On Doubly Nonnegative Relaxations of Standard Quadratic Programs

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Joint work with Yakup Görkem Gökmen

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$$\text{(StQP)} \quad \nu(Q) = \min\{x^T Q x : x \in \Delta_n\},$$

where

- $\Delta_n = \{x \in \mathbb{R}^n : e^T x = 1, \quad x \in \mathbb{R}^n_+\}$ (the unit simplex),
- $Q \in \mathcal{S}^n$, where $\mathcal{S}^n$ denotes the space of $n \times n$ real symmetric matrices,
- $x \in \mathbb{R}^n$,
- $e \in \mathbb{R}^n$ denotes the vector of all ones, and
- $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$.

Applications

- Portfolio optimization [Markowitz, 1952]
- Quadratic resource allocation problem [Ibaraki and Katoh, 1988]
- Population genetics [Kingman, 1961]
- Evolutionary game theory [Bomze, 2002]
- Social network analysis [Bomze et al., 2018]
- Copositivity detection (a matrix $M \in \mathcal{S}^n$ is copositive iff $\nu(M) = \min\{x^T Mx : x \in \Delta_n\} \geq 0$)
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- NP-hard in general
- Can have at least $(1.4933)^n$ strict local minimizers! [Bomze et al., 2018]

Motivation and Focus

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- **Main Goal:** To shed light on instances of (StQP) that admit exact convex relaxations.

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$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathcal{K}^{n+m}.$$

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In particular, $B = 0$ can be chosen.

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- (DN) is referred to as the *doubly nonnegative relaxation*.

Basic Relations and Our Focus

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- For $n \leq 4$, we have $\ell(Q) = \nu(Q)$ by Diananda's result.
- **Question:** For $n \geq 5$, can we give a characterization of instances of (StQP) for which $\ell(Q) = \nu(Q)$?

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Theorem (Bomze, 1997)

Let $Q \in \mathcal{S}^n$ and let $x^* \in \Delta_n$. Then,

$$\nu(Q) = (x^*)^T Q x^* \iff Q - \underbrace{((x^*)^T Q x^*)}_{\nu(Q)} E \in \mathcal{COP}^n.$$

A General Characterization

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Theorem

Let $Q \in \mathcal{S}^n$. We have

$$\ell(Q) = \nu(Q) \Longleftrightarrow Q - \nu(Q)E \in \mathcal{SPN}^n.$$

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- We may have $Q + \lambda E \in \mathcal{PSD}^n$ for some $\lambda \in \mathbb{R}$ even if $Q \notin \mathcal{PSD}^n$:

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Let $Q \in \mathcal{S}^n$*. Then,* $Q + \lambda E \in \mathcal{PSD}^n$ *for some* $\lambda \in \mathbb{R}$ *iff*

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or, equivalently, $U^T Q U \in \mathcal{PSD}^{n-1}$, *where* $U \in \mathbb{R}^{n \times (n-1)}$ *is an orthonormal basis for* $e^\perp$.

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- We have $0 \le \ell(Q^s) \le \nu(Q^s)$.
- In particular, this implies that $\min_{1 \le i \le j \le n} Q_{ij} \le \ell(Q) \le \nu(Q)$.

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Corollary

If $Q \in \mathcal{S}^n$ satisfies $\min_{1 \le i \le j \le n} Q_{ij} = Q_{kk}$ for some $k = 1, \ldots, n$, then $\ell(Q) = \nu(Q) = Q_{kk}$.

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- We will slightly digress.

Maximum Weighted Stable Set Problem I

- Let $G = (V, E)$ be a simple, undirected graph with $V = \{1, \ldots, n\}$ and let $w \in \mathbb{R}^n_+$ be strictly positive, where w_k denotes the weight of vertex k, $k = 1, \ldots, n$.

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- The maximum weighted stable set problem is concerned with finding a stable set with the maximum weight, and its weight is denoted by $\alpha(G, w)$.

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Theorem (Gibbons et al., 1997)

Let $G = (V, E)$ be a simple, undirected graph and let $w \in \mathbb{R}^n_+$ be strictly positive. Then, for any $B \in \mathcal{M}(G, w)$,

$$\frac{1}{\alpha(G, w)} = \nu(B) = \min\{x^T B x : x \in \Delta_n\}.$$

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- Then,

$$\nu(Q) = \frac{1}{\alpha(G, w)}.$$

Establishing Connections

- Let $G = (V, E)$ be a simple, undirected graph with $V = \{1, \ldots, n\}$ and let $w \in \mathbb{R}^n_+$ be strictly positive.
- Let $Q \in \mathcal{M}(G, w)$, where

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Theorem

Let $G = (V, E)$ be a simple, undirected graph and let $w \in \mathbb{R}^n_+$ be strictly positive. For any $Q \in \mathcal{M}(G, w)$,

$$\ell(Q) = \frac{1}{\vartheta'(G, w)}.$$

Implications

- Recall **Case 2:** $Q^s_{kk} > 0$ for all $k = 1, \ldots, n$ and $Q^s_{ij} = Q^s_{ji} = 0$ for some $1 \leq i < j \leq n$.

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- Then, $Q^s \in \mathcal{M}(G(Q^s, w))$.

Corollary

Let $Q \in \mathcal{S}^n$ be such that Q^s has strictly diagonal entries, and $Q^s \in \mathcal{M}(G(Q^s, w))$, where $w_k = 1/Q^s_{kk}, \; k = 1, \ldots, n$. If $G(Q^s)$ is a perfect graph, then $\ell(Q^s) = \nu(Q^s)$ and therefore, $\ell(Q) = \nu(Q)$.

Summary

$$\begin{aligned}\nu(Q) &= \min\{x^T Q x : x \in \Delta_n\} = \min\{\langle Q, X\rangle : \langle E, X\rangle = 1, \quad X \in \mathcal{CP}^n\} \\ \ell(Q) &= \min\{\langle Q, X\rangle : \langle E, X\rangle = 1, \quad X \in \mathcal{DN}^n\}\end{aligned}$$

$$Q^s = Q - \left(\min_{1 \le i \le j \le n} Q_{ij}\right) E.$$

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- **Case 3:** If $Q + \lambda E \notin \mathcal{PSD}^n$ for any $\lambda \in \mathbb{R}$, $Q^s_{kk} > 0$ for all $k = 1, \ldots, n$ and $Q^s_{ij} = Q^s_{ji} = 0$ for some $1 \leq i < j \leq n$, then construct $G(Q^s)$ and define $w \in \mathbb{R}^n_+$, where $w_k = 1/Q^s_{kk}, \; k = 1, \ldots, n$:

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  - **Case 3a:** If $Q^s \in \mathcal{M}(G(Q^s, w))$ and $G(Q^s)$ is a perfect graph, then $\ell(Q) = \nu(Q)$.

Summary

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- **Case 3:** If $Q + \lambda E \notin \mathcal{PSD}^n$ for any $\lambda \in \mathbb{R}$, $Q^s_{kk} > 0$ for all $k = 1, \ldots, n$ and $Q^s_{ij} = Q^s_{ji} = 0$ for some $1 \le i < j \le n$, then construct $G(Q^s)$ and define $w \in \mathbb{R}^n_+$, where $w_k = 1/Q^s_{kk}, \; k = 1, \ldots, n$:
  - **Case 3a:** If $Q^s \in \mathcal{M}(G(Q^s, w))$ and $G(Q^s)$ is a perfect graph, then $\ell(Q) = \nu(Q)$.
  - **Case 3b:** Two further subcases:
    - **Case 3b-i:** If $Q^s \in \mathcal{M}(G(Q^s, w))$ but $G(Q^s)$ is not a perfect graph?

Summary

$$\begin{aligned} \nu(Q) &= \min\{x^T Q x : x \in \Delta_n\} = \min\{\langle Q, X\rangle : \langle E, X\rangle = 1, \quad X \in \mathcal{CP}^n\} \\ \ell(Q) &= \min\{\langle Q, X\rangle : \langle E, X\rangle = 1, \quad X \in \mathcal{DN}^n\} \end{aligned}$$

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- **Case 1:** If $Q + \lambda E \in \mathcal{PSD}^n$ for some $\lambda \in \mathbb{R}$, then $\ell(Q) = \nu(Q)$.
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- **Case 3:** If $Q + \lambda E \notin \mathcal{PSD}^n$ for any $\lambda \in \mathbb{R}$, $Q^s_{kk} > 0$ for all $k = 1, \ldots, n$ and $Q^s_{ij} = Q^s_{ji} = 0$ for some $1 \leq i < j \leq n$, then construct $G(Q^s)$ and define $w \in \mathbb{R}^n_+$, where $w_k = 1/Q^s_{kk}, \; k = 1, \ldots, n$:
  - **Case 3a:** If $Q^s \in \mathcal{M}(G(Q^s, w))$ and $G(Q^s)$ is a perfect graph, then $\ell(Q) = \nu(Q)$.
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    - **Case 3b-i:** If $Q^s \in \mathcal{M}(G(Q^s, w))$ but $G(Q^s)$ is not a perfect graph?
    - **Case 3b-ii:** If $Q^s \notin \mathcal{M}(G(Q^s, w))$?

Example 1 (Case 1)

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Example 1 (Case 1)

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

- $Q \in \mathcal{PSD}^5$. Therefore, $\nu(Q) = \ell(Q) = 0.4$.

Example 2 (Case 2)

$$Q = Q^s = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 3 & 0 & 2 \\ 0 & 3 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

Example 2 (Case 2)

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- $Q + \lambda E \notin \mathcal{PSD}^5$ for any $\lambda \in \mathbb{R}$.

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$$Q = Q^s = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 3 & 0 & 2 \\ 0 & 3 & 2 & 2 & 1 \\ 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

- $Q + \lambda E \notin \mathcal{PSD}^5$ for any $\lambda \in \mathbb{R}$.
- $Q^s_{11} = 0$. Therefore, $\nu(Q) = \ell(Q) = 0$.

Example 3 (Case 3a)

Recall $G(Q^s) = (V, E)$, where $V = \{1, \ldots, n\}$ and

$$E = \left\{(i,j) : 2Q^s_{ij} \geq Q^s_{ii} + Q^s_{jj}, \quad 1 \leq i < j \leq n\right\}.$$

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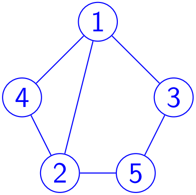

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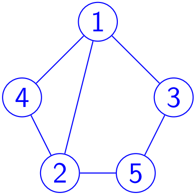

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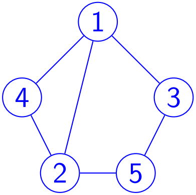

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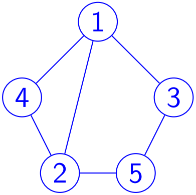

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Recall $G(Q^s) = (V, E)$, where $V = \{1, \ldots, n\}$ and

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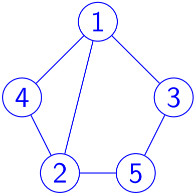

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- $G(Q^s)$ is a perfect graph. Therefore, $\ell(Q) = \nu(Q) = 1/2$.

Example 4 (Case 3b-i)

Recall $G(Q^s) = (V, E)$, where $V = \{1, \ldots, n\}$ and

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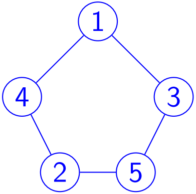

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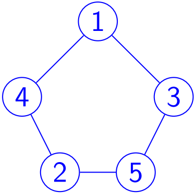

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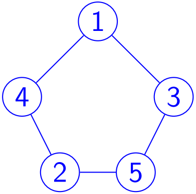

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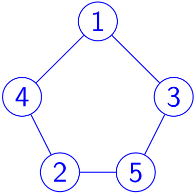

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Recall $G(Q^s) = (V, E)$, where $V = \{1, \ldots, n\}$ and

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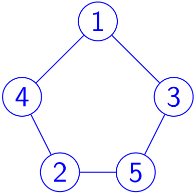

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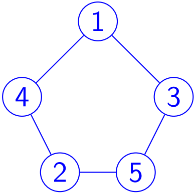

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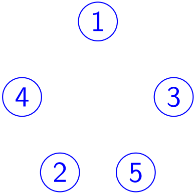

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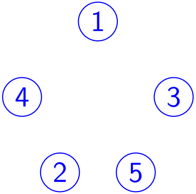

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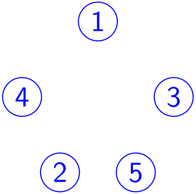

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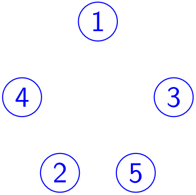

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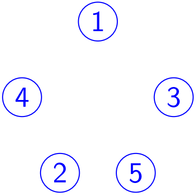

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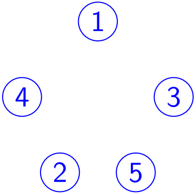

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- Can we identify further subcases of **Case 3b** for which the doubly nonnegative relaxation is exact?

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