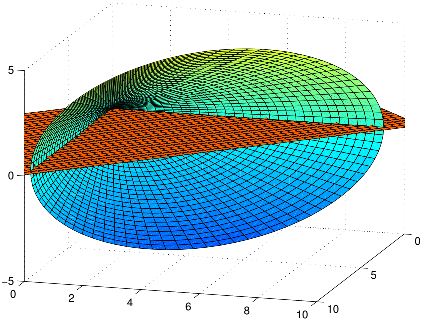
An Introduction to Conic and Semidefinite Programming with Applications

Christoph Helmberg
Technische Universität Chemnitz



Edinburgh, October 19, 2018

Contents

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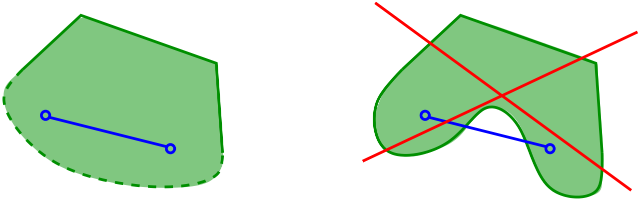
the straight line segment $\{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$ lies in C.



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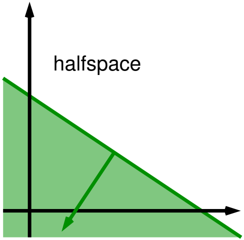
Examples:

- $\emptyset$, $\mathbb{R}^n$
- halfspaces: for given $z \in \mathbb{R}^n, \zeta \in \mathbb{R}$ the set $\{x \in \mathbb{R}^n : z^T x \geq \zeta\}$



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Note:

- the intersection of convex sets is convex
- any closed convex set is the intersection of the halfspaces containing it

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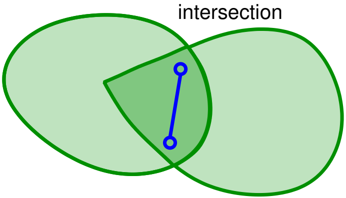

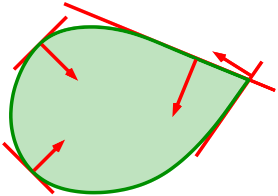

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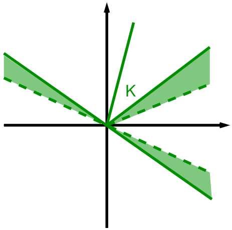

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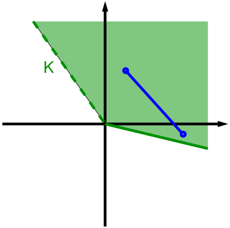

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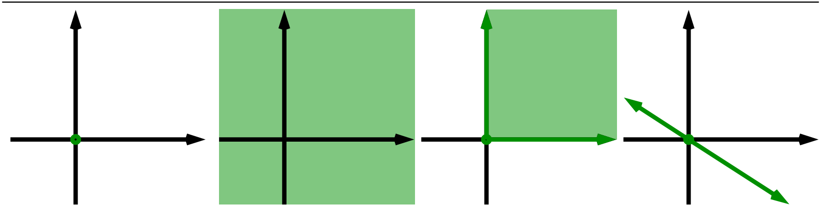

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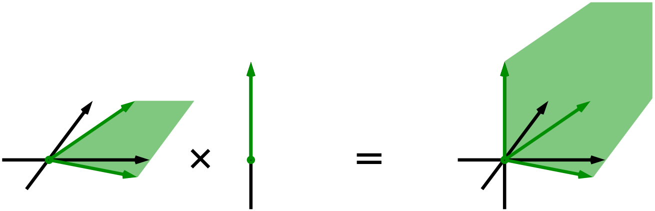

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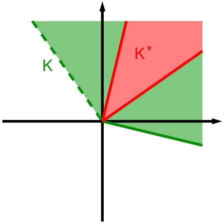

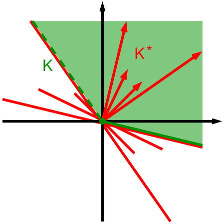

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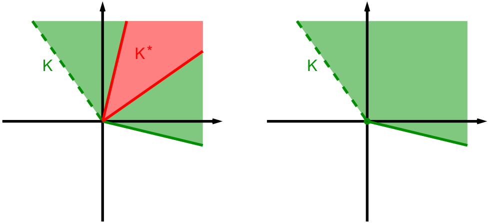

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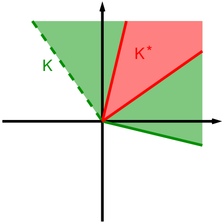

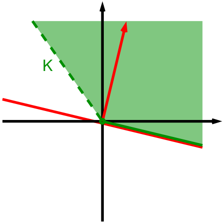

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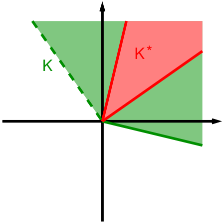

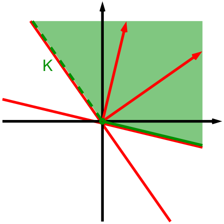

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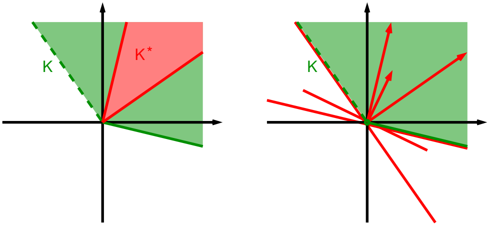

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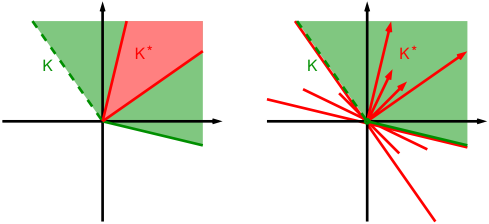

K^* is always closed!

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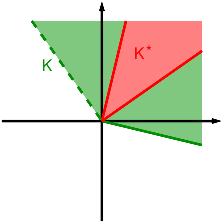

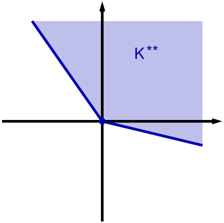

$(K^*)^* \supseteq K$,
equality holds iff
K is a closed convex cone!

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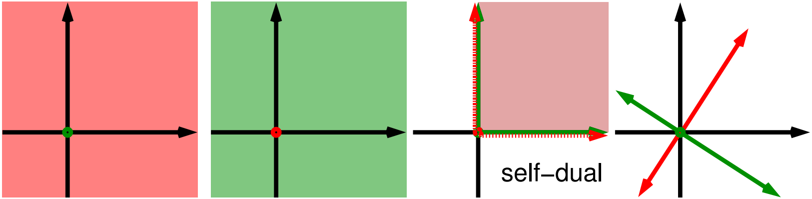

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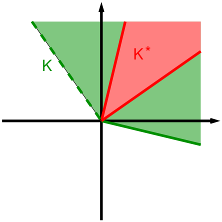

Important property for optimisation:

$$\inf_{x \in K} z^T x = \begin{cases} 0 & \Leftrightarrow z \in K^*, \\ -\infty & \textit{otherwise}. \end{cases}$$

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Replace $\mathbb{R}^n_+$ within linear optimisation by a convex cone $K \subseteq \mathbb{R}^n$:

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Define the Lagrange function

$$L(x, y) := c^T x + y^T (b - Ax) \quad \text{for } (x, y) \in K \times \mathbb{R}^m.$$

For $y \in \mathbb{R}^m$ and $Ax = b$ we have $(b - Ax)^T y = 0$, hence

$$\text{for all } y \in \mathbb{R}^m: \quad \inf_{x \in K} L(x, y) \leq \inf\{c^T x \colon Ax = b, x \in K\}.$$

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The best lower bound (Lagrangian relaxation) is

$$\sup_{y \in \mathbb{R}^m} \inf_{x \in K} L(x,y) = \sup_{y \in \mathbb{R}^m} [b^T y + \inf_{x \in K} x^T(c - A^T y)]$$

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The inner inf is finite only for $z = c - A^T y \in K^*$ giving the dual program

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + z = c \\ & y \in \mathbb{R}^m, z \in K^* \end{array}$$

Weak and Strong Duality

Let $K \subseteq \mathbb{R}^n$ be a closed convex cone.

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array} \qquad (D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + z = c \\ & y \in \mathbb{R}^m, z \in K^* \end{array}$$

Weak duality, i.e., $v(P) \geq v(D)$, always holds by construction.
Equality does NOT hold in general (see later examples)!

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To ensure strong duality we need to require additional properties:
A primal feasible $\bar{x}$ is strictly feasible for (P) if $\bar{x}$ lies in the interior of K,

$$\exists \rho > 0\colon B_\rho(\bar{x}) := \{x \in \mathbb{R}^n \colon \|x - \bar{x}\| \leq \rho\} \subseteq K$$

If such an $\bar{x}$ exists, (P) is strictly feasible.

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A dual feasible $(\bar{y}, \bar{z})$ is strictly feasible for (D), if $\bar{z}$ lies in the interior of K^* ($\exists \rho > 0\colon B_\rho(\bar{z}) \subseteq K^*$). If such $(\bar{y}, \bar{z})$ exist, (D) is strictly feasible.

Weak and Strong Duality

Let $K \subseteq \mathbb{R}^n$ be a closed convex cone.

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array} \qquad (D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + z = c \\ & y \in \mathbb{R}^m, z \in K^* \end{array}$$

Weak duality, i.e., $v(P) \geq v(D)$, always holds by construction.
Equality does NOT hold in general (see later examples)!

To ensure strong duality we need to require additional properties:
A primal feasible $\bar{x}$ is strictly feasible for (P) if $\bar{x}$ lies in the interior of K,

$$\exists \rho > 0 \colon B_\rho(\bar{x}) := \{x \in \mathbb{R}^n \colon \|x - \bar{x}\| \leq \rho\} \subseteq K$$

If such an $\bar{x}$ exists, (P) is strictly feasible.
A dual feasible $(\bar{y}, \bar{z})$ is strictly feasible for (D), if $\bar{z}$ lies in the interior of K^* ($\exists \rho > 0 \colon B_\rho(\bar{z}) \subseteq K^*$). If such $(\bar{y}, \bar{z})$ exist, (D) is strictly feasible.

Theorem (Strong Duality)

If (P) is strictly feasible, the dual optimum $v(D)$ is attained.
If (D) is strictly feasible, the primal optimum $v(P)$ is attained.
In both cases there holds $v(P) = v(D)$.

Self-dual Cones

Here we mainly consider three special types of cones K:

- $K = \mathbb{R}^n_+$, the nonnegative orthant
- $K = \mathcal{Q}^n$, the second order/quadratic/Lorentz/ice cream cone
- $K = S^n_+$ the cone of positive semidefinite matrices

The detailed definitions of $\mathcal{Q}^n$ and S^n_+ will be given soon.

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The most important properties of these three are:

- They are self-dual, i.e., $K = K^*$. [+ homogeneous $\rightarrow$ symmetric]

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array} \qquad (D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + z = c \\ & y \in \mathbb{R}^m, z \in K \end{array}$$

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In applications K is typically composed of several subcones,

$$K = \mathbb{R}^n_+ \times \mathcal{Q}^{m_1} \times \cdots \times \mathcal{Q}^{m_k} \times S^{n_1}_+ \times \cdots \times S^{n_h}_+$$

This will arise naturally and $K = K^*$ always holds for these combinations.

Contents

The Second Order Cone

The Second-Order-Cone (SOC)

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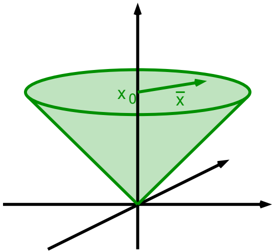

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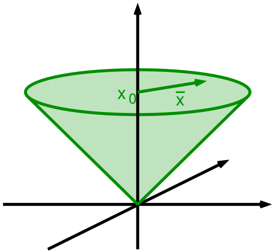

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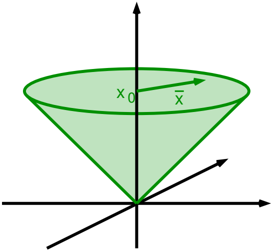

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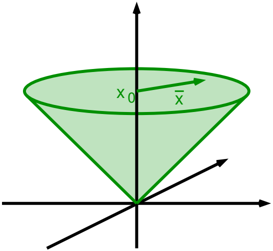

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A linear program that only uses cones $\mathbb{R}^{n_i}_+$ and at least one $\mathcal{Q}^n$ is a second-order-cone program (SOCP in short).
An SOCP with just one $\mathcal{Q}^n$ reads

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq_{\mathcal{Q}} 0 \end{array} \qquad (D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + z = c \\ & y \in \mathbb{R}^m, z \geq_{\mathcal{Q}} 0 \end{array}$$

[An SOCP with exactly one single SOC as here is solvable explicitly.]

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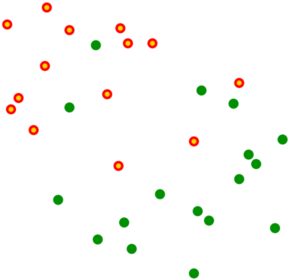
For data points in $\mathbb{R}^n$, that have or have not a certain property, we search for a hyperplane that separates the points according to this property as good as possible (goal: classify new points)



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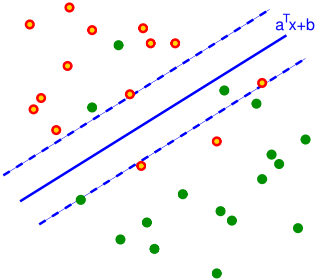

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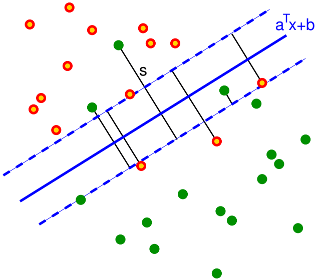

One approach: Minimise simultaneously $\|a\|$ and the sum of violations of the inequality constraints, scalarised by parameter $\gamma > 0$,

$$
\begin{array}{lll}
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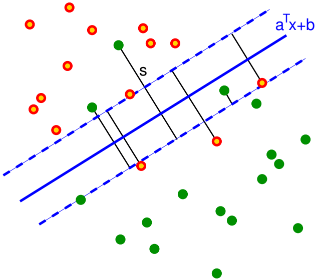

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The Markowitz Model

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$x \in \mathbb{R}^n_+$ with $\mathbf{1}^T x = 1$ represents the fraction of the budget invested into stock $1, \ldots, n$. The profit g per investment is a random variable with expectation $\bar{g} \in \mathbb{R}^n$ and covariance matrix $G \in S^n_+$ ($n \times n$, positive semidefinite). Let $s \in \mathbb{R}$ be a given profit threshold. As a risk measure the Markowitz model uses $x^T Gx$. [better measures exist]

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Because G is positive semidefinite this is a convex quadratic problem. [The two criteria profit against risk are now implemented by a constraint on one of the criteria.]

How to model this as an SOCP?

Quadratic Constraints with SOCP

Let $Q \in S_+^n$ be positive semidefinite, $q \in \mathbb{R}^n$, $\delta \in \mathbb{R}$. The convex quadratic constraint

$$x^T Qx + q^T x + \delta \leq 0$$

may be represented as an SOCP-constraint by (factor $Q = LL^T$)

$$\left\| \begin{array}{c} L^T x \\ \frac{1+(q^T x+\delta)}{2} \end{array} \right\| \leq \frac{1-(q^T x + \delta)}{2}$$

(proof: square both sides).

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For the Markowitz model just use $x_0 \geq \|L^T x\|$ with $G = LL^T$, then

$$\begin{array}{ll} \min & x_0 \\ \text{s.t.} & \bar{x} = L^T x \\ & \bar{g}^T x \geq s \\ & \mathbf{1}^T x = 1 \\ & \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \geq_{\mathcal{Q}} 0, x \geq 0 \end{array}$$

Probabilistic Constraints, Chance Constraint

Assume profit g is distributed normally with mean $\bar{g}$ and variance G. In addition to $\bar{g}^T x \geq s$ we now also require with probability at least $\eta \in (0, 1)$ that the profit is above a threshold value $\underline{s} < s$,

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This is modelled using a technique of robust optimisation:
$g^T x \geq \underline{s}$ is interpreted as an inequality with uncertain coefficients.

Linear Constraints with Uncertain Coefficients

If the coefficients of inequality $a^T x \le b$ are only known to lie inside the ellipsoid $a \in \{\bar{a} + Hu \colon \|u\| \le 1\}$ for given $H \in S^n_+$ (pos. semidef.) and if x has to satisfy this inequality for all such a, this requires

$$\max_{\|u\|=1} \bar{a}^T x + u^T Hx = \bar{a}^T x + \|Hx\| \le b$$

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For its probabilistic interpretation let g be distributed normally around $\bar{g}$ with covariance matrix $G = H^2$ and suppose $g^T x \ge \underline{s}$ needs to be satisfied with probability $0 < \eta < 1$. Then $\mathbb{P}(g^T x \ge \underline{s}) \ge \eta$ corresponds to the constraint $-\bar{g}^T x + \Phi^{-1}(\eta)\|Hx\| \le -\underline{s}$. [$\Phi$ ... normal distribution]

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For $A, B \in S^n$ we use the Frobenius inner product

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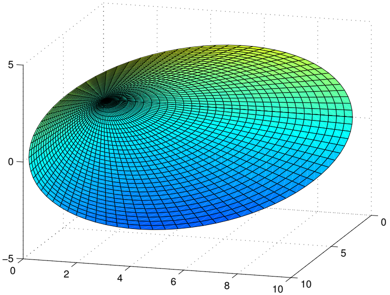
Theorem

For $A \in S^n$ the following are equivalent:

- $A \succeq 0$,
- $\lambda_i(A) \geq 0$, $i = 1, \dots, n$, [$\Rightarrow \det(A) \geq 0$]
- $A = C^T C$ for some $C \in \mathbb{R}^{k\times n}$, [there holds: $\mathrm{rank}(A) = \mathrm{rank}(C)$]
- $\langle A, B\rangle \geq 0 \quad \forall B \succeq 0$.

The Cone of Positive Semidefinite Matrices

The positive semidefinite matrices S_+^n form a convex cone, because for $X, Y \in S_+^n$, $\alpha \geq 0$ and all $v \in \mathbb{R}^n$
$v^T(\alpha(X+Y))v = \alpha(v^T X v + v^T Y v) \geq 0.$



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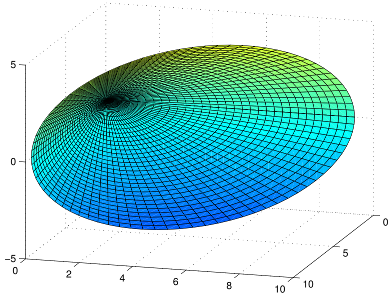

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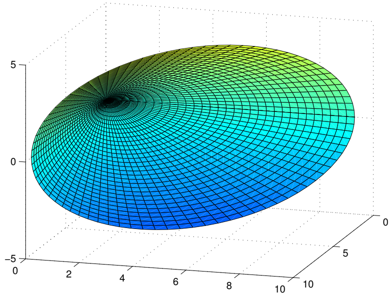

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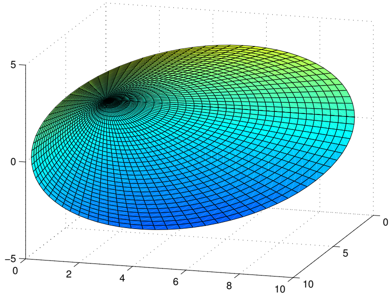

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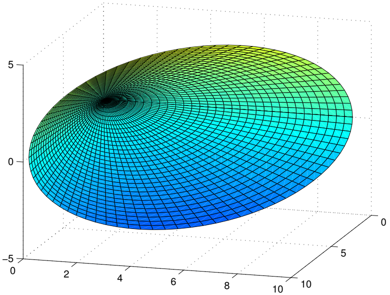

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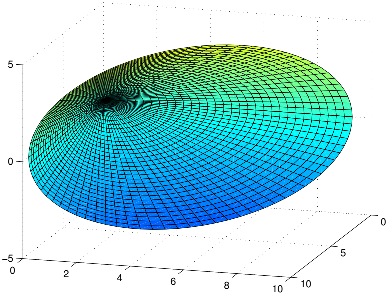

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Frequently used in formulating applications as semidefinite programs:

Theorem (Schur Complement)

For $A \in S^m_{++}$, $C \in S^n$ and $B \in \mathbb{R}^{m\times n}$ there holds

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \quad (\textit{resp.} \succ 0) \quad \Longleftrightarrow \quad C \succeq B^TA^{-1}B \quad (\textit{resp.} \succ 0)$$

LP $\leftrightarrow$ Semidefinite Programs (SDP)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \mathcal{A}X = b \\ & X \succeq 0 \end{aligned}$$

$$x \in \mathbb{R}^n_+ \qquad\qquad X \in S^n_+$$

$$c^T x = \sum_i c_i x_i \qquad\qquad \langle C, X \rangle = \sum_{i,j} C_{ij} X_{ij}$$

$$Ax = \begin{pmatrix} a_1^T x \\ \vdots \\ a_m^T x \end{pmatrix} \qquad\qquad \mathcal{A}X = \begin{pmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{pmatrix}$$

$$A^T y = \sum_i a_i y_i \qquad\qquad \mathcal{A}^T y = \sum_i A_i y_i$$

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y + z = c \\ & y \in \mathbb{R}^m, z \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & \mathcal{A}^T y + Z = C \\ & y \in \mathbb{R}^m, Z \succeq 0 \end{aligned}$$

Semidefinite Programs (SDP) in Normal Form

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(P) \quad \begin{array}{ll} \min & \langle C, X \rangle \\ \text{s.t.} & \mathcal{A}X = b \\ & X \succeq 0 \end{array}
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(D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & \mathcal{A}^T y + Z = C \\ & y \in \mathbb{R}^n, Z \succeq 0 \end{array}
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Semidefinite Optimisation also allows to formulate SOC-constraints:

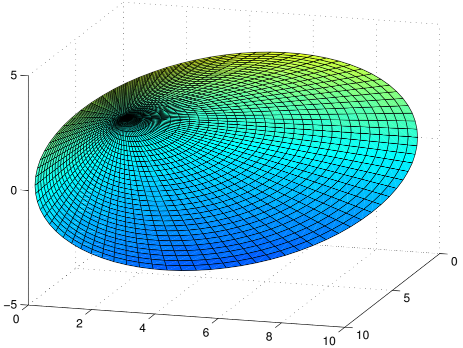
$$\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} \geq_{\mathcal{Q}} 0 \quad \overset{x_0 \geq 0}{\Leftrightarrow} \quad x_0 \geq \frac{1}{x_0} \bar{x}^T I \bar{x} \quad \overset{\text{Schur}}{\Leftrightarrow} \quad \begin{bmatrix} x_0 & \bar{x}^T \\ \bar{x} & x_0 I \end{bmatrix} \succeq 0.$$

[for $x_0 = 0$ this is checked directly]

Illustration: $X \in S^2_+$ intersected with $\langle A, X \rangle = \beta$

$X = \begin{bmatrix} x & z \\ z & y \end{bmatrix} \succeq 0.$

$\Rightarrow x \geq 0,\ y \geq 0,\ xy - z^2 \geq 0$



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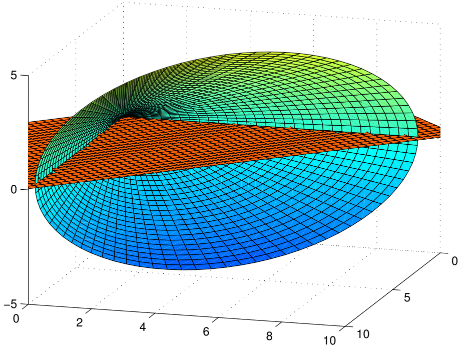

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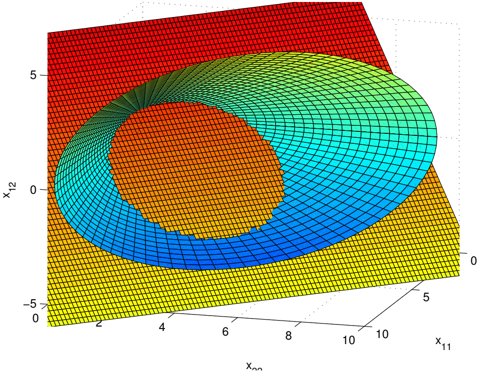

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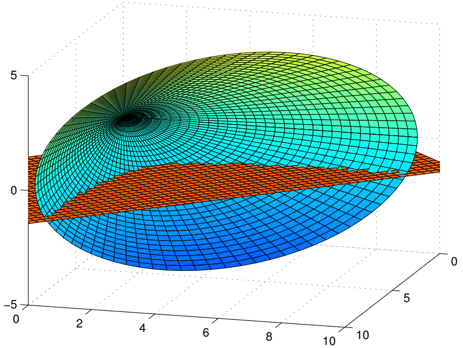

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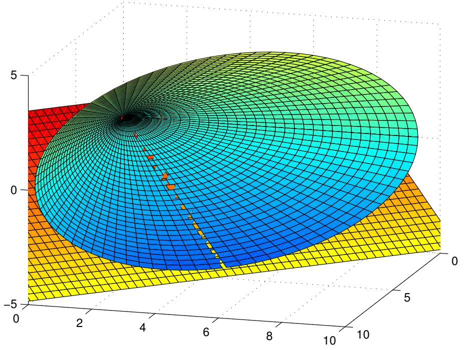
$A = vv^T, \beta = 0 \rightarrow v$ Evec to $\lambda_1 = 0$

boundary points, numerically difficult!



Linear Matrix Inequalities (LMI)

A constraint of the form

$$y_1 A_1 + y_2 A_2 + \cdots + y_m A_m \preceq C$$

with $A_i, C \in S^n$ is a Linear Matrix Inequality.

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Ex.: the Lyapunov inequality requires for fixed $P \in \mathbb{R}^{n \times n}$

$$P^T X + XP \prec 0, \quad X \succ 0.$$

In LMI-representation write $y = [x_{11}, x_{12}, \ldots, x_{1n}, x_{22}, x_{23}, \ldots, x_{nn}]^T$, but it is cumbersome/useless to list the A_i for this constraint. It is better to exploit the structure directly within SDP.

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For recognising LMIs it suffices to ensure that all matrices depend linearly on the corresponding variables:

the matrix multiplication $P^T X$ (XP resp.) is linear in X.

Applications of Semidefinite Optimisation

- optimal control
- eigenvalue optimisation
- experiment design in statistics
- combinatorial optimisation
- global optimisation over polynomials
- moment problems in probability theory
- signal processing
- robust truss topology design
- free material design
- robust optimisation
- optimisation (trust-region subproblems, quadratic relaxations)

Robust Stability of Dynamical Systems

Given a (homogenous linear) dynamical system with uncertain data,

$$(DS) \quad \dot{x} = P(t)x(t) \quad \text{with } P(t) \in \mathcal{P} := \text{conv}\{P_1, \ldots, P_k\} \subset \mathbb{R}^{n\times n},$$

where
- $x(t)$ . . . state of the system at time t.
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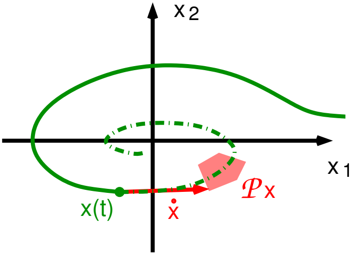
(DS) is stable if $x(t) \to 0$ for $t \to \infty$ and arbitrary $P(t) \in \mathcal{P}$.

[In optimal control, $\mathcal{P}$ would comprise the possible effects of imperfect implementations of the control. Does it do its job anyways even with tiny mistakes?]



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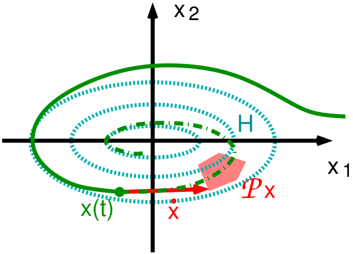
sufficient condition: there is a norm

$$\|x\|_H := \sqrt{x^T H x} \quad \text{with } H \succ 0$$

so that $\frac{d}{dt}\|x(t)\|_H^2 < 0$ for all trajectories

(the system is quadratically stable, $x^T Hx$ a quadratic Lyapunov Function).



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has feasible solutions, because for any such H also each convex combination $P \in \mathcal{P}$ satisfies the condition $P^T H + HP \prec 0$.

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Search for H by eigenvalue optimisation:

$$\max\ \lambda \text{ s.t. } H \succeq \lambda I, \quad P_i^T H + HP_i \preceq -\lambda I \quad \text{for } i = 1, \dots, k.$$

SDP and Eigenvalue Optimisation

For $A \in S^n$ let $\lambda_{\min}(A) := \lambda_1(A) \leq \cdots \leq \lambda_n(A) =: \lambda_{\max}(A)$.
There holds $\lambda_i(A + y_0 I) = \lambda_i(A) + y_0$ for $i = 1, \ldots, n$ and $y_0 \in \mathbb{R}$.

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For affine $A(y)$, e.g. $A(y) := C - \sum_{i=1}^m y_i A_i$ with $C, A_i \in S^n$, this leads to

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To model this as SDP: $\lambda_{\max}(A) = -\lambda_{\min}(-A)$ and

$$y_0 \geq \lambda_{\max}(C - \mathcal{A}^T y) \;\Leftrightarrow\; y_0 + \lambda_{\min}(\mathcal{A}^T y - C) \geq 0 \;\Leftrightarrow\; \lambda_{\min}(y_0 I + \mathcal{A}^T y - C) \geq 0$$

Because $Z \succeq 0 \;\Leftrightarrow\; \lambda_{\min}(Z) \geq 0$ we have

$$\min_{y \in \mathbb{R}^m} \lambda_{\max}(C - \mathcal{A}^T y) \quad \Leftrightarrow \quad \begin{array}{ll} \min & y_0 \\ \text{s.t.} & Z = y_0 I + \mathcal{A}^T y - C \\ & y \in \mathbb{R}^m, Z \succeq 0 \end{array}$$

Design of Experiments

In order to estimate the value of some parameter vector $\xi \in \mathbb{R}^p$,
a set $\mathcal{R} = \{r_i \in \mathbb{R}^p \colon i = 1, \ldots, n\}$ of possible experiments are available.
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If m experiments $a_j \in \mathcal{R}$ (repetitions are allowed) are performed resulting in measurements $\eta_j = a_j^T \xi + \rho_j$, the maximum-likelihood estimate (for $\text{rank}[a_1, \ldots, a_m] = p$) yields an estimated

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Let G and G' be two covariance matrices of this kind and suppose $G \preceq G'$, then the experiments of G are better, because variance of the estimation error is smaller.
$\rightarrow$ Find a minimal (w.r.t. $\preceq$) element of

$$\left\{ G = \left(\sum_{i=1}^{n} m_i r_i r_i^T \right)^{-1} \colon m_i \in \mathbb{N}_0, \sum_i m_i = m \right\}.$$

Relaxations

Rather than selecting m experiments, determine their relative contribution,

$$\left\{ G = \left(\sum_{i=1}^{n} \alpha_i r_i r_i^T \right)^{-1} : \mathbf{1}^T \alpha = 1, \alpha \geq 0 \right\}.$$

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There are several approaches for finding a $\preceq$-minimal G. For this, interpret G as a "confidence ellipsoid" with semi axes of length $\lambda_j(G)$,

$$\mathcal{E} = \{\zeta \colon (\zeta - \hat{\xi})^T G^{-1} (\zeta - \hat{\xi}) \leq \beta\}.$$

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D-**optimal design:** the volume is proportional to $\det G = \prod \lambda_j(G)$.
Because $\det(G^{-1}) = \det(G)^{-1} \Leftrightarrow$ maximise the determinant of G^{-1},

$$\begin{array}{ll} \min & -\log\det X \\ \text{s.t.} & X = \sum_{i=1}^{n} \alpha_i r_i r_i^T \\ & \mathbf{1}^T\alpha = 1 \\ & \alpha \geq 0, [X \succ 0] \end{array}$$

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E-**optimal design:** the longest semi axis is $\lambda_{\max}(G)$.
Because $\lambda_{\min}(G^{-1}) = \lambda_{\max}(G)^{-1} \Leftrightarrow$ maximise $\lambda_{\min}(G^{-1})$,

$$\begin{array}{ll} \max & -\lambda \\ \text{s.t.} & \sum_{i=1}^{n} \alpha_i r_i r_i^T \succeq \lambda I \\ & \mathbf{1}^T \alpha = 1 \\ & \alpha \geq 0, \lambda \in \mathbb{R} \end{array}$$

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A-**optimal Design:** $\sum_{j=1}^p \lambda_j(G) = \sum_{j=1}^p G_{jj} = \sum_{j=1}^p e_j^T G e_j$.
For each j represent the inequality $u_j \succeq e_j^T G e_j$ by its Schur complement:

$$\begin{aligned} \min \quad & \mathbf{1}^T u \\ \text{s.t.} \quad & \begin{bmatrix} \sum_{i=1}^n \alpha_i r_i r_i^T & e_j \\ e_j^T & u_j \end{bmatrix} \succeq 0, \quad j = 1, \ldots, p \\ & \mathbf{1}^T \alpha = 1, \alpha \geq 0, u \in \mathbb{R}^p \end{aligned}$$

Graph Partition: Max-Cut

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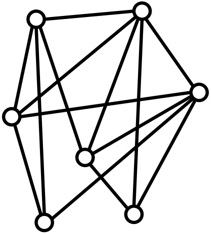
Given: graph $G = (V, E)$, $V = \{1, \dots, n\}$,
$E \subseteq \{ij \colon i, j \in V, i < j\}$, edge weights a_{ij}

Find: $S \subset V$ with maximum weight cut
$\delta(S) := \{ij \in E \colon i \in S, j \in V \setminus S\}$



$$\text{(MC)} \qquad \max_{S \subseteq V} \sum_{ij \in \delta(S)} a_{ij} \qquad [NP\text{-compl.}]$$

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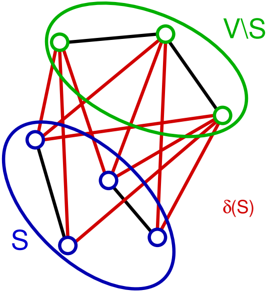

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$$\text{then } x_i x_j = \begin{cases} -1 & ij \in \delta(S) \\ 1 & \text{otherwise} \end{cases}, \quad \text{or} \quad \frac{1 - x_i x_j}{2} = \begin{cases} 1 & ij \in \delta(S) \\ 0 & \text{otherwise} \end{cases}$$

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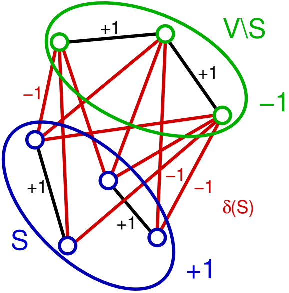

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Model: represent the partition by

$$x \in \{-1, 1\}^n \quad \text{with} \quad x_i = \begin{cases} 1 & i \in S \\ -1 & i \in V \setminus S \end{cases}$$

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$$\max_{S \subseteq V} \sum_{ij \in \delta(S)} a_{ij} = \max_{x \in \{-1,1\}^n} \sum_{ij \in E} a_{ij} \frac{1 - x_i x_j}{2} \qquad \to \qquad \max_{x \in \{-1,1\}^n} x^T C x$$

[$C \in S^n$: $C_{ii} = \frac{1}{4} \sum_{j:\, ij \in E} a_{ij}$ (for $i \in V$), $C_{ij} = -\frac{1}{4} a_{ij}$ (for $ij \in E$), 0 otherw.]

Equivalent to quadratic 0-1 optimisation!

Semidefinite Max-Cut Relaxation

Observe: $x^T C x = \langle Cx, x \rangle = \left\langle C, xx^T \right\rangle$

Properties of $xx^T = [x_i x_j]$ for $x \in \{-1, 1\}^n$:

- $x_i^2 = 1 \quad \Rightarrow \quad \text{diag}(xx^T) = \mathbf{1}$
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Illustration for $n = 3$:

boundary described by

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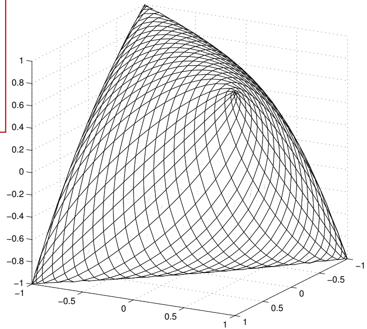

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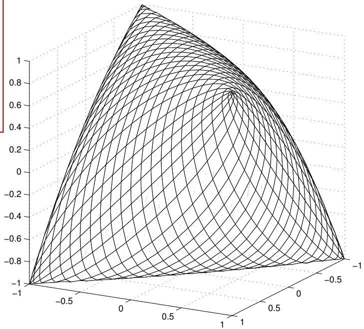

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[Approx.-alg. of GW95: factorise X, use randomised hyperplane rounding]

Moment Matrices and Optimisation Over Polynomials

[Lasserre]

Polynomial $p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} > -\infty$ of degree $2m$. Find

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exact $\Leftrightarrow$ $p(x) - p_*$ is a sum of squares of polynomials (SOS).

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Try to find $\min_x p(x) = p_*$ via maximising p_0,

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$$\begin{aligned} \max \quad & \langle B_0, H \rangle \ [= -H_{0,0} = -p_0] \\ \text{s.t.} \quad & \langle B_\alpha, H \rangle = p_\alpha \quad \alpha \neq 0 \\ & H \succeq 0. \end{aligned}$$

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The approach is extendable to sets restricted by polynomial (in)equalities
$\rightarrow$ GloptiPoly

If $p(x) - p_*$ is not SOS ...

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For equality restrictions $x_i^2 = x_i$ ($x \in \{0,1\}^n$) the initial relaxation is equivalent to the semidefinite Max-Cut relaxation.

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The complex hermitian case provided excellent results for optimal power flow [LavaeiLow2012]

Contents

Duality Gap Example

$$
\begin{array}{ll}
\min & x_{12} \\
\text{s.t.} & \begin{bmatrix} 0 & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1 + x_{12} \end{bmatrix} \succeq 0
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corresponding coefficient matrices:

$$C = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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$x_{11} = 0 \quad \Rightarrow \quad x_{12} = 0,$ primal optimal value is 0.

$z_{22} = 0 \quad \Rightarrow \quad \frac{1+y_1}{2} = 0,$ dual optimal value is -1.

Difficulty: Primal Problem is Unstable

$$\min \quad x_{12}$$
$$\text{s.t.} \begin{bmatrix} \varepsilon & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1+x_{12} \end{bmatrix} \succeq 0$$

$$\max \quad y_1 + \varepsilon y_2$$
$$\text{s.t.} \quad Z = \begin{bmatrix} -y_2 & \frac{1+y_1}{2} & -y_3 \\ \frac{1+y_1}{2} & 0 & -y_4 \\ -y_3 & -y_4 & -y_1 \end{bmatrix} \succeq 0$$

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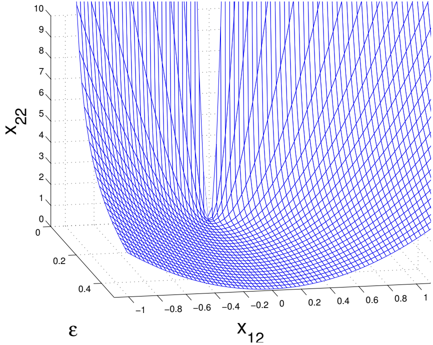
$$\begin{aligned} \max \quad & \langle C, X \rangle \\ & \langle A_1, X \rangle = 1 \\ & \langle A_2, X \rangle = \varepsilon \\ & \langle A_3, X \rangle = 0 \\ & \langle A_4, X \rangle = 0 \\ & X \succeq 0 \end{aligned}$$

$x_{33} \geq 0 \;\Rightarrow\; x_{12} \geq -1,\; x_{22} \geq \frac{x_{12}^2}{\varepsilon}$, primal optimal value is -1.

$z_{22} = 0 \;\Rightarrow\; \frac{1+y_1}{2} = 0,\; y_2 = 0$, dual optimal value is -1.

$$\begin{bmatrix} \varepsilon & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1 + x_{12} \end{bmatrix} \succeq 0 \quad \Leftrightarrow \quad \begin{cases} \varepsilon > 0 & x_{12} \geq -1 \quad x_{22} \geq \frac{x_{12}^2}{\varepsilon} \\ \varepsilon = 0 & x_{12} = 0 \quad\quad x_{22} \geq 0 \end{cases}$$



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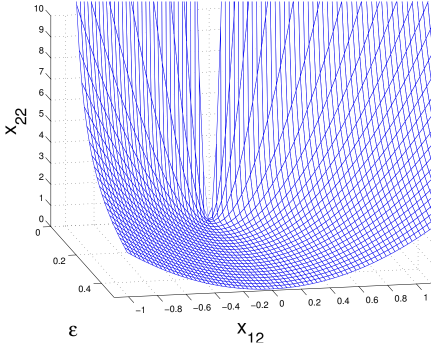

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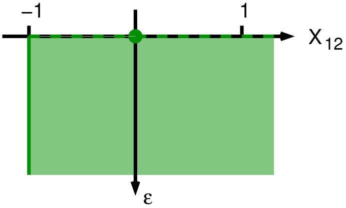

For $\varepsilon > 0$ all $x_{12} \in [-1, -\infty)$,
for $\varepsilon = 0$ only $x_{12} \in \{0\}$
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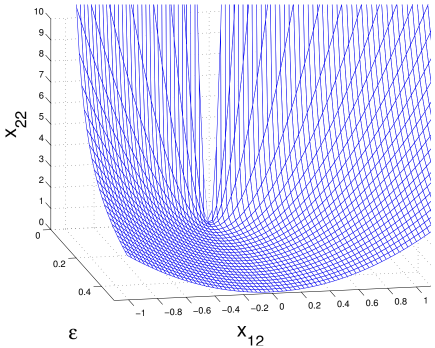

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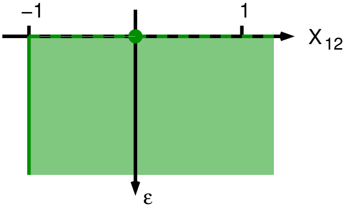

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mathematical reason: the set $\left\{ \begin{bmatrix} \langle C, X \rangle \\ \mathcal{A}X \end{bmatrix} : X \succeq 0 \right\}$ is not closed.

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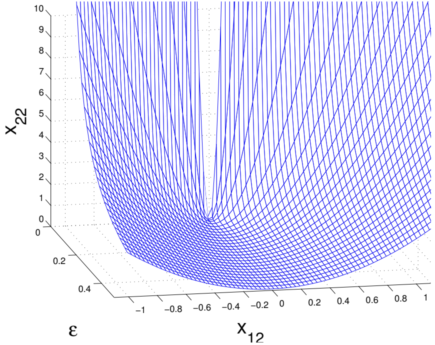

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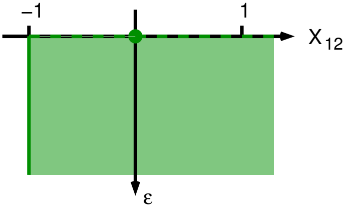

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To avoid this, require strictly feasible points or apply facial reduction.

Facial Structure and Facial Reduction

The faces of S^n_+ are: $\emptyset$, $\{\mathbf{0}\}$ and

for each r-dim. linear subspace $\mathcal{L}$ of $\mathbb{R}^n$ represented by some basis $P \in \mathbb{R}^{n\times r}$,

$F_{\mathcal{L}} = \{X = PUP^T : U \in S^r_+\}$

[Barker and Carlson 1975]

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Feasible $\begin{bmatrix} 0 & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1+x_{12} \end{bmatrix} \succeq 0$ live on the face $F_{\mathcal{L}}$ for $P = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$,
the dual $Z \in F^*_{\mathcal{L}}$ requires positive semidefiniteness on this subspace

$$\begin{aligned} \min \quad & x_{12} \\ \text{s.t.} \quad & \begin{bmatrix} 0 & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1+x_{12} \end{bmatrix} \in F_{\mathcal{L}} \end{aligned} \qquad \begin{aligned} \max \quad & y_1 \\ \text{s.t.} \quad & P^TZP = \begin{bmatrix} 0 & -y_4 \\ -y_4 & -y_1 \end{bmatrix} \succeq 0 \end{aligned}$$

Facial reduction ensures primal strict feasibility $\rightarrow$ both optima 0

Complexity of SDP

The feasibility problem in the real number model is in $NP \cap Co\text{-}NP$, in the Turing- or bit-model the status is still open!

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doubly exponential values!

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polynomial for ε-solutions in bounded regions (ellipsoid method) [Grötschel Lovász Schrijver 1988]

Contents

Solution Methods

- interior-point methods ("polynomial")
  codes: SDPT3, Sedumi, SDPA, CSDP, Mosek, ...
- penalty methods
  code: Pennon
- spectral bundle method ($f(y) := \lambda_{\max}(C - \mathcal{A}^T y) + b^T y$)
  code: ConicBundle
- quadratic reformulations (replace $0 \preceq X = LL^T$)
  code: SDPLR

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based on the barrier idea take from *nonlinear optimisation*:

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barrier-function for $Z \in S_n^+$: $-\log\det Z$

- $\det Z = \prod \lambda_i(Z)$ is > 0 in the interior of S_n^+ and 0 on the boundary
- $\log\det Z = \sum \log \lambda_i(Z)$
- if some $\lambda_i(Z) \to 0$ then $-\log\det Z \to \infty$
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SUMT, Fiacco and McCormick 1968: solve a sequence of barrier-problems

$$\min_y \; b^T y - \mu \log\det(\underbrace{\mathcal{A}^T y - C}_{=Z})$$

by Newton's method for $\mu > 0$, $\mu \to 0$.

Minimise $f(y) = b^T y - \mu \log\det(\mathcal{A}^T y - C)$ by Newton:

1. first order necessary (here sufficient) conditions

$$\nabla f(y) = 0$$

2. determine step Δy so that the linearisation in the current point y_c,

$$\nabla f(y_c) + \nabla^2 f(y_c)^T \Delta y = 0,$$

   satisfies the conditions for $y_c + \Delta y$.
   [= minimises the quadratic model of f]

3. damped Newton step: $y_+ = y_c + \alpha\Delta y$ with $\alpha \in (0, 1]$
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Assumptions:

(A) $\mathcal{A}$ has full row rank. [w.l.o.g.]

(S) There exist primal and dual strictly feasible solutions.

Minimise $f(y) = b^T y - \mu \log\det(\mathcal{A}^T y - C)$ by Newton:

1. Sufficient first order optimality conditions:

$\nabla f(y) = 0$, use $\nabla_Z \log\det Z = Z^{-1}$

$$\boxed{b - \mathcal{A}[\mu(\mathcal{A}^T y - C)^{-1}] = 0}$$

yields methods of Jarre 1993, Nesterov and Nemirovskii 1994

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[primal feasibility]
[dale feasibility]
[perturbed complementarity]

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[Maximising the determinant ($\mu = 1$) is relevant on its own!]

2. Primal-dual Linearisation

$$
\begin{array}{rc}
\mathrm{I} & b - \mathcal{A}(X + \Delta X) = 0 \\
\mathrm{II} & Z + \Delta Z = \mathcal{A}^T(y + \Delta y) - C \\
\mathrm{III} & XZ + X\Delta Z + \Delta X Z = \mu I
\end{array}
$$

Solution $(\Delta X, \Delta y, \Delta Z)$ is the step direction/Newton step

Difficulty: ΔZ is symmetric [II] but i.g. ΔX is not [III]

2. Primal-dual Linearisation

$$\begin{array}{ll} \text{I} & b - \mathcal{A}(X + \Delta X) = 0 \\ \text{II} & Z + \Delta Z = \mathcal{A}^T(y + \Delta y) - C \\ \text{III} & XZ + X\Delta Z + \Delta X Z = \mu I \end{array}$$

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Difficulty: ΔZ is symmetric [II] but i.g. ΔX is not [III]
Suggestions:

(a) HRVW96/KSH97/M97: use the symmetric part,

$$\frac{1}{2}(\Delta X + \Delta X^T)$$

(b) NT97: scale III by the matrix $W = X^{\frac{1}{2}}(X^{\frac{1}{2}} Z X^{\frac{1}{2}})^{-\frac{1}{2}} X^{\frac{1}{2}}$,

$$W^{-1} \Delta X W^{-1} + \Delta Z = \mu X^{-1} - Z$$

(c) AHO98: symmetrise III directly,

$$X\Delta Z + \Delta Z X + \Delta X Z + Z \Delta X = 2\mu I - XZ - ZX$$

Several others exist, all differ slightly.(Todd 1999).

Algor. Scheme for Primal-Dual Interior-Point Methods

Input: $\mathcal{A}$, b, C, starting point (X^0, y^0, Z^0) with $X^0 \succ 0$ and $Z^0 \succ 0$

1. Choose $\mu = \sigma \frac{\langle X,Z \rangle}{n}$ with $\sigma \in (0, 1]$.
2. Compute $(\Delta X, \Delta y, \Delta Z)$.
3. Line search: determine $\alpha \in (0, 1]$ with $X + \alpha \Delta X \succ 0$ and $Z + \alpha \Delta Z \succ 0$.
4. Put $(X, y, Z) := (X + \alpha \Delta X, y + \alpha \Delta y, Z + \alpha \Delta Z)$.
5. If $\|\mathcal{A}X - b\|$, $\|\mathcal{A}^T y + Z - C\|$ and $\langle X, Z \rangle$ "sufficiently small", **stop**, otherwise **goto** 1.

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Theorem (Kojima, Shindoh and Hara 1997)

(X^0, y^0, Z^0) *feasible and "centred". Choose* $\sigma = 1 - \frac{0.35}{\sqrt{n}}$ *and* $\alpha = 1$, *then each step is feasible and* $\langle X, Z \rangle < \varepsilon$ *in*

$$\boxed{O\left(\sqrt{n} \log \frac{\langle X^0, Z^0 \rangle}{\varepsilon}\right)} \text{ iterations.}$$

[$O(n^2)$ Variable!]

Work per Iteration

Computation of the HRVW/KSH/M step:

$$\Delta Z = \mathcal{A}^T(y + \Delta y) - C - Z$$
$$\Delta X = \mu Z^{-1} - X - X \Delta Z Z^{-1}$$

solve $\qquad \mathcal{A}(X \mathcal{A}^T(\Delta y) Z^{-1}) = M \Delta y = \ldots$

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X and Z^{-1} are in general dense.

$\Rightarrow$ M is a dense positive definite matrix of order m.
Cholesky factorisation needs $m^3/3$ flops and $O(m^2)$ memory.

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In practice: split matrices into blocks exploiting semidefinite completion techniques, exploit structure induced by these blocks, solve the big system without M via iterative methods, ...

Some General References

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