega_0 \), belongs to \( G_{017}, G_{028}, G_{034}, G_{056} \).

11. One has now accounted for all the operations of \( H \) save the 216, shown in Kirkman's partitioning, of period 7. These must compose, with identity, 36 subgroups of order 7; indeed each of the 36 two-fold stabilisers \( H_{ij} \) will be found to include one such subgroup.

Take \( i = 7, j = 8 \). If \( \Omega_7 \) and \( \Omega_8 \) are both stable, the triads

\[
356 \quad 246 \quad 016 \quad 145 \quad 025 \quad 034 \quad 123
\]

that accompany 7 in tetrads of \( \Omega_8 \) must be permuted among themselves; so must the triads

\[
013 \quad 024 \quad 056 \quad 125 \quad 146 \quad 236 \quad 345
\]

that accompany 8 in tetrads of \( \Omega_7 \). So the two-fold stabiliser \( H_{78} \) belongs to the intersection of two Klein groups, the groups of projectivities of the two seven-point planes depicted thus:

![Diagrams of seven-point planes]

it being understood that 145 is rectilinear in the left-hand, 125 in the right-hand diagram, and that all the visually apparent Euclidean collinearities are valid in the finite planes.
Choose any point in either plane. The other six lie in threes on four lines and one, and only one, of these lines is such that those three of the six points that are not on it are collinear in the other plane. Take, for instance, 0 in the left-hand plane. Those four lines that consist of three points other than 0 are

$$\begin{align*}
123, & \quad 246, & \quad 356, & \quad 145 \\
456, & \quad 135, & \quad 124, & \quad 236
\end{align*}$$

of which only 236 is a collinear triad in the right-hand plane; the permutations

$$(145)(632) \quad \text{and} \quad (154)(623)$$

belong to both Klein groups, and indeed to the threefold stabiliser $H_{078}$. So one obtains, common to $H_7$ and $H_8$,

$$\begin{align*}
(145)(263), & \quad (024)(356), & \quad (016)(345), & \quad (025)(164), \\
(123)(056), & \quad (013)(246), & \quad (034)(152)
\end{align*}$$

and their inverses: indeed the operations of the seven groups $H_{178}$. The product of any two of these permutations belongs to $H_{78}$ and has period 7;

$$\text{e.g. } (024)(356) \cdot (145)(263) = (0234651) = S$$

say; then if $T = (123)(056)$, $ST = TS^2$ and the defining relations [2; p. 134] for a group of order 21 are satisfied. As $H$ includes 36 two-fold stabilisers they account for 216 operations of period 7.

A stabiliser $H_i$ is of order 168; it has, like Klein’s group of this order, 8 subgroups of order 21, namely the two-fold stabilisers $H_{ij}$; but $H_i$ is not a Klein group. $H_i$ includes only 7 involutions, whereas a Klein group includes 21. Nor does a Klein group possess any operations of period 6, whereas the operations $RS$ and $SR$ mentioned in §9 belong one to $H_3$ and the other to $H_6$. Moreover, Klein’s group is a simple group, whereas $H_i$ has a normal elementary Abelian subgroup composed of its 7 involutions and the identity. If publication dates are to decide priority, Kirkman has been, if only just, anticipated in encountering this group of order 168, since it appears, along with “Klein’s group”, on p. 292 of Mathieu’s memoir [7].

12. It was explained in §15 of [5] how the 135 $m$ consist, in relation to any of the 960 $\mathfrak{S}$, of the nine vertices of $\mathfrak{d}$ and 126 other $m$; these latter each supplement one of the 126 tetrads of vertices of $\mathfrak{d}$ in that the solid $\lambda$ spanned by the tetrad meets $\mathfrak{S}$ further in the supplementary $m$, and in no other point. No two of the five $m$ in $\lambda$ are conjugate. The polar solid $\lambda'$ of $\lambda$ also meets $\mathfrak{S}$ in five $m$, no two conjugate; when $\lambda$ is spanned by four vertices of $\mathfrak{d}$, the $m$ in $\lambda'$ are those supplementing the tetrads among the residual pentad of vertices of $\mathfrak{d}$. It was also seen that the third point on the join of those $m$ which supplement disjoint tetrads in $\mathfrak{d}$ is that point of $\mathfrak{S}$ not in either tetrad.

The triality on $\mathfrak{S}$ imposes a (1, 1) correspondence between points $m$ and solids $\omega$; if two $m$ are conjugate their correspondents have a common line, or are incident; to an ennead $\mathfrak{S}$, nine $m$ no two of them conjugate, corresponds an enneagram $\eta$ of mutually skew $\omega$. The 135 $\omega$ consist, in relation to any of the 960 $\eta$, of the nine $\omega$ in $\eta$ and 126 others; these latter each supplement one of the 126 sets of four $\omega$ in $\eta$ in the sense of being skew to all these four and incident with all the other five—transversal to these five one might say. Those $\omega$ which so supplement disjoint sets of four in $\eta$
are incident, and the third $\omega$ through their common line is the ninth member of $\eta$. These facts, consequences by triality of properties of $m$, can be established directly by using the tableau of Table 1.

A solid transversal to each of five $\omega$ in $\eta$ has, by three points on each of five skew lines, all its 15 points accounted for; it is thus ipso facto skew to the other four $\omega$ in $\eta$. That such a solid exists is seen by constructing the scheme for it.

Suppose that one requires an $\omega$ transversal to $\omega_4, \omega_5, \omega_6, \omega_7, \omega_8$. It is in $\Omega_6$ that the tetrad 0123 occurs and one links, in one of the six possible ways, the bisections of 0123 with those of 7845. The linking in $\Omega_6$ was

$$0.23 \text{ with } 47.58, \quad 0.31 \text{ with } 48.57, \quad 0.21 \text{ with } 45.78;$$

since an $\omega$ distinct from $\omega_6$ is required, one shifts the three bisections of 7845 cyclically. But 0148 is in $\Omega_3$, 0157 in $\Omega_2$, 2348 in $\Omega_0$, 2357 in $\Omega_1$, and the scheme that would evolve from any linking of 01.23 with 48.57 would give an $\omega$ incident with $\omega_0, \omega_1, \omega_2, \omega_3$. So one links

$$0.23 \text{ with } 45.78, \quad 0.31 \text{ with } 48.57, \quad 0.21 \text{ with } 47.58$$

and arrives at

$$\begin{array}{cccccccc}
0123 & 0145 & 0178 & 0248 & 0257 & 0347 & 0358 \\
7845 & 8732 & 3245 & 5713 & 4813 & 1285 & 2147 \\
\end{array} \quad (\Omega_{0123})$$

The primes here symbolised contain, respectively (see Table 1)

$$\begin{array}{cccccccc}
\omega_6 & \omega_7 & \omega_4 & \omega_7 & \omega_8 & \omega_4 & \\
\omega_6 & \omega_5 & \omega_8 & \omega_5 & \omega_7 & \omega_5 & \\
\end{array}$$

so that, with $\omega_6$ lying in 01234578, each of $\omega_4, \omega_5, \omega_6, \omega_7, \omega_8$ lies with this new $\omega$ in three primes of which two are independent. So the $\omega$ thus obtained is indeed transversal to $\omega_4, \omega_5, \omega_6, \omega_7, \omega_8$ and (therefore) skew to $\omega_0, \omega_1, \omega_2, \omega_3$.

13. The fifth power of the cyclic permutation $\Psi$ applied to $\Omega_{0123}$ produces a scheme $\Omega_{5678}$ that identifies an $\omega$ transversal to $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4$ and skew to $\omega_5, \omega_6, \omega_7, \omega_8$; this scheme is

$$\begin{array}{cccccccc}
5678 & 5601 & 5634 & 5704 & 5713 & 5803 & 5814 \\
3401 & 4387 & 8701 & 1368 & 0468 & 6741 & 7603 \\
\end{array} \quad (\Omega_{5678})$$

This $\omega$, supplementing $\omega_5, \omega_6, \omega_7, \omega_8$, and the previous $\omega$, supplementing $\omega_0, \omega_1, \omega_2, \omega_3$ ought to have in common a line in $\omega_4$. So, indeed, they do. The three tetrads

$$0178 \quad 0358 \quad 1357$$

are common to $\Omega_{0123}, \Omega_{5678}$ and $\Omega_4$; the three solids are all in the [5] common to these three linearly dependent primes and, being all their own polars with respect to $\mathcal{S}$, contain the polar line of this [5].

References


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