Solitary Waves for the Nonlinear Schrödinger Equation

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Abstract

This report is dedicated to the study of the solitary waves for the nonlinear Schrödinger equation (NLS). The NLS equation is known as the dispersive partial differential equation. By dispersion, we mean that waves of different wavelengths propagate at different phase velocities.

Our study is focused on the solitary wave solutions of NLS. The solution is a shape-preserving wave when propagating at a constant velocity. It results from a balance of dispersive and nonlinear effects. To achieve our study, we shall divide it into two parts: (i) the existence of solitary wave solutions and (ii) the stability of solitary wave solutions.

Before we start discussing the solitary waves, we will first briefly go over the well-posedness results of NLS. These results provide assurance of the existence of solutions to NLS in certain function spaces. Then, by viewing the well-posedness results, we shall specifically look into the existence of solitary wave solutions. Moreover, we also study their stability, which by considering the nonlinear interaction, we will split into stable and unstable cases.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Yan Ding*)

I would like to express my deepest gratitude to the supervisor, Dr.Guopeng Li, whose guidance, expertise, and unwavering support have been invaluable throughout this journey.
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Contents

Chapter 1

Introduction

1.1 Background

The phenomenon of solitary waves was initially uncovered in 1834 by the renowned civil engineer John Scott Russel[1]. His discovery was prompted by the observation of a swiftly moving boat in a canal, where he discerned that the abrupt cessation of the boat's movement generated a wave that persisted in its trajectory along the canal with seemingly unaltered shape and speed. In 1895, the first theoretical justification for the existence of solitary waves was elucidated by Korteweg[2] and De Vries[3], who derived an equation accounting for the motion of water that admitting solitary wave solutions. This equation has since become widely known as the Korteweg-De Vries (KDV) equation.

In fact, solitary waves emerge as a consequence of the balance of two competing effects: the dispersive effect of the linear component and the focusing effect of the nonlinear component. The former effect strives to flatten the solution over time, whereas the latter aims to concentrate the solution. So far, numerous nonlinear model equations have since been demonstrated to possess solitary wave solutions, including the Klein-Gordon, Kadomtsev-Petviashvili and Nonlinear Schrödinger equations (NLS).

This report focuses on the study of solitary waves for NLS. This topic has garnered significant attention owing to its broad-ranging applications in both Physics and Mathematics, including the study of light propagation in nonlinear optical fibers, fluid dynamics, and plasma physics. See [1, 2, 4, 12] for more examples.

To study solitary waves of NLS, we begin by considering the Cauchy problem of the following NLS:

$$\begin{cases} i\partial_t v + \Delta v = f(v) \\ v(x,0) = v_0 \end{cases} \qquad (x,t) \in \mathbb{R}^n \times I, \tag{1.1}$$

[1]John Scott Russell (9 May 1808, Parkhead, Glasgow – 8 June 1882, Ventnor, Isle of Wight).
[2]Diederik Johannes Korteweg (31 March 1848 – 10 May 1941) was a Dutch mathematician.
[3]Gustav de Vries (22 January 1866 – 16 December 1934) was a Dutch mathematician.

where $i = \sqrt{-1}, \Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ and $I = (-T, T)$ with $T > 0$. Here, the function $f(v)$ represents the nonlinearity of the equation. Note that the nonlinearity of the NLS can result in either a focusing or defocusing effect. For instance, Let us consider the case when f takes the form $f(v) = \lambda |v|^{p-1} v$. In this scenario, the NLS is referred to as defocusing when $\lambda > 0$, whereas it is called focusing when $\lambda < 0$. As previously discussed, the solitary waves arise from a balance between the dispersive and focusing effects of the equation. Therefore, we will restrict our attention to the focusing NLS in order to study the solitary waves.

It is well-known that when f satisfies certain conditions, NLS (1.1) admits two conserved quantities, namely mass and energy. Specifically, let v be a smooth solution of NLS (1.1) and f satisfies certain assumptions [See Lemma 2.2.1], the following two quantities (mass and energy) are conserved for v:

$$M(v) := \int |v|^2 dx = \|v\|_{L^2}^2,$$
$$E(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 + \int_{\mathbb{R}^n} F(|v(x)|) dx,$$

where F is defined as

$$F(s) = \int_0^s f(\tau) d\tau, \quad s \in \mathbb{R}.$$

By means of conservation, we have $\partial_t M(v) = 0$ and $\partial_t E(v) = 0$. In particular,

$$M(v) = M(v_0), \qquad E(v) = E(v_0). \tag{1.2}$$

The use of these conservation laws enables us to study the global-in-time dynamics. In particular, let v be a solution to NLS (1.3) defined on a specific function space over the time interval I. If v is the unique solution to NLS (1.3) and is continuous with respect to the initial condition v_0, then we can say that NLS (1.3) is locally well-posed in this function space [See proposition 2.2.2]. The conservation laws (1.2) now would allow us to extend this local solution globally in time. Roughly speaking for some appropriate nonlinearity, if the solution v lies in the Sobolev space H^1, we have $M + E \sim H^1$. Therefore they provide a global control on the H^1-solution, which implies that our solution exists for any $T > 0$. This is known as the global well-posedness of NLS (1.3) [See proposition 2.2.4].

After verifying the existence of solutions to NLS (1.1), we discuss our main topic: **solitary wave solutions of NLS**.

Let $f(v) = -|v|^{p-1} v$ with $p > 1$. From (1.1), we have the following NLS associated with a polynomial nonlinearity:

$$i\partial_t v + \Delta v = -|v|^{p-1} v. \tag{1.3}$$

In studying the nonlinear dispersive equations, the behaviours of solutions are determined by the interplay between nonlinear and dispersive effects. As the parameter p increases, the nonlinear effect becomes stronger and the behaviour of solutions changes accordingly. Therefore, when studying solitary wave solutions, different ranges of p can yield different results. In particular, Strass [10] established the existence of non-trivial solitary wave solutions to NLS (1.3) when $1 < p < p^*(n)$, where $p^*(n) = \infty$ for $n = 1, 2$ and $p^*(n) = \frac{n+2}{n-2}$ for $n \geqslant 3$, and showed that no non-trivial solitary wave solution exists when $p \geqslant p^*(n)$. For stability, Cazenave and Lions [3] demonstrated that the ground state solution of NLS (1.3) is stable when $1 < p < 1 + \frac{4}{n}$ and unstable when $1 + \frac{4}{n} < p < p^*(n)$. Weinstein [14] proved the instability also holds when $p = 1 + \frac{4}{n}$. In this report, we concentrate on two cases: $1 < p < 1 + \frac{4}{n}$ and $1 + \frac{4}{n} < p < p^*(n)$.

Solitary wave solutions refer to a specific type of solutions represented by the form:

$$v(x, t) = e^{i\eta t} u(x), \tag{1.4}$$

where $u(x)$ denotes the solutions of the following nonlinear elliptic problem:

$$-\eta u + \Delta u = -|u|^{p-1} u, \tag{1.5}$$

for $u \not\equiv 0$ and $\eta > 0$. We can verify $v(x, t)$ satisfy NLS (1.3) by straightforward calculations. Substituting (1.4) into (1.3) yields

$$\begin{aligned} -\eta e^{i\eta t} u(x) + e^{i\eta t} \Delta u(x) &= -|e^{i\eta t} u(x)|^{p-1} e^{i\eta t} u(x), \\ -\eta u(x) + \Delta u(x) &= -|e^{i\eta t} u(x)|^{p-1} u(x), \end{aligned}$$

which is equivalent to (1.5) since $|e^{i\eta t}| = |\cos(\eta t) + i \sin(\eta t)| = \sqrt{\cos^2(\eta t) + \sin^2(\eta t)} = 1$. Moreover,these solutions are also called the **standing wave solutions of NLS (1.3).** As a consequence, we can prove the existence of standing wave solutions to NLS (1.3) by demonstrating the existence of the solutions to the nonlinear elliptic problem (1.5).

1.2 Main results

The first goal of this report is to establish the existence of solutions to (1.5). Our approach to proving the existence involves using variational methods. Specifically, we convert the elliptic problem (1.5) into a constrained minimization problem and then prove the existence of the solution to this minimization problem while subject to specific constraints.

We consider two different minimization problems in our analysis, depending on the range of p values. When $1 < p < 1 + \frac{4}{n}$, the minimization problem we investigate is

$$b_\alpha = \inf_{u \in K_\alpha} S_\eta(u), \tag{1.6}$$

where $\eta > 0$, $\alpha > 0$ and the definitions of S_η and K_α are given as

$$S_\eta(u) = \|\nabla u\|_{L^2}^2 + \eta \|u\|_{L^2}^2 - \frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1}, \tag{1.7}$$

$$K_\alpha = \{u \in H^1; \|u\|_{L^2} = \sqrt{\alpha}\}. \tag{1.8}$$

On the other hand, when $p < 1 + \frac{4}{n} < p^*(n)$, we consider a different minimization problem with the same functional $S_\eta(u)$ but with different constraints:

$$c_\eta = \inf_{u \in K} S_\eta(u), \tag{1.9}$$

where $\eta > 0$ and K is defined as

$$K = \{u \in H^1;\ u \neq 0,\, T(u) = 0\},$$

with the expression of $T(u)$ given as follows:

$$T(u) = 2\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Our main result about the existence is the following:

Theorem 1 (existence)**.** Let $n \geqslant 2$, $\eta > 0$. Then, the following statements hold.

(i) Let $1 < p < 1 + \frac{4}{n}$ and $\alpha > 0$. Then, there exists $w \in K_\alpha \cap H_r^1$ such that $0 > b_\alpha > -\infty$. Moreover, this function w is a weak solution to (1.5) for some $\eta > 0$.

(ii) Let $1 + \frac{4}{n} < p < p^*(n)$. Then, $c_\eta > 0$ and there exists $w \in K \cap H_r^1$ attaining the minimum value c_η in (1.9). In this case, the function w is a weak solution to (1.5).

Theorem 1 shows that, for both $1 < p < 1 + \frac{4}{n}$ and $1 + \frac{4}{n} < p < p^*(n)$, there exists a standing wave solution to NLS (1.3). After establishing the existence of the standing wave solutions, we examine their stability. For this purpose, we will need the following definitions of stability:

Definition 1.2.1 ([13] Definition of stability restricted to H_r^1)**.** A ground state solution v is stable under radially symmetric perturbations if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &u_0 \in H_r^1,\ \|u_0 - v(0)\|_{H^1} < \delta \\ \Rightarrow &\inf_{\theta \in \mathbb{R}} \|(u(t) - e^{i\theta} v(t)\|_{H^1} < \epsilon, \quad t > 0. \end{aligned}$$

Definition 1.2.2 ([13] Orbital stablity)**.** A standing wave solution v is orbitally stable if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$u_0 \in H^1,\ \|u_0 - v(\cdot, 0)\|_{H^1} < \delta$$

$$\Rightarrow \inf_{\theta \in \mathbb{R}; y \in \mathbb{R}^n} \|(u(\cdot, t) - e^{i\theta} v(\cdot + y, t)\|_{H^1} < \epsilon, \quad t > 0,$$

where u denotes the solution to NLS (1.3) with initial condition $u\,|_{t=0} = u_0$.

Now, we are ready to present the main result about the stability of standing wave solutions to NLS (1.3).

Theorem 2 (Stability)**.** Let $n \geqslant 2$, $\eta > 0$. Then, the following statements hold.

(i) Let $1 < p < 1 + \frac{4}{n}$. Then, a ground state solution v to NLS (1.3) is stable in the sense of Definition 1.2.1.

(ii) Let $1 + \frac{4}{n} < p < p^*(n)$. Then, a ground state solution v to NLS (1.3) is orbitally unstable.

Theorem 2 states that a ground state solution to NLS (1.3) is stable under radially symmetric perturbations for $1 < p < 1 + \frac{4}{n}$ and orbitally unstable for $1 + \frac{4}{n} < p < p^*(n)$. To prove the stability, we utilize various lemmas, including Rellich's compactness theorem, the uniqueness of the standing wave solutions, and the conservation laws satisfied by solutions to NLS (1.3). In the case where instability occurs, we demonstrate the non-existence of global-in-time solutions to NLS (1.3) and conclude that the standing wave solutions are unstable by blow-up.

The report is organized as follows:

Chapter 2 serves as an introduction to the preliminaries required for the subsequent discussion on standing wave solutions. The chapter is divided into three sections:

- In the first section, we display notations and some important background materials from functional analysis.
- The second section focuses on the Cauchy problem of NLS (1.1). We present known results regarding the local/global well-posedness of the problem, which are essential for comprehending the subsequent discussion on standing wave solutions. Furthermore, these results are also used to prove our main theorems.
- The final section of the chapter is dedicated to the study of variational methods. Here, we demonstrate the propositions and lemmas that elucidate the process of transforming the nonlinear elliptic problem (1.5) into a minimization problem.

In chapter 3, we investigate the existence and stability of standing wave solutions to NLS (1.3). Our study is divided into two distinct sections, based on the range of the parameter p:

- Section 3.1 focuses on the case where $1 < p < 1 + \frac{4}{n}$ and aims to prove the first part of both main theorems.
- In Section 3.2, we consider the case where $1+\frac{4}{n} < p < p^*(n)$ and concentrate on proving the second part of the main theorems.

Chapter 2

Preliminaries

In this chapter, we first introduce the notations and the background materials from functional analysis. We then focus on the Cauchy problem of NLS (1.1) and associated lemmas, which are essential to present the proof of the main theorems accurately. Additionally, the variational method is presented. It is a powerful method for proving the existence of solutions to a wide range of nonlinear partial differential equations and has been used extensively in the study of nonlinear dispersive equations. A general overview of variational methods can be found in [11].

2.1 Notations and Functional analysis

First of all, we introduce the main notation that will be used throughout.

$C > 0$ stands for the universal constant, which varies at different occurrences. We denote $a \lesssim b$ if $a \leqslant Cb$ for some constant $C > 0$.

$L^p(\Omega)$ denotes the Banach space of measurable functions $u : \Omega \to \mathbb{R}$(or $\mathbb{C}$) such that the norm $\|u\|_{L^p(\Omega)} < \infty$ where for $1 \leqslant p < \infty$

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}},$$

and for $p = \infty$

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{Ess\,sup}_{x\in\Omega} |u|.$$

We use the notation $L^p \equiv L^p(\mathbb{R}^n)$. Moreover, it is a Hilbert space when $p = 2$ and for $u, v \in L^2$ the inner product is given as

$$(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}dx.$$

$H^1(\Omega)$ denotes the L^2-based Sobolev space on Ω, which is defined as

$$H^1(\Omega) = \{u \in L^2(\Omega); \partial_x u \in L^2(\Omega)\}.$$

It is equipped with the norm

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u(x)|^2 + |\partial_x u|^2 dx\right)^{\frac{1}{2}}.$$

and the scalar product

$$(u,v)_{H^1(\Omega)} = \int_\Omega \operatorname{Re}\left(u(x)\overline{v(x)}\right) dx.$$

We use the notation $H^1 \equiv H^1(\mathbb{R}^n)$, and $H^{-1} = (H^1)^*$ denotes the dual of H^1.

Additionally, a real Hilbert space H^1_{real} and its inner product are defined as $H^1_{\text{real}} = \{u : u \in H^1\}$ and

$$(u,v)_{H^1_{\text{real}}} = \operatorname{Re}\int_{\mathbb{R}^n} \left(u(x)\overline{v(x)} + \nabla u(x)\cdot\overline{\nabla v(x)}\right) dx, \quad u,v \in H^1_{\text{real}}.$$

Note that although the Hilbert space H^1_{real} is comprised of complex-valued functions in H^1, it is considered to be a real Hilbert space, as the coefficients are taken to be real numbers. It is analogous to regarding $\mathbb{C}$ as $\mathbb{R}^2$ by separating into the real and imaginary parts.

Moreover, we use the notation H^1_r and L^p_r to denote the spaces H^1 and L^p with the property of radial symmetry, respectively. That is,

$$H^1_r = \{u \in H^1; u \text{ is radially symmetric}\},$$
$$L^p_r = \{u \in L^p; u \text{ is radially symmetric}\}.$$

Here, a function u defined on $\mathbb{R}^n$ is considered radially symmetric if it satisfies the condition $u(x) = u(|x|)$, meaning that u depends solely on the radial component $|x|$.

Lastly, given a Banach space X and an open interval $(-T,T)$ with $T > 0$, we let

$$C_b((-T,T);X) = \left\{v \in C((-T,T);X);\ \sup_{t\in I}\|v(t)\|_X < \infty\right\},$$
$$C_b^m((-T,T);X) = \left\{v \in C^m((-T,T);X);\ \sum_{j=0}^m \sup_{t\in I}\left\|\frac{d^j}{dt^j}(v(t))\right\|_X < \infty\right\}.$$

We use the notation $C_TX \equiv C_b((-T,T);X)$ and $C_T^mX \equiv C_b^m((-T,T);X)$. Moreover, $C_tX \equiv C_b(\mathbb{R};X)$ and $C_t^mX \equiv C_b^m(\mathbb{R};X)$.

Similarly, we denote $L^r((-T,T);X)$ and $L^r([-T',T'];X)$ by L^r_TX and $L^r_{T'}X$ respectively, where $[-T',T']$ is any bounded closed subinterval of $(-T,T)$.

We then recall some important results from functional analysis [9].

Proposition 2.1.1 (Lebesgue dominated convergence theorem)**.** Let f be measurable in Ω, and $\{f_n\}$ be a sequence of measurable functions on Ω such that

$$\lim_{n\to\infty} f_n(x) = f(x),$$

for every $x \in \Omega$. Suppose that there exists nonnegative $g \in L^1(\Omega)$ such that

$$|f_n(x)| \leqslant g(x),$$

for every $x \in \Omega$ and $n \geqslant 1$. Then, we have

$$\lim_{n\to\infty} \int_\Omega |f_n(x) - f(x)|dx = 0,$$
$$\text{i.e.} \quad \lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

Proposition 2.1.2 (Hölder's inequality)**.** Let p and q be constants such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where $p, q \in (1, \infty)$. Then, for $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have

$$\int_\Omega |f(x)g(x)|dx \leqslant \|f\|_{L^p}\|g\|_{L^q}.$$

In particular, when $p = q = 2$, it is called Schwarz's inequality.

The following proposition is well-known as Sobolev embedding theorem. It plays an important role in the proof of the main theorems.

Proposition 2.1.3 ([5] Theorem 2 in Section 5.6 and Theorem 1 in Section 5.7)**.** Let $2 \leqslant p \leqslant \infty$ and $0 \leqslant a \leqslant 1$ such that

$$\frac{1}{p} = \frac{1}{2}(1-a) + \left(\frac{1}{2} - \frac{1}{n}\right) a.$$

Then, except for $(n, p) = (2, \infty)$, the following inequality holds:

$$\|u\|_{L^p} \lesssim \|u\|_{L^2}^{1-a}\|\nabla u\|_{L^2}^{a}, \quad u \in H^1.$$

In particular, the following embedding holds:

$$H^1 \subset L^p,$$

except for $(n, p) = (2, \infty)$. Moreover, with $p = p^*(n) + 1, n \geqslant 3$, we have

$$\|u\|_{L^{p^*(n)+1}} \leqslant C\|u\|_{H^1}.$$

Based on Proposition 2.1.3, we expand the statement by considering the functional spaces H_r^1 and L_r^p. Note that the embedding $H_r^1 \subset L_r^p$ holds except for $(n, p) = (2, \infty)$, and the following proposition states this embedding $H_r^1 \subset L_r^p$ is compact, except for $p = 2$ or $p = p^*(n) + 1$.

Proposition 2.1.4 ([5] Theorem 1 in Section 5.7)**.** Let $n \geqslant 2$ and $2 < p < p^*(n) + 1$. Then the embedding $H_r^1 \subset L_r^p$ is compact. That is, for any bounded sequence $\{u_m\}$ in H_r^1, there exists a subsequence $\{u_{m_k}\}$ convergent in L_r^p.

We end the section by recalling some definitions.

Definition 2.1.5 ([13] The differentiation of operators in Banach spaces)**.** Let $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be real Banach spaces. $B(X_1, X_2)$ denotes the collection of bounded, linear operators from X_1 into X_2. Let F be a map from X_1 into X_2. (i) F is said to be Fréchet differentiable at $x \in X_1$ if, for given $x \in X_1$, there exists $A \in B(X_1, X_2)$ such that

$$\lim_{\|h\|_{X_1} \to 0} \frac{\|F(x+h) - F(x) - Ah\|_{X_2}}{\|h\|_{X_1}} = 0. \tag{2.1}$$

Such an operator A is called the Fréchet derivative of F at x, denoted by $dF(x)$. (ii) F is said to be Gâteaux differentiable at $x \in X_1$ if the limit

$$\lim_{t \to 0} \frac{F(x+ty) - F(x)}{t},$$

exists for all $y \in X_1$. Then the limit is called the Gâteaux derivative of F at x, denoted by $dF(x, y)$.

From the definition, we can conclude that if F is Fréchet differentiable, then it is Gâteaux differentiable. Conversely, when F is Gâteaux differentiable at $x \in X_1$, it is Fréchet differentiable at x if the Gâteaux derivative $dF(x, y)$ is linear in y and $dF(x, \cdot)$ is a continuous map from $x \in X_1$ to $dF(x, \cdot) \in B(X_1, X_2)$.

Definition 2.1.6 ([13] Definition 5.2.7 and 5.3.1)**.** Let S be a functional mapping from H^1 into $\mathbb{R}$ and $u \in H^1$

(i) u is called a critical point of S if $dS(u, v) = 0$ for all $v \in H^1$. It follows that $S(u)$ is called a critical value.

(ii) u is a local maximum point of S if there exists $\epsilon > 0$ such that

$$S(u) > S(v), \tag{2.2}$$

for all $v \in H^1$ with $0 < \|u - v\|_{H^1} < \epsilon$. Here the corresponding value $S(u)$ is called a local maximum value. On the other hand, if the inequality (2.2) holds in the opposite direction, then u is a local minimum point of S_η and its value $S(u)$ is a local minimum value.

Definition 2.1.7 ([13] Definition 5.2.8)**.** Let $n \geqslant 1, 1 < p < p^*(n)$. $w \in H^1$ is called a weak solution to (1.5), if it satisfies (1.5) in distributional sense,

$$i.e. \quad (\nabla v, \nabla w) + \eta(v, w) - (v, |w|^{p-1} w) = 0, \quad v \in H^1.$$

Weak convergence

Definition 2.1.8 ([13] Definition A.3.13)**.** Let X be a normed space over $\mathbb{C}$. A mapping from X into $\mathbb{C}$ is called a functional. The collection of bounded linear functionals on X is called the dual space of X, denoted by X^*. For $f \in X^*$ and $x \in X$, we write $\langle f, x \rangle$ for $f(x)$. We say that X is reflexive if $X^{**} = X$ where X^{**} denotes the dual space of X^*.

Remark. X^* is a Banach space since $\mathbb{C}$ is a Banach space with the norm given by the absolute value.

Remark. For $x \in X$, a mapping

$$f \in X^* \longmapsto \langle f, x \rangle,$$

is a bounded linear functional on X^*. Thus, $x \in X$ can be regarded as an element in X^{**}. Hence, $X \subset X^{**}$.

Definition 2.1.9 ([13] Definition A.3.16)**.** Let X be a Banach space. We say that a sequence $\{x_n\} \subset X$ converges weakly to $x \in X$ and write

$$x_n \longrightarrow x \quad \text{weakly},$$

if, for any $f \in X^*$, we have

$$\langle f, x_n \rangle \longmapsto \langle f, x \rangle,$$

as $n \to \infty$.

Proposition 2.1.10 ([13] Theorem A.3.18)**.** Let X be a Banach space. For $x \in X$ and $\{x_n\} \subset X$, if $x_n \to x$ strongly, then $x_n \to x$ weakly.

Moreover, if $x_n \to x$ weakly, then $\{x_n\}$ is bounded in X and

$$\|x\|_X \leqslant \liminf_{n \to \infty} \|x_n\|_X.$$

Proposition 2.1.11 ([13] Theorem A.3.19)**.** Let X be a reflexive Banach space. A bounded sequence $\{x_n\}$ in X always has a weakly convergent subsequence.

2.2 Cauchy problem of Nonlinear Schrödinger Equations

In this section, we present some known results for the Cauchy problem of NLS (1.1). First, the following assumptions for the nonlinearity f are stated:

A1 f is a function from $\mathbb{C}$ to $\mathbb{C}$ with $f(0) = 0$. By regarding $\mathbb{C} \cong \mathbb{R}^2$, we have $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$.

A2 For $1 < p < p^*(n)$, there exists $K > 0$ such that

$$|f(z_1) - f(z_2)| \leqslant K(1 + |z_1| + |z_2|)^{p-1}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C}.$$

A3 $f(s) \in \mathbb{R}$ for $s \in \mathbb{R}$ and

$$f(e^{i\theta} z) = e^{i\theta} f(z), \quad z \in \mathbb{C},\ \theta \in \mathbb{R}.$$

A4 Define $F(z)$ by

$$F(z) = \int_0^z f(s)ds.$$

Then we have

$$F(|z|) \geqslant -L_1|z|^{q+1} - L_2|z|^2, \quad z \in \mathbb{C},\ 1 < q < 1 + \frac{4}{n},$$

where L_1, L_2 and q are positive constants that independent of z.

A5

$$f(z) = \lambda |z|^{p-1} z, \quad \lambda < 0,\ 1 + \frac{4}{n} \leqslant p < p^*(n).$$

Remark. The conditions A1 and A2 guarantee the existence and uniqueness of local-in-time solutions to the Cauchy problem (1.1). The condition A3 is well known as the gauge condition, which ensures the solutions to (1.1) satisfy physical principles, such as the conservation of energy.

Lemma 2.2.1 (The conservation laws)**.** Assume A1 - A3 hold for f, there exists a smooth solution v to (1.1). If it satisfies $v(x,t) \longrightarrow 0$ as $|x| \to \infty$, then the conservation laws (1.2) hold. Moreover, we can rewrite them as

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2}, \qquad E(v(t)) = E(v_0). \tag{2.3}$$

Local well-posedness

Proposition 2.2.2 ([12] Theorem 3.13)**.** Let $n \geqslant 1$ and f satisfies A1-A2. Given $v_0 \in H^1$, there exists $T > 0$ and a unique solution v, satisfying (1.1) on $(-T, T)$, such that

$$\begin{aligned} v &\in C_T H^1 \cap C_T^1 H^{-1}, \\ \nabla v &\in L_T^r L^{p+1}, \end{aligned}$$

where $1 < p < p^*(n)$, and r satisfies

$$r\left(\frac{n}{2} - \frac{n}{p+1}\right) = 2. \tag{2.4}$$

Here T depends on $K, p, n,$ and $\|v_0\|_{H^1}$, where K is as in A2.

Global well-posedness

By Proposition 2.2.2, we construct the local-in-time solutions to the Cauchy problem (1.1), and the next step is to extend them globally in time.

Lemma 2.2.3. Assume A1 - A3 hold for f. For an open interval $(-T, T)$ with $T > 0$. Let v be a solution to (1.1) such that

$$\begin{aligned} v &\in C_T H^1 \cap C_T^1 H^{-1}, \\ \nabla v &\in L_{T'}^r L^{p+1}, \end{aligned} \tag{2.5}$$

where $1 < p < p^*(n)$, and r satisfies (2.4). Then, the conservation laws (2.3) hold for any $t \in I$.

Proposition 2.2.4 ([12] Theorem 3.14)**.** Let $n \geqslant 1$. Suppose f satisfies A1 - A4. Then, for any $v_0 \in H^1$, the local-in-time solution v to (1.1) constructed in Proposition 2.2.2 can be extended globally in time such that it is unique and

$$\begin{aligned} v &\in C_t H^1 \cap C_t^1 H^{-1}, \\ \nabla v &\in L_T^r L^{p+1}. \end{aligned}$$

Moreover, by Lemma 2.2.3, the global-in-time solution v follows the conservation laws (2.3).

However, it is not always the case that we can find global-in-time solutions. If we only focus on whether we can extend a local-in-time solution to $[0, \infty)$, the following proposition holds.

Proposition 2.2.5 ([13] Theorem 4.5.1)**.** Let $n \geqslant 1$. Assume A5 holds for f and $v_0 \in H^1, xv_0 \in L^2$. If $E(v_0) < 0$, then the local-in-time solution v to (1.1) constructed in Proposition 2.2.2 can not be extended globally in time.

Moreover, when A5 is satisfied by f, we have the following lemma.

Lemma 2.2.6 ([13] Lemma 4.5.3)**.** Let $n \geqslant 1$. Suppose A5 holds for f, $v_0 \in H$ and $xv_0 \in L^2$. Let v be a solution to (1.1) on $[0, T)$ that satisfies (2.5). Then

$$xv \in C([0, T); L^2),$$

and the following identity holds for $t \in [0, T)$:

$$\begin{aligned} \|xv(t)\|_{L^2}^2 = \|xv_0\|_{L^2}^2 &+ 4tIm \int_{\mathbb{R}^n} \overline{v_0} x \cdot \nabla v_0 \, dx \\ &+ 4 \int_0^t \int_0^s \left[2\|\nabla v(\tau)\|_{L^2}^2 + \frac{n\lambda(p-1)}{p+1} \|v(\tau)\|_{L^{p+1}}^{p+1} \right] d\tau ds. \end{aligned}$$

We end this section by recalling the nonlinearity $f(z)$ in NLS (1.3):

$$f(z) = -|z|^{p+1} z.$$

Note that it satisfies all the conditions A1 - A5 for $1 < p < p^*(n)$.

2.3 Variational formulation

To study the existence of standing wave solutions, it is important to comprehend the variational methods. Following this section, one can understand the principle of converting the elliptic problem into a minimization problem.

Let us recall the definition of $S_\eta(u)$ (1.7) and consider $S_\eta(u) : H^1_{\text{real}} \to \mathbb{R}$ as a functional on H^1_{real}. Moreover, if it is Gâteaux differentiable, its Gâteaux derivative is denoted by $dS_\eta(u, v)$, $v \in H^1_{\text{real}}$.

Proposition 2.3.1 ([13] Theorem 5.2.5)**.** Let $n \geqslant 1, 1 < p < p^*(n)$ and $\eta > 0$. Then the functional S_η is defined on all of H^1 and its Gâteaux derivative $dS_\eta(u,v)$, $v \in H^1_{\text{real}}$ exists for all $u \in H^1_{\text{real}}$. Furthermore, for $u, v \in H^1$, the Gâteaux derivative $dS_\eta(u,v)$ can be written as

$$\begin{aligned} dS_\eta(u,v) &= \partial_u S_\eta(u)v + \partial_{\overline{u}} S_\eta(u)\overline{v} \\ &= 2\mathrm{Re}[\partial_u S_\eta(u)v], \end{aligned}$$

where, for each $u \in H^1$, $\partial_u S_\eta(u)$ and $\partial_{\overline{u}} S_\eta(u)$ are bounded linear functionals from H^1 into $\mathbb{C}$ given as

$$\begin{aligned} \partial_u S_\eta(u)v &= (\nabla v, \nabla u) + \eta(v,u) - (v, |u|^{p-1}u), \\ \partial_{\overline{u}} S_\eta(u)v &= \overline{\partial_u S_\eta(u)\overline{v}}, \end{aligned} \tag{2.6}$$

Proof. By the Proposition 2.1.3, we obtain that for $u \in H^1$,

$$\|u\|_{L^{p+1}} \leqslant C\|u\|_{H^1}, \tag{2.7}$$

which implies that S_η maps H^1 into $\mathbb{R}$.

Given $u, v \in H^1_{\text{real}}$, we define

$$R(t) = S_\eta(u+tv) - S_\eta(u) - t^2\left(\|\nabla v\|^2_{L^2} + \eta\|v\|^2_{L^2}\right),$$

for $t \in \mathbb{R}$ with $|t| < 1$. It follows from the definition of S_η (1.7) that

$$\begin{aligned} R(t) = {} & \|\nabla(u+tv)\|^2_{L^2} + \eta\|u+tv\|^2_{L^2} - \frac{2}{p+1}\|u+tv\|^{p+1}_{L^{p+1}} \\ & - \|\nabla u\|^2_{L^2} - \eta\|u\|^2_{L^2} + \frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}} - t^2\left(\|\nabla v\|^2_{L^2} + \eta\|v\|^2_{L^2}\right). \end{aligned}$$

Rearranging it and evaluating the norms, we have

$$\begin{aligned} R(t) = {} & \int_{\mathbb{R}^n} \left(|\nabla u + t\nabla v|^2 - |\nabla u|^2 - t^2|\nabla v|^2 \right) dx \\ & + \eta \int_{\mathbb{R}^n} \left(|u+tv|^2 - |u|^2 - t^2|v|^2 \right) dx \\ = {} & \int_{\mathbb{R}^n} \left(|\nabla u|^2 + 2t\mathrm{Re}(\nabla u\overline{\nabla v}) + t^2|\nabla v|^2 - |\nabla u|^2 - t^2|\nabla v|^2 \right) dx \\ & + \eta \int_{\mathbb{R}^n} \left(|u|^2 + 2t\mathrm{Re}(u\overline{v}) + t^2|v|^2 - |u|^2 - t^2|v|^2 \right) dx \\ & - \frac{2}{p+1}\int_{\mathbb{R}^n}\int_0^1 \frac{d}{d\theta}(\theta|u+tv|^2 + (1-\theta)|u|^2)^{\frac{p+1}{2}} d\theta dx \\ = {} & 2t\mathrm{Re}\left(\int_{\mathbb{R}^n} (\nabla u\overline{\nabla v})dx + \eta\int_{\mathbb{R}^n}(u\overline{v})dx \right) \\ & - \int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2 + (1-\theta)|u|^2)^{\frac{p+1}{2}-1} d\theta(|u+tv|^2 - |u|^2)dx. \end{aligned}$$

Expanding $|u+tv|^2-|u|^2$ yields $2t\mathrm{Re}+(u\overline{v})+t^2|v|^2$. Consequently,

$$\begin{aligned} R(t) &= 2t\mathrm{Re}[(\nabla v, \nabla u)+\eta(v,u)] \\ &\quad - 2t\int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta \mathrm{Re}(u\overline{v})dx \\ &\quad - t^2\int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta |v|^2dx \\ &= 2t\mathrm{Re}[(\nabla v, \nabla u)+\eta(v,u)] - tI(t) - t^2J(t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} I(t) &= 2\int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta \mathrm{Re}(u\overline{v})dx, \\ J(t) &= \int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta |v|^2dx. \end{aligned} \tag{2.9}$$

For estimating $I(t)$, we notice that for $x\in\mathbb{R}^n$,

$$\begin{aligned} &\lim_{t\to 0}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta \mathrm{Re}(u\overline{v}) \\ &\qquad = |u|^{p-1}\mathrm{Re}(u\overline{v}), \end{aligned} \tag{2.10}$$

and, for all t such that $|t|<1$, the following inequality holds:

$$\begin{aligned} &\left|\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta \mathrm{Re}(u\overline{v})\right| \\ &\qquad \lesssim (|u|^{p-1}+|v|^{p-1})|u||v|. \end{aligned} \tag{2.11}$$

From Proposition 2.1.3, it follows $u,v\in L^{p+1}$. Therefore, by (2.10)-(2.11) and Lebesque dominated convergence theorem 2.1.1, we derive

$$\begin{aligned} &\lim_{t\to 0}\int_{\mathbb{R}^n}\int_0^1 (\theta|u+tv|^2+(1-\theta)|u|^2)^{\frac{p-1}{2}}d\theta \mathrm{Re}(u\overline{v})dx \\ &\qquad = \int_{\mathbb{R}^n} |u|^{p-1}\mathrm{Re}(u\overline{v})dx. \end{aligned}$$

Then, from (2.9), it follows that

$$\begin{aligned} &\lim_{t\to 0} I(t) = 2\mathrm{Re}\int_{\mathbb{R}^n}(|u|^{p-1}u\overline{v})dx, \\ i.e. \quad &\lim_{t\to 0} I(t) = 2\mathrm{Re}(|u|^{p-1}u, v). \end{aligned}$$

For estimating $J(t)$, it follows from Proposition 2.1.2 and (2.7) that, for all t such that $|t|<1$,

$$\begin{aligned} |J(t)| &\lesssim \int_{\mathbb{R}^n}\left(\ |u|^{p-1}+|v|^{p-1}\right)\ |v|^2dx \\ &\lesssim (\|u\|_{L^{p+1}}^{p-1}+\|v\|_{L^{p+1}}^{p-1})\|v\|_{L^{p+1}}^2 \\ &\lesssim (\|u\|_{H^1}^{p-1}+\|v\|_{H^1}^{p-1})\|v\|_{H^1}^2. \end{aligned} \tag{2.12}$$

Finally, we can conclude the Gâteaux derivative $dS_\eta(u,v)$ by (2.8)-(2.12) that

$$\begin{aligned} dS_\eta(u,v) &= \lim_{t\to 0}\frac{S_\eta(u+tv)-S_\eta(u)}{t} = \lim_{t\to 0}\frac{R(t)}{t} \\ &= 2\mathrm{Re}\left[\ (\nabla v,\nabla u)+\eta(v,u)\right] - 2\mathrm{Re}(|u|^{p-1}u,v)\big] \\ &= 2\mathrm{Re}\left[\ (\nabla v,\nabla u)+\eta(v,u)-(|u|^{p-1}u,v)\right] \\ &= \partial_u S_\eta(u)v + \partial_{\overline{u}}S_\eta(u)\overline{v}, \end{aligned}$$

where, by (2.6), the operators $\partial_u S_\eta(u)$ and $\partial_{\overline{u}} S_\eta(u)$ are clearly linear for each $u \in H^1$. Moreover, from (2.7), we have

$$\begin{aligned} |\partial_u S_\eta(u)v|, |\partial_{\overline{u}} S_\eta(u)v| &= |(\nabla v,\nabla u)+\eta(v,u)-(v,|u|^{p-1}u)| \\ &\lesssim (\|u\|_{H^1}+\|u\|_{H^1}^p)\|v\|_{H^1}, \end{aligned}$$

which implies that the operators $\partial_u S_\eta(u)$ and $\partial_{\overline{u}} S_\eta(u)$ are also bounded for each $u \in H^1$. □

Remark. As $\partial_u S_\eta(u)$ and $\partial_{\overline{u}} S_\eta(u)$ are operators from H^1 into $\mathbb{C}$, we regard $dS_\eta(u,v)$ as an operator from $H^1 \times H^1$ into $\mathbb{C}$.

Proposition 2.3.2 ([13] Theorem 5.2.10)**.** Let $n \geqslant 1, 1 < p < p^*(n)$. $u \in H^1$ is a critical point of S_η if and only if $u \in H^1$ is a weak solution to (1.5). Moreover, for $u, v \in H^1$,

$$dS_\eta(u,v) = 0,$$

if and only if

$$\partial_u S_\eta(u)v = 0.$$

Proof. If $u \in H^1$ is a critical point of S_η, then by Proposition 2.3.1 and definition 2.1.6(i) we have

$$dS_\eta(u,v) = 2\mathrm{Re}[\partial_u S_\eta(u)v] = 0,$$

for $v \in H^1$. It is followed by

$$\mathrm{Re}[(\nabla v,\nabla u)+\eta(v,u)-(v,|u|^{p-1}u)] = 0. \tag{2.13}$$

Replacing v by iv yields

$$\mathrm{Im}[(\nabla v,\nabla u)+\eta(v,u)-(v,|u|^{p-1}u)] = 0. \tag{2.14}$$

Combining (2.13) and (2.14) gives

$$\begin{aligned} &(\nabla v,\nabla u)+\eta(v,u)-(v,|u|^{p-1}u) = 0, \\ i.e.\quad &\partial_u S_\eta(u)v = 0. \end{aligned}$$

Thus, by definition 2.1.7, u is a weak solution to (1.5).

Conversely, if $u \in H^1$ is a weak solution of S_η, then we have

$$\begin{aligned} \partial_u S_\eta(u)v &= 0, \\ dS_\eta(u,v) &= 2\mathrm{Re}[\partial_u S_\eta(u)v] \\ &= 0, \end{aligned}$$

for $v \in H^1$, which implies $u \in H^1$ is a critical point of S_η. □

In summary, the nonlinear elliptic problem (1.5) can be solved in distributional sense by determining the critical point of the functional S_η. As per the standard understanding that the derivative of a function is zero at local extremum points, namely local maximum or minimum points. Moreover, from Definition 2.1.6(i), we have $dS_\eta(u,v) = 0$ at critical points of S_η. Therefore, it is logical to explore the relationship between critical points of S_η and its local extremum points.

Proposition 2.3.3 ([13] Proposition 5.3.3)**.** Let $n > 1, 1 < p < p^*(n)$. If $u \in H^1$ is a local extremum point for the functional S_η, then we have $\partial_u S_\eta(u)v = 0$ for $v \in H^1$.

Proof. Suppose u is a local maximum point for the functional S_η. Then, by Definition 2.1.6(ii), we have

$$S_\eta(u+tv) - S_\eta(u) < 0,$$

for $v \in H^1$ and $|t| \ll 1$. Let $t > 0$, divide both sides by t and take limits as $t \to 0$:

$$\lim_{t\to 0} \frac{S_\eta(u+tv) - S_\eta(u)}{t} \leqslant 0.$$

Thus, by Definition 2.1.5(ii) and Proposition 2.3.1, we obtain

$$dS_\eta(u,v) \leqslant 0. \tag{2.15}$$

The one for $t < 0$ is treated similarly, then we have

$$dS_\eta(u,v) \geqslant 0. \tag{2.16}$$

From (2.15)-(2.16) and Proposition 2.3.2, it follows that

$$\partial_u S_\eta(u)v = 0,$$

for $v \in H^1$. A similar argument is applied for u being a local minimum point. □

Thus, it is clear that the local extremum points for the functional S_η are the weak solutions of (1.5).

Regarding extremum points, we usually think of absolute maximum or minimum points. However, since u ranges over the entire space, the functional S_η is not bounded H^1. Therefore, in finding the local extremum values of an unbounded functional, we must impose some constraints and then find the maximum or minimum points. Such problems are known as conditional extremum problems.

Next, we present a useful proposition in considering the conditional extremum problems.

Proposition 2.3.4 ([13] Theorem 5.3.5)**.** Let $n \geqslant 1, 1 < p < p^*(n)$, and $u \in H^1$,

$$T(u) := a\|\nabla u\|_{L^2}^2 + b\|u\|_{L^2}^2 + c\|u\|_{L^{p+1}}^{p+1},$$

for some $a, b, c \in \mathbb{R}$.

(i) By regarding $T(u)$ as a functional on H^1_{real}, the Gâteaux derivative $dT(u,v))$, $v \in H^1_{\text{real}}$, exists for any $u \in H^1_{\text{real}}$ and can be expressed as

$$\begin{aligned} dT(u,v) &= \partial_u T(u)v + \partial_{\overline{u}} T(u)\overline{v} \\ &= 2\text{Re}[\partial_u T(u)v], \end{aligned}$$

where $u, v \in H^1$ and, for each $u \in H^1$, $\partial_u T(u)$ and $\partial_{\overline{u}} T(u)$ are bounded linear operators from H^1 into $\mathbb{C}$ with expressions

$$\begin{aligned} &\partial_u T(u)v = a(\nabla v, \nabla u) + b(v,u) + \frac{c(p+1)}{2}(v, |u|^{p-1}u), \\ &\partial_{\overline{u}} T(u)v = \overline{\partial_u T(u)\overline{v}}, \end{aligned} \tag{2.17}$$

(ii) For $\alpha \in \mathbb{R}$, define a subset K of H^1 by $K = \{v \in H^1;\ T(v) = \alpha\}$. If $u \in K$ is a local extremum point of S_η on K and $\partial_u T(u) \neq 0$, then the following holds :

$$\partial_u S_\eta(u) - \lambda \partial_u T(u) = 0,$$
$$\lambda = \frac{\text{Re}[\partial_u S_\eta(u)v]}{\text{Re}[\partial_u T(u)v]},$$

where $v \in H^1$, $\text{Re}[\partial_u T(u)v] \neq 0$.

To prove it, we need the following lemma which is known as the Lagrange multiplier method.

Lemma 2.3.5 ([13] Theorem A.1.3)**.** Let D be a domain in $\mathbb{R}^2$ and $f(x,y), g(x,y) \in C^1(D)$. Also, let $E = \{(x,y) : f(x,y) = 0\}$. Suppose that $f(x,y)$ takes an extreme value at $(x_0,y_0) \in E$ under the condition $g(x,y) = 0$. Then there exists a constant $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} f_x((x_0,y_0)) - \lambda g_x(x_0,y_0) = 0, \\ f_y((x_0,y_0)) - \lambda g_y(x_0,y_0) = 0. \end{aligned}$$

Now it is possible to present the proof of Proposition 2.3.4.

Proof. The proof of part (i) is analogous to that of Proposition 2.3.1. Thus we provide a sketch proof here. First, we have T maps H^1 into $\mathbb{R}$ from Proposition 2.1.3.

Given $u, v \in H^1_{\text{real}}$, define

$$R(t) = T(u+tv) - T(u) - t^2\left(a\|\nabla v\|_{L^2}^2 + b\|v\|_{L^2}^2\right),$$

for $t \in \mathbb{R}$ with $|t| < 1$. It follows that

$$R(t) = 2t\mathrm{Re}[a(\nabla v, \nabla u) + b(v, u)] + \frac{ct(p+1)}{2} I(t) + \frac{ct^2(p+1)}{2} J(t),$$

where

$$I(t) = 2 \int_{\mathbb{R}^n} \int_0^1 (\theta|u + tv|^2 + (1-\theta)|u|^2)^{\frac{p-1}{2}} d\theta \mathrm{Re}(u\overline{v}) dx,$$
$$J(t) = \int_{\mathbb{R}^n} \int_0^1 (\theta|u + tv|^2 + (1-\theta)|u|^2)^{\frac{p-1}{2}} d\theta |v|^2 dx.$$

Then, we deduce

$$\lim_{t \to 0} I(t) = 2\mathrm{Re}(|u|^{p-1}u, v),$$

and, for $|t| < 1$, we have

$$|J(t)| \leqslant C(\|u\|_{H^1}^{p-1} + \|v\|_{H^1}^{p-1})\|v\|_{H^1}^2.$$

Consequently, we obtain

$$\begin{aligned} dT(u, v) &= \lim_{t \to 0} \frac{T(u + tv) - T(u)}{t} = \lim_{t \to 0} \frac{R(t)}{t} \\ &= 2\mathrm{Re}\left[a(\nabla v, \nabla u) + b(v, u) + \frac{c(p+1)}{2}(|u|^{p-1}u, v) \right] \\ &= \partial_u T(u)v + \partial_{\overline{u}} T(u)\overline{v}, \end{aligned}$$

for $u, v \in H^1$, where $\partial_u T(u)$ and $\partial_{\overline{u}} T(u)$ are bounded linear operators from H^1 into $\mathbb{C}$.

Next, we prove part (ii). In view of the assumption $\partial_u T(u) \neq 0$, there exists $w \in H^1$ such that

$$\mathrm{Re}[\partial_u T(u)w] \neq 0.$$

Fix $v \in H^1$ and let

$$F(s, t) = S_\eta(u + sv + tw),$$
$$G(s, t) = T(u + sv + tw) - \alpha.$$

It follows from Proposition 2.3.1 and part (i) that $F, G \in C^1(\mathbb{R}^2)$. Moreover, since u is a local extremum point of S_η on the set K where $T(u) = \alpha$, the point $(s, t) = (0, 0)$ is a local extremum point of F on the set

$$\widetilde{K} = \{(s, t) \in \mathbb{R}^2;\ G(s, t) = 0\}.$$

From Lemma 2.3.5, it follows that there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial F}{\partial s}(0, 0) - \lambda \frac{\partial G}{\partial s}(0, 0) = 0, \\ \frac{\partial F}{\partial t}(0, 0) - \lambda \frac{\partial G}{\partial t}(0, 0) = 0. \end{aligned} \tag{2.18}$$

Moreover, by taking the partial derivative of $G(s,t)$ with respect to t, we deduce that

$$\begin{aligned}\frac{\partial G}{\partial t}(0,0) &= \partial_u T(u)w + \partial_{\overline{u}} T(u)\overline{w}\\ &= 2\mathrm{Re}[\partial_u T(u)w] \neq 0.\end{aligned}$$

Since $\frac{\partial G}{\partial t}(0,0) \neq 0$, we can rewrite (2.18) as

$$\begin{aligned}&\mathrm{Re}[\partial_u S_\eta(u)v - \lambda\partial_u T(u)v] = 0,\\ &\lambda = \frac{Re[\partial_u S_\eta(u)w]}{Re[\partial_u T(u)w]}.\end{aligned}$$

As $\partial_u S_\eta(u)$ and $\partial_u T(u)$ are linear operators, we can replace v by iv which then leads to

$$\begin{aligned}&\mathrm{Im}[\partial_u S_\eta(u)v - \lambda\partial_u T(u)v]\\ &\quad = -\mathrm{Re}[i\partial_u S_\eta(u)v - i\lambda\partial_u T(u)v]\\ &\quad = -\mathrm{Re}[\partial_u S_\eta(u)(iv) - \lambda\partial_u T(u)(iv)]\\ &\quad = 0.\end{aligned}$$

Therefore, we have

$$\partial_u S_\eta(u)v - \lambda\partial_u T(u)v = 0.$$

which completes the proof. $\square$

Chapter 3

Standing wave solutions of the Nonlinear Schrödinger Equations

In this chapter, we give the proof of the main theorems. As mentioned before, we first let $1 < p < 1 + \frac{4}{n}$ and prove the first part of both theorems, and then prove the second part where $1 + \frac{4}{n} < p < p^*(n)$.

3.1 Existence of standing wave solutions and their stability for $1 < p < 1 + \frac{n}{4}$

In this section, we let $n \geqslant 2$ and $1 < p < 1 + \frac{4}{n}$ throughout and investigate the existence of standing wave solutions along with their stability by proving Theorem 1(i) and 2(i).

3.1.1 Proof of Theorem 1(i)

Let us recall the conditional minimization problem (1.6) for $1 < p < 1 + \frac{n}{4}$:

$$b_\alpha = \inf_{u \in K_\alpha} S_\eta(u),$$

where $\eta > 0$, $\alpha > 0$ and the definitions of S_η and K_α are given as

$$S_\eta(u) = \|\nabla u\|_{L^2}^2 + \eta \|u\|_{L^2}^2 - \frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1},$$
$$K_\alpha = \{u \in H^1; \|u\|_{L^2} = \sqrt{\alpha}\}.$$

Remark. If there exists $u \in K_\alpha$ such that $b_\alpha = S_\eta(u) < \infty$, then we say it is a solution of the minimization problem (1.6) and the corresponding b_α is a minimum of the problem.

To prove the theorem, we will use the following lemmas.

Lemma 3.1.1. The minimization problem (1.6) is equivalent to the following conditional extremum problem:

$$c_\alpha = \inf_{u \in K_\alpha} E(u), \tag{3.1}$$

where $\alpha > 0$ and $E(u)$ denotes the energy functional with expression

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{3.2}$$

Proof. Suppose $w \in K_\alpha$ is a solution to the problem (3.1), then

$$\|w\|_{L^2} = \sqrt{\alpha},$$
$$c_\alpha = E(w) = \inf_{u \in K_\alpha} E(u).$$

By Definition 1.7, we have

$$\begin{aligned} S_\eta(u) &= \|\nabla u\|_{L^2}^2 + \eta\|u\|_{L^2}^2 - \frac{2}{p+1}\|u\|_{L^{p+1}}^{p+1} \\ &= 2\left(\frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) + \eta\|u\|_{L^2}^2 \\ &= 2E(u) + \eta\alpha, \end{aligned} \tag{3.3}$$

for $u \in K_\alpha$. Take the infimum on both sides:

$$\begin{aligned} \inf_{u \in K_\alpha} S_\eta(u) &= 2\inf_{u \in K_\alpha} E(u) + \eta\alpha \\ &= 2E(w) + \eta\alpha \\ &= S_\eta(w), \end{aligned}$$

which implies w is also a solution of problem (1.6).

Moreover, the following equality holds:

$$b_\alpha = 2c_\alpha + \eta\alpha.$$

□

Lemma 3.1.2 ([13] Lemma 5.3.8)**.** Let $v \in H^1$, then there exists $v^* \in H^1$ with the following properties:

i. $v^* \geqslant 0$, a.e. $x \in \mathbb{R}^n$.

ii. radial symmetry: $v^*(x) = v^*(|x|)$.

iii. $|x| \leqslant |y| \Rightarrow v^*(x) \leqslant v^*(y)$.

iv. $\text{meas}\{x; v^*(x) > t\} = \text{meas}\{x; |v(x)| > t\}, \quad t > 0$.

v. $\|\nabla v^*\|_{L^2} \leqslant \|\nabla v\|_{L^2}$.

Here meas denotes the Lebesque measure on $\mathbb{R}^n$ and (iv) is equivalent to

$$\|v\|_{L^q} = \|v^*\|_{L^q},$$

for $1 \leqslant q \leqslant \infty$. We call such $v^*(x)$ the symmetric decreasing rearrangement of v. Moreover, for a given $v \in H^1$, if there exist two symmetric decreasing rearrangements v_1^* and v_2^*, then $v_1^*(x) = v_2^*(x)$ for almost every $x \in \mathbb{R}^n$.

Lemma 3.1.3 ([13] Lemma 5.4.2)**.** Let $n \geqslant 1, 1 < p < 1 + \frac{4}{n}$. Given $\alpha > 0$, suppose c_α is a finite negative value. Let $\{u_m\}$ be a sequence in H_r^1. If the following holds:

$$\|u_m\|_{L^2} \to \sqrt{\alpha},$$
$$E(u_m) \to c_\alpha,$$

as $m \to \infty$. Then there exists a subsequence $\{u_{m_k}\}$ of $\{u_m\}$ and $w \in H_r^1$ such that, as $m_k \to \infty$,

$$\{u_{m_k}\} \longrightarrow w \quad in\ H^1,$$
$$c_\alpha = E(w), \quad \|w\|_{L^2} = \sqrt{\alpha}.$$

Furthermore, w is a weak solution to (1.5) for some $\eta > 0$.

The proof for Lemma 3.1.3 is postponed to the end of this section. We can easily prove Theorem 1(i) and Theorem 2(i) by assuming this lemma.

Now, we are able to present the proof of Theorem 1(i).

Proof. First, in view of Lemma 3.1.1, it suffices to prove there exists $w \in K_\alpha \cap H_r^1$ such that $0 > c_\alpha > -\infty$, and w is a weak solution to (1.5) for some $\eta > 0$.

Step 1: We claim $0 > c_\alpha > -\infty$.
Given $u \in H^1$, define

$$u_\lambda(x) = \lambda^\beta u(\lambda x), \tag{3.4}$$

where $\lambda > 0$ and $\beta \in \mathbb{R}$. Then we have

$$\|u_\lambda\|_{L^2}^2 = \int_{\mathbb{R}^n} |u_\lambda(x)|^2 dx = \int_{\mathbb{R}^n} \lambda^{2\beta} |u(\lambda x)|^2 dx,$$

by the change of variables,

$$\begin{aligned} \|u_\lambda\|_{L^2}^2 &= \lambda^{2\beta} \int_{\mathbb{R}^n} \frac{|u(y)|^2}{\lambda^n} dy \\ &= \lambda^{2\beta - n} \int_{\mathbb{R}^n} |u(y)|^2 dy \\ &= \lambda^{2\beta - n} \|u\|_{L^2}^2. \end{aligned} \tag{3.5}$$

In like manner, we obtain

$$\begin{aligned} \|\nabla u_\lambda\|_{L^2}^2 &= \lambda^{2\beta} \int_{\mathbb{R}^n} |\nabla(u(\lambda x))|^2 dx \\ &= \lambda^{2\beta+2} \int_{\mathbb{R}^n} |(\nabla u)(\lambda x)|^2 dx \\ &= \lambda^{2\beta+2-n} \int_{\mathbb{R}^n} |(\nabla u)(y)|^2 dy \\ &= \lambda^{2\beta+2-n} \|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.6}$$

and

$$\|u_\lambda\|_{L^{p+1}}^{p+1} = \lambda^{(p+1)\beta-n}\|u\|_{L^{p+1}}^{p+1}. \tag{3.7}$$

Let $\beta = \frac{n}{2}$ and $u \in K_\alpha$. Substituting (3.5)-(3.7) into the energy functional (3.2) yields

$$\begin{aligned} E(u_\lambda) &= \frac{\lambda^2}{2}\|\nabla u_\lambda\|_{L^2}^2 - \frac{\lambda^{\frac{n(p+1)}{2}-n}}{p+1}\|u\|_{L^{p+1}}^{p+1} \\ &= \frac{\lambda^2}{2}\|\nabla u_\lambda\|_{L^2}^2 - \frac{\lambda^{\frac{n(p-1)}{2}}}{p+1}\|u\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Since $1 < p < 1 + \frac{4}{n}$, $E(u_\lambda) < 0$ if and only if

$$0 < \lambda < \left(\frac{2\|u\|_{L^{p+1}}^{p+1}}{(p+1)\|\nabla u\|_{L^2}^2} \right)^{\frac{1}{2-\frac{n(p-1)}{2}}}.$$

Moreover, $\|u_\lambda\|_{L^2}^2 = \|u\|_{L^2}^2 = \sqrt{\alpha}$ indicates that $u_\lambda \in K_\alpha$. Since c_α is defined as the infimum of the energy on the set $u \in K_\alpha$, $E(u_\lambda) < 0$ is equivalent to $c_\alpha < 0$.

Next, we use Proposition 2.1.3 to show $c_\alpha > -\infty$. Recall the proposition, we have $2 \leqslant p+1 \leqslant \infty$ and evaluate $a = \frac{n(p-1)}{2(p+1)}$ from

$$\frac{1}{p+1} = \frac{1}{2}(1-a) + \left(\frac{1}{2} - \frac{1}{n}\right) a.$$

Since $n \geqslant 2$, we have $0 \leqslant a \leqslant 1$. Thus, the following inequality holds:

$$\begin{aligned} &\|u\|_{L^{p+1}} \leqslant C\|u\|_{L^2}^{1-\frac{n(p-1)}{2(p+1)}}\|\nabla u\|_{L^2}^{\frac{n(p-1)}{2(p+1)}}, \\ i.e. \quad &\|u\|_{L^{p+1}}^{p+1}, \leqslant C\|u\|_{L^2}^{\frac{n+2-p(n-2)}{2}}\|\nabla u\|_{L^2}^{\frac{n(p-1)}{2}}, \end{aligned}$$

for $u \in H^1$. Then, we deduce that

$$E(u) \geqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^{\frac{n+2-p(n-2)}{2}}\|\nabla u\|_{L^2}^{\frac{n(p-1)}{2}}. \tag{3.8}$$

By $\|u\|_{L^2} = \sqrt{\alpha}$ and $\frac{n(p-1)}{2} < 2$, we can conclude that

$$\begin{aligned} 0 > c_\alpha &\geqslant \inf_{s>0}\left[\frac{1}{2}s^2 - C\alpha^{\frac{n+2-p(n-2)}{4}}s^{\frac{n(p-1)}{2}}\right] \\ &> -\infty. \end{aligned}$$

Step 2: We show w is a weak solution to (1.5) for some $\eta > 0$.
By the definition (3.1) of c_α, there exists a sequence $\{u_m\}$ in K_α such that

$$E(u_m) \to c_\alpha,$$

as $m \to \infty$, and for all $m \geqslant 1$, we have

$$c_\alpha \leqslant E(u_m).$$

From Lemma 3.1.2, it follows that there exists a symmetric decreasing rearrangement $\{u_m^*\}$ in $K_\alpha \cap H_r^1$ such that, as $m \to \infty$,

$$\begin{aligned} E(u_m^*) &\longrightarrow c_\alpha, \\ \|u_m^*\|_{L^2} &= \sqrt{\alpha}. \end{aligned}$$

It is clear that $\{u_m^*\}$ satisfies the hypothesis in Lemma 3.1.3, therefore, by Lemma 3.1.3, there exists a subsequence $\{u_{m_k}^*\}$ and $w \in H_r^1$ such that, as $m_k \to \infty$,

$$\begin{aligned} &\{u_{m_k}^*\} \longrightarrow w \quad \text{in } H^1, \\ &c_\alpha = E(w), \quad \|w\|_{L^2} = \sqrt{\alpha}, \end{aligned}$$

Furthermore, w is a weak solution to (1.5) for some $\eta > 0$. □

The Theorem 1(i) states that there exists at least one weak solution to (1.5) for some $\eta > 0$. Then, for any $\widetilde{\eta} > 0$, define $\widetilde{w}$ by

$$\widetilde{w}(x) = \left(\frac{\widetilde{\eta}}{\eta}\right)^{\frac{1}{p-1}} w\left(\sqrt{\frac{\widetilde{\eta}}{\eta}}x\right). \tag{3.9}$$

Let θ denote $\frac{\widetilde{\eta}}{\eta}$. Substituting (3.9) into

$$-\Delta\widetilde{w} + \widetilde{\eta}\widetilde{w} - |\widetilde{w}|^{p-1}\widetilde{w},$$

yields

$$\theta^{\frac{p}{p-1}}\left(-\Delta w(\sqrt{\theta}x) + \eta w(\sqrt{\theta}x) - \left|w(\sqrt{\theta}x)\right|^{p-1} w(\sqrt{\theta}x)\right).$$

Since w is a weak solution to (1.5), we have

$$-\Delta w(\sqrt{\theta}x) + \eta w(\sqrt{\theta}x) - \left|w(\sqrt{\theta}x)\right|^{p-1} w(\sqrt{\theta}x) = 0,$$

$$\text{i.e.} \quad -\Delta\widetilde{w} + \widetilde{\eta}\widetilde{w} - |\widetilde{w}|^{p-1}\widetilde{w} = 0,$$

which implies that $\widetilde{w}$ is a weak solution to (1.5) for $\eta = \widetilde{\eta}$. As a consequence, (1.5) has weak solutions for any $\eta > 0$.

Uniqueness of the standing wave solutions After verifying the existence, we fix a value of $\eta > 0$ to derive the uniqueness of standing wave solutions.

Proposition 3.1.4 ([13] Proposition 5.4.3)**.** Let $n \geqslant 2, 1 < p < p^*(n)$ and $\eta > 0$. Suppose that $w \in H_r^1$ is a weak solution to (1.5) such that

$$w(x) \geqslant 0, \tag{3.10}$$

for every $x \in \mathbb{R}^n$. Let G_η be the collection of weak solutions v to (1.5) satisfying $S_\eta(v) = S_\eta(w)$. Then we have

$$G_\eta = \{v \in H^1; v(x) = e^{i\theta}w(x+a),\ \theta \in \mathbb{R},\ a \in \mathbb{R}^n\}.$$

In other words, the proposition states that all weak solutions v such that $S_\eta(v) = S_\eta(w)$ are equal to w, modulo translations and modulations, which implies the uniqueness of standing wave solutions.

Remark. Recall the solution $w(x)$ we constructed in the proof of Theorem 1(i). It appears as a limit of the symmetric decreasing rearrangement $u^*_{m_k}$. Therefore, it must satisfy (3.10). Moreover, if w is a solution to (1.5), $v(\cdot) = e^{i\theta}w(\cdot + a)$ is a solution to (1.5) for all $\theta \in \mathbb{R}$, $a \in \mathbb{R}^n$.

The notable thing is there are still infinitely many solutions to (1.5) that are not contained in G_η and we call them excited state solutions or excited states. Meanwhile, the solutions contained in G_η are called ground state solutions or ground states. In particular, the solution w given by Theorem 1(i) is a ground state solution as $w \in G_\eta$.

Furthermore, since the ground states minimize the functional S_η among all non-zero solutions, the values of S_η for excited states are always greater than that of ground states. Physically, the functional S_η corresponds to the quantity called action. Hence, ground states are known as least-action solutions.

If w is a ground state solution to (1.5), then the standing wave solution v to (1.3) constructed by $v(x,t) = e^{i\eta t}w(x)$ is called a ground state solution. If w is an excited state solution to (1.5), then v is called an excited state solution.

3.1.2 Proof of Theorem 2(i)

To prove the Theorem 2(i), we let w denote the solution to (1.5) constructed in Theorem 1(i) and show that the standing wave solution v given by $v(x,t) = e^{i\eta t}w(x)$ is stable under small radially symmetric perturbations.

As we know, w is a ground state solution to (1.5), then $v(x,t) = e^{i\eta t}w(x)$ is a ground state solution to (1.3). Note that the ground state solutions are radially symmetric for each t, then we restrict ourselves to H^1_r when considering the stability of ground state solutions. Thus, we shall prove the stability of the standing wave solution v in the sense of Definition 1.2.1.

Since the Laplacian in NLS (1.3) is invariant under rotations, even if a solution $u(x,t)$ to (1.3) is rotated in spatial variables, it remains a solution to (1.3). Thus, if an initial condition $u_0 \in H^1_r$, from Proposition 2.2.2, we have the corresponding solution $u(x,t)$ is also radially symmetric for each t. More precisely, if we take a solution u that has a radially symmetric initial condition and let u_θ denote the solution obtained from rotating u by angle θ around some axis of rotation for all t. As the initial condition $u_0 \in H^1_r$, we have

$$u(x,0) = u_\theta(x,0),$$

for $x \in \mathbb{R}$ and $t = 0$. Then according to the uniqueness of solutions, we have $u \equiv u_\theta$ for all t, which implies that both the angle and the axis of rotation are arbitrary. That is to say, the solution u is radially symmetric.

In Definition 1.2.1, we do not need translations in spatial variables since only radially symmetric solutions are being considered.

Now, let us prove the standing wave solution v is stable under small radially symmetric perturbations, i.e. Theorem 2(i).

Proof. We prove this theorem by contradiction. Let v be a ground state solution to (1.3) and we assume it is unstable in the sense of Definition 1.2.1. Then there exists $\epsilon_0 > 0$, a sequence $\{t_k\} \subset \mathbb{R}_+$, and a sequence $\{u_{0,k}\} \subset H_r^1$ such that

$$\begin{aligned} \|u_{0,k} - v(0)\|_{H^1} &< \frac{1}{k}, \\ \inf_{\theta \in \mathbb{R}} \|(u_k(t_k) - e^{i\theta} v(t_k)\|_{H^1} &\geqslant \epsilon_0, \end{aligned} \tag{3.11}$$

for all $k \in \mathbb{N}$, where u_k denotes the solution to (1.3) with i.c. $u_k(0) = u_{0,k}$. From Proposition 2.2.4, it follows that $u_k(t)$ exists for all $0 \leqslant t < \infty$ and

$$\begin{aligned} \|u_k(t)\|_{L^2} &= \|u_{0,k}\|_{L^2}, \\ E(u_k(t)) &= E(u_{0,k}). \end{aligned}$$

Now, let $\alpha = \|v(0)\|_{L^2}^2$, then, as k goes to infinity, we have

$$\|u_k(t_k)\|_{L^2} = \|u_{0,k}\|_{L^2} \to \sqrt{\alpha}. \tag{3.12}$$

Moreover, by Proposition 3.1.4, we have

$$\begin{aligned} E(v(x)) &= E(e^{i\theta} w(x+a)) \\ &= \frac{1}{2}\|\nabla e^{i\theta} w(x+a)\|_{L^2}^2 - \frac{1}{p+1}\|e^{i\theta} w(x+a)\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2}\|\nabla w\|_{L^2}^2 - \frac{1}{p+1}\|w\|_{L^{p+1}}^{p+1} \\ &= E(w) = c_\alpha, \end{aligned}$$

for all $t \geqslant 0$. Together with the assumption $u_{0,k} \to v(0)$ in H^1 and Proposition 2.1.3, we obtain that, as $k \to \infty$,

$$\begin{aligned} E(u_{0,k}) &\to c_\alpha, \\ \text{i.e.} \quad E(u_k(t_k)) &\to c_\alpha, \end{aligned} \tag{3.13}$$

for all $t \geqslant 0$. From (3.12)-(3.13), we can apply the Lemma 3.1.3 to $(u_k(t_k))$, then there exists a subsequence $\{u_{k_l}(t_{k_l})\}$ of $\{u_k(t_k)\}$ and $\widetilde{w} \in H_r^1$ such that as $k_l \to \infty$,

$$\begin{aligned} &\{u_{k_l}(t_{k_l})\} \longrightarrow \widetilde{w} \quad \text{in } H^1, \\ &E(\widetilde{w}) = c_\alpha, \quad \|\widetilde{w}\|_{L^2} = \sqrt{\alpha}. \end{aligned}$$

Therefore, we have $\widetilde{w} \in G_\eta$. It is followed by

$$\{u_{k_l}(t_{k_l})\} \longrightarrow e^{i\theta} w \quad \text{in } H^1,$$

as $k_l \to \infty$, which contradicts (3.11). This completes the proof. $\square$

The final part of this section is devoted to proving Lemma 3.1.3. In the following, we introduce one more lemma to present the proof of Lemma 3.1.3.

Lemma 3.1.5 ([13] Lemma 5.4.11)**.** Let $n \geqslant 2, 1 < p < 1 + \frac{4}{n}$. For $\alpha > 0$, suppose that we have

$$\lim_{m\to\infty} \|u_m\|_{L^2} = \sqrt{\alpha}.$$

Then,

$$\liminf_{m\to\infty} E(u_m) \geqslant c_\alpha.$$

Proof. Since we can easily see that the lemma holds when $\liminf_{m\to\infty} E(u_m) = \infty$, we just need to consider the case $\liminf_{m\to\infty} E(u_m) < \infty$. Assume by contradiction that there exists a subsequence $\{u_{m_k}\}$ such that

$$E(u_{m_k}) \leqslant c_\alpha - \epsilon_0, \tag{3.14}$$

for any m_k. Define

$$\begin{aligned} u^\lambda_{m_k} &= (1+\lambda)u_{m_k}, \\ \lambda &= \frac{\alpha}{\|u_{m_k}\|^2_{L^2}} - 1. \end{aligned}$$

Then, we obtain

$$\|u^\lambda_{m_k}\|^2_{L^2} = (1+\lambda)\|u_{m_k}\|^2_{L^2} = \alpha,$$

which implies $u^\lambda_{m_k} \in k_\alpha$. From (3.8) and $\frac{n(p-1)}{2} < 2$, it follows that

$$\begin{aligned} E(u_{m_k}) &\geqslant \frac{1}{2}\|\nabla u_{m_k}\|^2_{L^2} - C\|u_{m_k}\|_{L^2}^{\frac{n+2-p(n-2)}{2}} \|\nabla u_{m_k}\|_{L^2}^{\frac{n(p-1)}{2}} \\ &> -\infty, \end{aligned}$$

for any $u_{m_k} \in H^1$. That is saying the sequence $\{u_{m_k}\}$ is bounded in H^1_r. Thus,

$$\lim_{m_k\to\infty} \|u^\lambda_{m_k} - u_{m_k}\|_{H^1} = 0.$$

Then there must exist some $N \geqslant 0$ such that

$$E(u^\lambda_{m_k}) \leqslant E(u_{m_k}) + \frac{1}{2}\epsilon_0 \leqslant c_\alpha - \frac{1}{2}\epsilon_0, \tag{3.15}$$

for all $m_k \geqslant N$. However, by definition of c_α and $u^\lambda_{m_k} \in k_\alpha$, we shall have $E(u^\lambda_{m_k}) \geqslant c_\alpha$. Therefore, (3.15) is a contradiction which completes the proof. □

Proof of Lemma 3.1.3 : The proof consists of four steps. The first step employs the compactness property of the embedding $H^1_r \subset L^p_r$ to demonstrate that $w \neq 0$. In step 2, we establish that $\|w\|_{L^2} = \liminf_{m_k\to\infty} \|u_{m_k}\|_{L^2} = \sqrt{\alpha}$. Step 3 demonstrates the convergence of u_{m_k} to w in the H^1 space. Lastly, in step 4, we show w is a weak solution to (1.5) for some $\eta > 0$.

Step 1: By the hypothesis $\|u_m\|_{L^2} \to \sqrt{\alpha}$ and (3.8) with $\frac{n(p-1)}{2} < 2$, we have $\{u_m\}$ is bounded H^1_r. Then, by Lemma 2.1.4, there exists a subsequence $\{u_{m_k}\}$,

$w \in H_r^1$ and $\delta \geqslant 0$ such that

$$u_{m_k} \longrightarrow w \quad \text{weakly in } H_r^1, \tag{3.16}$$
$$u_{m_k} \longrightarrow w \quad \text{in } L^{p+1}, \tag{3.17}$$
$$\|\nabla u_{m_k}\|_{L^2} \longrightarrow \delta, \tag{3.18}$$
$$\|w\|_{L^2} \leqslant \liminf_{m_k \to \infty} \|u_{m_k}\|_{L^2} = \sqrt{\alpha}. \tag{3.19}$$

We claim $w \neq 0$ and prove it by contradiction. Suppose $w = 0$. Recall the expression of $E(u)$ and substitute (3.18) into it, then we have

$$\begin{aligned}\lim_{m_k \to \infty} E(u_{m_k}) &= \lim_{m_k \to \infty} (\frac{1}{2}\|\nabla u_{m_k}\|_{L^2}^2 - \frac{1}{p+1}\|u_{m_k}\|_{L^{p+1}}^{p+1}) \\ &= \frac{1}{2}\delta^2 - \frac{1}{p+1}\|w\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2}\delta^2.\end{aligned}$$

On the other hand, $\{u_{m_k}\}$ is a subsequence of $\{u_m\}$, thus

$$E(u_{m_k}) \longrightarrow c_\alpha, \tag{3.20}$$

as $m_k \to \infty$. By the uniqueness of limits,

$$c_\alpha = \frac{1}{2}\delta^2 \geqslant 0,$$

which contradicts the assumption $c_\alpha < 0$. Therefore, we have $w \neq 0$.

Step 2: Let $\gamma = \|w\|_{L^2}^2$, then $\gamma \leqslant \alpha$ by (3.19). We aim to show $\gamma = \alpha$ in this step. First we assume $\gamma < \alpha$, and then let

$$\widetilde{u}_{m_k} := u_{m_k} - w.$$

From (3.16)-(3.17), it follows that

$$\begin{aligned}\widetilde{u}_{m_k} &\longrightarrow 0 \quad \text{weakly in } H_r^1, \\ \widetilde{u}_{m_k} &\longrightarrow 0 \quad \text{in } L^{p+1},\end{aligned} \tag{3.21}$$

as $m_k \to \infty$. Moreover, we derive that

$$\begin{aligned}(\widetilde{u}_{m_k}, w) &\longrightarrow 0, \\ \|u_{m_k}\|_{L^2}^2 &= \|\widetilde{u}_{m_k} + w\|_{L^2}^2 \\ &= \|\widetilde{u}_{m_k}\|_{L^2}^2 + \|w\|_{L^2}^2 + 2\mathrm{Re}(\widetilde{u}_{m_k}, w) \\ &\longrightarrow \alpha,\end{aligned}$$

Thus, we obtain

$$\|\widetilde{u}_{m_k}\|_{L^2}^2 \longrightarrow \alpha - \gamma, \tag{3.22}$$

as $m_k \to \infty$.

Recall the proof of Theorem 1(i), we define $u_\lambda(x) = \lambda^\beta u(\lambda x)$, with $\lambda > 0$ and $\beta \in \mathbb{R}$. Now let $\beta = \frac{2}{(p-1)}$. Then, by (3.5)-(3.7), we have

$$\begin{aligned}
\|u_\lambda\|_{L^2}^2 &= \lambda^{\frac{4}{p-1}-n}\|u\|_{L^2}^2 = \lambda^{\frac{4-n(p-1)}{p-1}-}\|u\|_{L^2}^2, \\
E(u_\lambda) &= \frac{1}{2}\|\nabla u_\lambda\|_{L^2}^2 - \frac{1}{p+1}\|u_\lambda\|_{L^{p+1}}^{p+1} \\
&= \frac{1}{2}\lambda^{\frac{4}{p-1}-n+2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\lambda^{\frac{2(p+1)}{p-1}-n}\|u\|_{L^{p+1}}^{p+1} \\
&= \lambda^{\frac{n+2-(n-2)p}{p-1}} E(u).
\end{aligned} \tag{3.23}$$

For $0 < \gamma < \alpha$, let

$$\lambda = \left(\frac{\gamma}{\alpha}\right)^{\frac{p-1}{4-n(p-1)}},$$

then

$$\|u_\lambda\|_{L^2}^2 = \left(\frac{\gamma}{\alpha}\right)\|u\|_{L^2}^2 = \gamma,$$

which means $u_\lambda \in K_\gamma$. Taking infimum on both sides of (3.23) yields

$$\begin{aligned}
\inf_{u \in K_\alpha} E(u_\lambda) &= \inf_{u_\lambda \in K_\gamma} E(u_\lambda) \\
&= \lambda^{\frac{n+2-(n-2)p}{p-1}} \inf_{u \in K_\alpha} E(u_\lambda).
\end{aligned}$$

By the definition of c_α, it can be rewritten as

$$c_\gamma = \lambda^{\frac{n+2-(n-2)p}{p-1}} c_\alpha = \left(\frac{\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}} c_\alpha.$$

Replacing γ by $\alpha - \gamma$ gives us

$$c_{\alpha-\gamma} = \left(\frac{\alpha-\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}} c_\alpha.$$

Recall the following technique: A function $\Phi : \mathbb{R} \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$ and $a \in [0,1]$, we have

$$\Phi(ax + (1-a)y) \leqslant a\Phi(x) + (1-a)\Phi(y).$$

Moreover, it has the following property:

$$\Phi(x) + \Phi(y) \leqslant \Phi(x+y).$$

Thus, from convexity of the function $\Phi(s) = s^q$, $s > 0$ with $q > 1$, it follows that

$$\begin{aligned}
\Phi(\theta) + \Phi(1-\theta) &< \Phi(\theta + 1 - \theta) \\
\theta^q + (1-\theta)^q &< 1.
\end{aligned}$$

where $0 < \theta < 1$.

By the assumption $1 < p < \frac{4}{n}$, we have $\frac{n+2-(n-2)p}{4-n(p-1)} > 1$. Using the above technique, we derive that

$$\left(\frac{\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}} + \left(\frac{\alpha-\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}} < 1.$$

Thus, we obtain

$$\begin{aligned} c_\gamma + c_{\alpha-\gamma} &= \left[\left(\frac{\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}} + \left(\frac{\alpha-\gamma}{\alpha}\right)^{\frac{n+2-(n-2)p}{4-n(p-1)}}\right] c_\alpha \\ &> c_\alpha. \end{aligned} \tag{3.24}$$

On the other hand, we have

$$\begin{aligned} E(u_{m_k}) &= E(\widetilde{u}_{m_k} + w) \\ &= \frac{1}{2}\|\nabla \widetilde{u}_{m_k} + \nabla w\|_{L^2}^2 - \frac{1}{p+1}\|\widetilde{u}_{m_k} + w\|_{L^{p+1}}^{p+1} \\ &= E(\widetilde{u}_{m_k}) + E(w) + A(\widetilde{u}_{m_k}, w), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} A(\widetilde{u}_{m_k}, w) &= \mathrm{Re}(\nabla \widetilde{u}_{m_k}, \nabla w) \\ &\quad - \frac{1}{p+1}[\|\widetilde{u}_{m_k} + w\|_{L^{p+1}}^{p+1} - \|\widetilde{u}_{m_k}\|_{L^{p+1}}^{p+1} - \|w\|_{L^{p+1}}^{p+1}]. \end{aligned}$$

By (3.21), we have that, as $m_k \to \infty$,

$$\begin{aligned} \mathrm{Re}(\nabla \widetilde{u}_{m_k}, \nabla w) &\to 0, \\ \|\widetilde{u}_{m_k}\|_{L^{p+1}} &\to 0. \end{aligned}$$

Moreover, we apply Dominated convergence theorem to derive

$$\lim_{m_k\to\infty} \|\widetilde{u}_{m_k} + w\|_{L^{p+1}} = \|\lim_{m_k\to\infty}(\widetilde{u}_{m_k} + w)\|_{L^{p+1}} = \|w\|_{L^{p+1}}.$$

Thus, as $m_k \to \infty$, we have

$$A(\widetilde{u}_{m_k}, w) \longrightarrow 0.$$

By (3.22) and Lemma 3.1.5, taking limit infimum on both sides of (3.25) yields

$$\begin{aligned} c_\alpha &= \liminf_{m_k\to\infty} E(u_{m_k}) \\ &\geqslant \liminf_{m_k\to\infty} E(\widetilde{u}_{m_k}) + E(w) + \liminf_{m_k\to\infty} A(\widetilde{u}_{m_k}, w) \\ &\geqslant c_{\alpha-\gamma} + c_\gamma, \end{aligned}$$

which contradicts to (3.24). Therefore, we conclude $\gamma = \alpha$.

Step 3: In this step, we show $u_{m_k} \to w$ in H^1 as $m_k \to \infty$.

First, we have (3.16) which is equivalent to the following:

$$u_{m_k} \to w \quad \text{weakly in } L^2, \tag{3.26}$$

$$\nabla u_{m_k} \to \nabla w \quad \text{weakly in } L^2. \tag{3.27}$$

Then, we deduce from (3.26) that

$$\begin{aligned}\|u_{m_k} - w\|_{L^2}^2 &= \|u_{m_k}\|_{L^2}^2 - 2\mathrm{Re}(u_{m_k}, w) + \|w\|_{L^2}^2 \\ &\to \|u_{m_k}\|_{L^2}^2 - 2\|w\|_{L^2}^2 + \|w\|_{L^2}^2. \\ &\to \alpha - \gamma,\end{aligned}$$

as $m_k \to \infty$. Since we have $\gamma = \alpha$, then $\lim_{m_k \to \infty} \|u_{m_k} - w\|_{L^2}^2 = 0$. It is known as a strong convergence:

$$u_{m_k} \to w \quad \text{in } L^2.$$

Next, from (3.16), (3.17) and (3.20), it follows that

$$\begin{aligned}c_\alpha &= \lim_{m_k \to \infty} E(u_{m_k}) \\ &\geqslant \frac{1}{2} \liminf_{m_k \to \infty} \|\nabla u_{m_k}\|_{L^2}^2 - \frac{1}{p+1}\|u_{m_k}\|_{L^{p+1}}^{p+1} \\ &\geqslant E(w).\end{aligned}$$

From step 2, we have $\|w\|_{L^2} = \sqrt{\gamma} = \sqrt{\alpha}$ which implies $w \in k_\alpha$. Thus, by the definition of c_α, we have $c_\alpha \leqslant E(w)$. Then, we conclude that

$$c_\alpha = E(w). \tag{3.28}$$

From (3.16), (3.20) and (3.28), we deduce

$$\begin{aligned}\lim_{m_k \to \infty} \|\nabla u_{m_k}\|_{L^2}^2 &= 2 \lim_{m_k \to \infty} \left[E(u_{m_k}) + \frac{1}{p+1}\|u_{m_k}\|_{L^{p+1}}^{p+1} \right] \\ &= 2\left(c_\alpha + \frac{1}{p+1}\|w\|_{L^{p+1}}^{p+1} \right) \\ &= \|\nabla w\|_{L^2}^2,\end{aligned}$$

which together with (3.27), implies

$$\begin{aligned}&\lim_{m_k \to \infty} \|\nabla u_{m_k} - \nabla w\|_{L^2}^2 \\ &= \lim_{m_k \to \infty} (\|\nabla u_{m_k}\|_{L^2}^2 - 2\mathrm{Re}(\nabla u_{m_k}, \nabla w) + \|\nabla w\|_{L^2}^2) \\ &= \|\nabla w\|_{L^2}^2 - 2\|\nabla w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \\ &= 0.\end{aligned}$$

It is followed by

$$\nabla u_{m_k} \to \nabla w \quad \text{in } L^2.$$

As a consequence, $u_{m_k} \longrightarrow w$ in H^1.

Step 4: The final step of the proof is to show the function w is a weak solution to (1.5) for some $\eta > 0$.
From step 3, we have $w \in K_\alpha$ and $E(w) = c_\alpha$. That is to say, w solves the minimization problem (3.1):

$$E(w) = \inf_{u \in K_\alpha} E(u).$$

By Lemma 3.1.1, we know that the problem (3.1) is equivalent to the problem (1.6). Thus,

$$S_\eta(w) = b_\alpha = \inf_{u \in K_\alpha} S_\eta(u).$$

It indicates that $w \in K_\alpha$ is a local extremum point of S_η on K_α.
Now, we apply Proposition 2.3.4(ii) with $a = c = 0$ and $b = 1$ such that

$$T(w) = \|w\|_{L^2}^2 = \alpha, \tag{3.29}$$
$$\partial_w T(w)v = (v, w), \tag{3.30}$$
$$K = \{v \in H^1; \|v\|_{L^2}^2 = \alpha\} = K_\alpha. \tag{3.31}$$

Then it follows from $\|w\|_{L^2}^2 = \sqrt{\alpha} \neq 0$ that, for $v \in H^1$,

$$\partial_w S_\eta(w)v - \lambda \partial_w T(w)v = 0,$$

where

$$\lambda = \frac{\mathrm{Re}[\partial_w S_\eta(w)w]}{\mathrm{Re}[\partial_w T(w)w]} = \frac{\|\nabla w\|_{L^2}^2 - \|w\|_{L^{p+1}}^{p+1}}{\|w\|_{L^2}^2}.$$

Substitute (2.6) and (3.30) into the equation, we get

$$(\nabla v, \nabla w) + \eta(v, w) - (v, |w|^{p-1}w) = \lambda(v, w),$$
$$(\nabla v, \nabla w) + (\eta - \lambda)(v, w) - (v, |w|^{p-1}w) = 0,$$

where $\eta > 0$ and $v \in H^1$. Since $E(w) = \frac{1}{2}\|\nabla w\|_{L^2}^2 - \frac{1}{p+1}\|w\|_{L^{p+1}}^{p+1} = c_\alpha < 0$, we have $\lambda < 0$. Thus, w is a weak solution to (1.5) for some $\eta' = \eta - \lambda > 0$.

This completes the proof of Lemma 3.1.3. □

3.2 Existence of standing wave solutions and their stability for $1 + \frac{4}{n} < p < p^*(n)$

In this section, we discuss the existence and instability of standing wave solutions when $n \geqslant 2$ and $1 + \frac{4}{n} < p < p^*(n)$.

3.2.1 Proof of Theorem 1(ii)

Let us recall the conditional minimization problem (1.7) for $1 < p < 1 + \frac{n}{4}$:

$$c_\eta = \inf_{u \in K} S_\eta(u),$$

where $\eta > 0$ and K is defined as

$$K = \{u \in H^1;\ u \neq 0,\ T(u) = 0\},$$

with

$$T(u) = 2\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{3.32}$$

For simplicity, we use another equivalent minimization problem to prove the theorem:

$$c_\eta = \inf_{u \in K} J_\eta(u), \tag{3.33}$$

where

$$J_\eta(u) = \frac{n(p-1)-4}{n(p-1)}\|\nabla u\|_{L^2}^2 + \eta\|u\|_{L^2}^2. \tag{3.34}$$

It follows from $p > 1 + \frac{4}{n}$ that the coefficient of the first term in (3.34) is positive. Hence, c_η is a finite non-negative number.

Proof. Since the minimization problem (1.9) is equivalent to (3.33), it suffices to show there exists $w \in K \cap H_r^1$ attaining the minimum value c_η in (3.33) and this function w is a weak solution to (1.5).

The proof of Theorem 1(ii) is made of two steps. In step 1, we find a solution $w \in K \cap H_r^1$ to the minimization problem (3.33). Then, we prove w is also a weak solution to (1.5) in step 2.

Step 1: By the definition of c_η, we can find a sequence $\{u_m\}$ in K such that, as $m \to \infty$,

$$J_\eta(u_m) \longrightarrow c_\eta, \tag{3.35}$$

From the definition of the functional J_η, it follows that $\{u_m\}$ is bounded in H^1. In view of Lemma 3.1.2, there exists a bounded sequence $\{u_m^*\}$ in H^1 satisfying

$$\begin{aligned} \|u_m^*\|_{L^p} &= \|u_m\|_{L^p}, \quad 2 \leqslant p < p^*(n), \\ \|\nabla u_m^*\|_{L^2} &\leqslant \|\nabla u_m\|_{L^2}. \end{aligned}$$

Then, by (3.32), (3.34) and the fact that $\{u_m\} \in K$, we have

$$\begin{aligned} J_\eta(u_m^*) &\leqslant J_\eta(u_m), \\ T(u_m^*) &\leqslant T(u_m) = 0. \end{aligned} \tag{3.36}$$

When $T(u_m^*) < 0$, we choose $\lambda \in (0,1)$ such that

$$T(\lambda u_m^*) = 2\lambda^2\|\nabla u_m^*\|_{L^2}^2 - \lambda^{p+1}\frac{n(p-1)}{p+1}\|u_m^*\|_{L^{p+1}}^{p+1} = 0.$$

There exists unique such λ since $p+1 > 2$. Then, we define a new sequence $\{v_m\}$ such that $v_m = u_m^*$ when $T(u_m^*) = 0$, and $v_m = \lambda u_m^*$ when $T(u_m^*) < 0$. Thus, we have $T(v_m) = 0$ which implies that $\{v_m\} \in K$.

By definition of c_η, we have

$$c_\eta \leqslant J_\eta(v_m). \tag{3.37}$$

Note that $\lambda \in (0,1)$. Hence,

$$\begin{aligned}
J_\eta(\lambda u_m^*) &= \lambda^2 \frac{n(p-1)-4}{n(p-1)} \|\nabla u_m^*\|_{L^2}^2 + \lambda^2 \eta \|u_m^*\|_{L^2}^2 \\
&\leqslant \frac{n(p-1)-4}{n(p-1)} \|\nabla u_m^*\|_{L^2}^2 + \eta \|u_m^*\|_{L^2}^2 \\
&= J_\eta(u_m^*), \\
i.e. \quad J_\eta(v_m) &\leqslant J_\eta(u_m^*).
\end{aligned} \tag{3.38}$$

Combining (3.36),(3.37) with (3.38) yields

$$c_\eta \leqslant J_\eta(v_m) \leqslant J_\eta(u_m^*) \leqslant J_\eta(u_m).$$

for all $m \geqslant 1$. Hence, v_m is bounded in H_r^1. Moreover, by (3.35), we have

$$J_\eta(v_m) \longrightarrow c_\eta, \tag{3.39}$$

as $m \to \infty$. By Lemma 2.1.4 and Proposition 2.1.11, there exists a subsequence $\{v_{m_k}\} \in K$ and $w \in H_r^1$ such that

$$v_{m_k} \longrightarrow w \quad \text{weakly in } H^1, \tag{3.40}$$

$$v_{m_k} \longrightarrow w \quad \text{in } L^{p+1}. \tag{3.41}$$

Now we claim that $w \in K$. It suffices to prove $T(w) = 0$ and $w \neq 0$. From (3.39),(3.40) and Proposition 2.1.10, it follows that

$$\begin{aligned}
c_\eta &= \lim_{m_k \to \infty} J_\eta(v_{m_k}) \\
&\geqslant \frac{n(p-1)-4}{n(p-1)} \liminf_{m_k \to \infty} \|\nabla v_{m_k}\|_{L^2}^2 + \eta \lim_{m_k \to \infty} \|v_{m_k}\|_{L^2}^2 \\
&\geqslant \frac{n(p-1)-4}{n(p-1)} \|\nabla w\|_{L^2}^2 + \eta \|w\|_{L^2}^2 \\
&= J_\eta(w).
\end{aligned} \tag{3.42}$$

In a like manner, we obtain the following inequality:

$$\begin{aligned}
\lim_{m_k \to \infty} T(v_{m_k}) &\geqslant 2 \liminf_{m_k \to \infty} \|\nabla v_{m_k}\|_{L^2}^2 - \frac{n(p-1)}{p+1} \lim_{m_k \to \infty} \|v_{m_k}\|_{L^{p+1}}^{p+1} \\
&\geqslant 2\|\nabla w\|_{L^2}^2 - \frac{n(p-1)}{p+1} \|w\|_{L^{p+1}}^{p+1} \\
&= T(w).
\end{aligned}$$

Since $\{v_{m_k}\} \in K$, we have $T(v_{m_k}) = 0$. Thus, $T(w) \leqslant 0$.

Let us prove $T(w) = 0$ by contradiction. Suppose $T(w) < 0$, then we choose $\lambda \in (0,1)$ such that

$$T(\lambda w) = 2\lambda^2 \|\nabla w\|_{L^2}^2 - \lambda^{p+1} \frac{n(p-1)}{p+1} \|w\|_{L^{p+1}}^{p+1} = 0.$$

Let $\widetilde{w} = \lambda w$. If $w \neq 0$, then $\widetilde{w} \in K$. Furthermore, by (3.42), we have

$$J_\eta(\widetilde{w}) = \lambda^2 J_\eta(w) < J_\eta(w) \leqslant c_\eta.$$

It contradicts the definition of c_η (3.33). Hence, $T(w) = 0$.

To complete the proof, we still need to show $w \neq 0$. From Proposition 2.1.3 and $T(v_{m_k}) = 0$, it follows that

$$\begin{aligned} \|\nabla v_{m_k}\|_{L^2}^2 &= \frac{n(p-1)}{2(p+1)} \|v_{m_k}\|_{L^{p+1}}^{p+1} \\ &\lesssim \|v_{m_k}\|_{L^2}^{(1-a)(p+1)} \|\nabla v_{m_k}\|_{L^2}^{a(p+1)}, \end{aligned}$$

where

$$a = \frac{n}{2} - \frac{n}{p+1} = \frac{n(p-1)}{2(p+1)}.$$

Since $v_{m_k} \neq 0$, dividing both sides by $\|\nabla v_{m_k}\|_{L^2}^2$ gives

$$1 \lesssim \|v_{m_k}\|_{L^2}^{\frac{(2-n)p+n+2}{2}} \|\nabla v_{m_k}\|_{L^2}^{\frac{n(p-1)}{2}-2}. \tag{3.43}$$

Now suppose $w = 0$, then from (3.41), we obtain

$$\begin{aligned} \lim_{m_k \to \infty} \|\nabla v_{m_k}\|_{L^2}^2 &= \lim_{m_k \to \infty} \left(\frac{n(p-1)}{2(p+1)} \|v_{m_k}\|_{L^{p+1}}^{p+1} \right) \\ &= \frac{n(p-1)}{2(p+1)} \|w\|_{L^{p+1}}^{p+1} \\ &= 0. \end{aligned} \tag{3.44}$$

Since $\{v_{m_k}\}$ is bounded in H_r^1, we have $\|v_{m_k}\|_{L^2}$ bounded. Also, by the hypothesis of p, we deduce $\frac{n(p-1)}{2} - 2 > 0$. Moreover, we have $\|\nabla v_{m_k}\|_{L^2} \to 0$ from (3.44). Therefore,

$$\lim_{m_k \to \infty} \left(\|v_{m_k}\|_{L^2}^{\frac{(2-n)p+n+2}{2}} \|\nabla v_{m_k}\|_{L^2}^{\frac{n(p-1)}{2}-2} \right) = 0.$$

This contradicts (3.43). Hence, $w \neq 0$. Now, we conclude $w \in K$ and it is followed by $c_\eta \leqslant J_\eta(w)$. Combining this with (3.42) yields

$$J_\eta(w) = c_\eta.$$

Step 1 illustrates that there exists $w \in K \cap H_r^1$ attaining the minimum value c_η in (3.33). In step 2, we shall prove it is a weak solution to (1.5).

Step 2: Notice that w is also a solution to the minimization problem (1.9). Then, by Proposition 2.3.4(ii), we have

$$\partial_w S_\eta(w)v - \mu \partial_w T(w)v = 0, \tag{3.45}$$

for $v \in H^1$, where

$$\mu = \frac{\mathrm{Re}[\partial_w S_\eta(w)w]}{\mathrm{Re}[\partial_u T(w)w]}. \tag{3.46}$$

To satisfy the hypothesis of Proposition 2.3.4, we claim that $\mathrm{Re}[\partial_u T(w)w] \neq 0$. From (2.17) and (3.32), we obtain

$$\begin{aligned}\mathrm{Re}[\partial_u T(w)w] &= 2\|\nabla w\|_{L^2}^2 - \frac{n(p-1)(p+1)}{2(p+1)}\|w\|_{L^{p+1}}^{p+1} \\ &= 2\|\nabla w\|_{L^2}^2 - \frac{n(p-1)}{2}\|w\|_{L^{p+1}}^{p+1}.\end{aligned}$$

Moreover, in step 1 we have

$$T(w) = 2\|\nabla w\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|w\|_{L^{p+1}}^{p+1} = 0, \tag{3.47}$$

which together with $p+1 > 2$, implies that

$$\mathrm{Re}[\partial_u T(w)w] < T(w) = 0.$$

Next, we claim $\mu = 0$. Define

$$w_\lambda = \lambda^\beta w(\lambda x),$$

where $\lambda > 0$. Then, we deduce from (3.5)-(3.7) that

$$\begin{aligned}T(w_\lambda) &= 2\|\nabla w_\lambda\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|w_\lambda\|_{L^{p+1}}^{p+1} \\ &= 2\lambda^{2\beta-n+2}\|\nabla w\|_{L^2}^2 - \frac{n(p-1)}{p+1}\lambda^{\beta(p+1)-n}\|w\|_{L^{p+1}}^{p+1}.\end{aligned}$$

By letting $\beta = \frac{2}{p-1}$, we have

$$T(w_\lambda) = \lambda^{\frac{n+2-(n-2)p}{p-1}}T(w) = 0,$$

which implies that $w_\lambda \in K$.

In a like mannar, from (3.5)-(3.7), we have

$$\begin{aligned}S_\eta(w_\lambda) &= \|\nabla w_\lambda\|_{L^2}^2 + \eta\|w_\lambda\|_{L^2}^2 - \frac{2}{p+1}\|w_\lambda\|_{L^{p+1}}^{p+1} \\ &= \lambda^{\frac{4}{p-1}-n+2}\|\nabla w\|_{L^2}^2 + \lambda^{\frac{4}{p-1}-n}\eta\|w\|_{L^2}^2 - \frac{2}{p+1}\lambda^{\frac{2(p+1)}{p-1}-n}\|w\|_{L^{p+1}}^{p+1} \\ &= \lambda^{\frac{n+2-(n-2)p}{p-1}}\left(\|\nabla w\|_{L^2}^2 - \frac{2}{p+1}\|w\|_{L^{p+1}}^{p+1}\right) + \lambda^{\frac{4+n-np}{p-1}}\eta\|w\|_{L^2}^2.\end{aligned}$$

Since w attains the minimum of $S_\eta(u)$ on the set K and $w_\lambda \in K$, then

$$S_\eta(w_\lambda) \geqslant c_\eta = S_\eta(w).$$

We observe that $S_\eta(w_\lambda)$ attains its minimum at $\lambda = 1$.

Recall that the derivative of a function at an extremum point is zero. Therefore, let us take a derivative of $S_\eta(w_\lambda)$ at $\lambda = 1$,

$$\begin{aligned}\frac{d}{d\lambda}S_\eta(w_\lambda)\Big|_{\lambda=1} &= \frac{n+2-(n-2)p}{p-1}\left(\|\nabla w\|_{L^2}^2 - \frac{2}{p+1}\|w\|_{L^{p+1}}^{p+1}\right) \\ &+ \frac{4+n-np}{p-1}\eta\|w\|_{L^2}^2 \\ &= 0.\end{aligned} \tag{3.48}$$

Moreover, from (3.47), it follows that

$$\|\nabla w\|_{L^2}^2 = \frac{n(p-1)}{2(p+1)}\|w\|_{L^{p+1}}^{p+1}.$$

Substituting this into (3.48) yields

$$\begin{aligned}-(4+n-np)\,\eta\|w\|_{L^2}^2 &= (n+2-(n-2)p)\left(\frac{n(p-1)-4}{2(p+1)}\|w\|_{L^{p+1}}^{p+1}\right), \\ \eta\|w\|_{L^2}^2 &= \frac{n+2-(n-2)p}{np-n+4}\left(\frac{n(p-1)-4}{2(p+1)}\|w\|_{L^{p+1}}^{p+1}\right) \\ &= \frac{n+2-(n-2)p}{2(p+1)}\|w\|_{L^{p+1}}^{p+1}.\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}&\|\nabla w\|_{L^2}^2 + \eta\|w\|_{L^2}^2 - \|w\|_{L^{p+1}}^{p+1} \\ &= \frac{n(p-1)}{2(p+1)}\|w\|_{L^{p+1}}^{p+1} + \frac{n+2-(n-2)p}{2(p+1)}\|w\|_{L^{p+1}}^{p+1} - \|w\|_{L^{p+1}}^{p+1} \\ &= 0.\end{aligned}$$

It is equivalent to $\partial_w S_\eta(w)w = 0$ which implies $\mu = 0$ by (3.46). Then, it follows from (3.45) that

$$\partial_w S_\eta(w)v = 0,$$

for $v \in H^1$. Lastly, by Definition 2.1.7, we conclude $w \in K \cap H_r^1$ is a weak solution to (1.5).

□

From the proof of Theorem 1(ii), it follows that $w \in K \cap H_r^1$ is radially symmetric, non-increasing in $r = |x|$, and non-negative for all $x \in \mathbb{R}^n$. In other words, w is a ground state solution to (1.5). By Proposition 3.1.4, any weak solution z to (1.5) such that $S_\eta(z) = S_\eta(w)$ coincides with the solution w, modulo translations and modulations.

We then construct a ground state solution v to (1.3) by $v = e^{i\eta t}w$ and show it is indeed unstable.

3.2.2 Proof of Theorem 2(ii)

First, we need to understand the definition of orbital stability 1.2.2.

For a solution u to (1.3), the set $\{u(\cdot,t) \in H^1; t \geqslant 0\}$ is called the orbit of the solution u. Since a standing wave solution v is periodic with period $\frac{2\pi}{\eta}$, the orbit is given by

$$L = \{v(\cdot,t);\ 0 \leqslant t < \frac{2\pi}{\eta}\}.$$

Theoretically, Definition 1.2.2 is saying that for a stable standing wave solution v, if an initial condition u_0 is close to it, then the corresponding solution u to (1.3) stays close to its orbits.

In the following, we introduce two important lemmas that will appear in the proof of Theorem 2(ii).

Lemma 3.2.1 ([13] Lemma 5.5.4)**.** Let $n \geqslant 2, 1 + \frac{4}{n} < p < p^*(n)$ and $\eta > 0$. Suppose $u \in H^1$ satisfies

$$T(u) < 0,$$
$$S_\eta(u) < c_\eta.$$

Then, the following inequality holds:

$$T(u) \leqslant S_\eta(u) - c_\eta.$$

Proof. Define u_λ by

$$u_\lambda(x) = \lambda^\beta u(\lambda x),$$

where $\lambda > 0$ and $\beta \in \mathbb{R}$. Let $\beta = \frac{n}{2}$. It follows from (3.5)-(3.7) that

$$T(u_\lambda) = \lambda^2 \left(2\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{p+1}\lambda^{\frac{n(p-1-\frac{4}{n})}{2}}\|u\|_{L^{p+1}}^{p+1}\right), \tag{3.49}$$

$$S_\eta(u_\lambda) = \lambda^2\|\nabla u\|_{L^2}^2 + \eta\|u\|_{L^2}^2 - \frac{2}{p+1}\lambda^{\frac{n(p-1)}{2}}\|u\|_{L^{p+1}}^{p+1}. \tag{3.50}$$

Notice that, at $\lambda = 1$, we have the assumption $T(u) < 0$,

$$\text{i.e.}\quad 2\|\nabla u\|_{L^2}^2 < \frac{n(p-1)}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{3.51}$$

Since $p > 1 + \frac{4}{n}$, we have $\frac{n(p-1-\frac{4}{n})}{2} > 0$. Thus, there must exist a unique $\lambda^* \in (0,1)$ such that

$$2\|\nabla u\|_{L^2}^2 = \frac{n(p-1)}{p+1}\lambda^{*\frac{n(p-1-\frac{4}{n})}{2}}\|u\|_{L^{p+1}}^{p+1},$$

which implies that

$$T(u_{\lambda^*}) = 0.$$

Moreover, for all $\lambda \in (\lambda^*, 1]$,

$$T(u_\lambda) < 0.$$

Differentiating (3.50) in λ yields

$$\frac{d}{d\lambda}S_\eta(u_\lambda) = 2\lambda\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{(p+1)}\lambda^{\frac{n(p-1-\frac{2}{n})}{2}}\|u\|_{L^{p+1}}^{p+1},$$

Then, we derive the second derivative:

$$\frac{d^2}{d\lambda^2}S_\eta(u_\lambda) = 2\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{(p+1)}\frac{(n(p-1)-2)}{2}\lambda^{\frac{n(p-1-\frac{4}{n})}{2}}\|u\|_{L^{p+1}}^{p+1}. \tag{3.52}$$

Define $h(\lambda) := S_\eta(u_\lambda)$, then

$$\begin{aligned} h'(1) &= \frac{d}{d\lambda}S_\eta(u_\lambda)\big|_{\lambda=1} \\ &= 2\|\nabla u\|_{L^2}^2 - \frac{n(p-1)}{(p+1)}\|u\|_{L^{p+1}}^{p+1} \\ &= T(u). \end{aligned}$$

Furthermore, since $\frac{(n(p-1)-2)}{2} > 1$, we have

$$h''(\lambda) = \frac{d^2}{d\lambda^2}S_\eta(u_\lambda) < T(u_\lambda).$$

It is followed by

$$h''(\lambda) < 0, \quad \forall \lambda \in [\lambda^*, 1]. \tag{3.53}$$

By Taylor's theorem, there exists $\theta \in (0,1)$ such that

$$h(\lambda^*) = h(1) + (\lambda^* - 1)h'(1) + \frac{1}{2}(\lambda^* - 1)^2 h''(1 + \theta(\lambda^* - 1)), \tag{3.54}$$

where $1 + \theta(\lambda^* - 1)$ denotes every $\lambda \in (\lambda^*, 1)$. Then, it follows from (3.53) that

$$h''(1 + \theta(\lambda^* - 1)) < 0.$$

Thus, from (3.54), the following inequality holds:

$$S_\eta(u_{\lambda^*}) \leqslant S_\eta(u) + (\lambda^* - 1)T(u). \tag{3.55}$$

Since $T(u_{\lambda^*}) = 0$, we have $u_{\lambda^*} \in K$. Therefore, by the definition of c_η,

$$c_\eta \leqslant S_\eta(u_{\lambda^*}).$$

Then, from (3.55), it follows that

$$\begin{aligned} (\lambda^* - 1)T(u) &\geqslant S_\eta(u_{\lambda^*}) - S_\eta(u) \\ &\geqslant c_\eta - S_\eta(u). \end{aligned}$$

Moreover, by $\lambda^* \in (0,1)$ and $S_\eta(u) < c_\eta$, we derive that

$$\begin{aligned} T(u) &\leqslant \frac{1}{1-\lambda^*}(S_\eta(u) - c_\eta) \\ &\leqslant S_\eta(u) - c_\eta, \end{aligned}$$

which completes the proof. □

Lemma 3.2.2 ([13] Lemma 5.5.5)**.** Let $n \geq 2, 1 + \frac{4}{n} < p < p^*(n)$ and $\eta > 0$. For $d < c_\eta$, define

$$A_d := \{u \in H^1;\ T(u) < 0,\ S_\eta(u) \leqslant d\}.$$

Suppose that a solution u to (1.3) with initial condition $u_0 \in A_d$ at time $t = 0$ exists on $[0, T)$, satisfying

$$\begin{aligned} u &\in C([0,T); H^1) \cap C^1([0,T); H^{-1}), \\ \nabla u &\in L^r((0,T'); L^{p+1}), \end{aligned}$$

where $0 < T' < T$ and r satisfies

$$r\left(\frac{n}{2} - \frac{n}{p+1}\right) = 2.$$

Then, we have $u(t) \in A_d$ for all $t \in [0, T)$.

Proof. By Proposition 2.2.3, the conservation laws (2.3) hold for $t \in [0, T)$. Thus, we have

$$\begin{aligned} S_\eta(u(t)) &= 2E(u(t)) + \eta\|u(t)\|_{L^2}^2 \\ &= 2E(u_0) + \eta\|u_0\|_{L^2}^2 \\ &= S_\eta(u_0) \leqslant d, \end{aligned} \tag{3.56}$$

for $t \in [0, T)$. Next, set

$$t_0 = \sup\{t \in [0,T);\ T(u(t)) < 0\}.$$

Since $T(u(t))$ is a continuous function in t and $T(u_0) < 0$, we have $t_0 > 0$. Suppose $t_0 < T$, then by the continuity of $T(u(t))$ in t,

$$T(u(t_0)) = 0.$$

Moreover, it is followed from the L^2-conservation that $u(t_0) \neq 0$. Thus, $u(t_0) \in K$. Then the following holds by the definition of c_η:

$$d < c_\eta \leqslant S_\eta(u(t_0)).$$

This is a contradiction with (3.56). Therefore, we have $t_0 = T$ which indicates $T(u(t)) < 0$ for all $t \in [0, T)$. As a consequence,

$$u(t) \in A_d, \quad \forall\, t \in [0, T).$$

□

Now we can prove the final theorem of this report which states the instability of the standing wave solutions for $1 + \frac{4}{n} < p < p^*(n)$, i.e. Theorem 2(ii).

Proof. In general, we know that to demonstrate the instability of a ground state solution v to (1.3), it suffices to show, in any neighbourhood of $v(0)$, there exists an initial condition such that the corresponding solution blows up in a finite amount of time.

Step 1: Let $d < c_\eta$. Suppose $u_0 \in A_d$ and $xu_0 \in L^2$, where $A_d \subset H^1$ as defined in Lemma 3.2.2. Let u be the corresponding solution to (1.3) with initial condition $u(0) = u_0$.

We claim the solution $u(t)$ can not be extended to $[0, \infty)$ and prove it by contradiction. Suppose the solution u exists on $[0, \infty)$. By Lemma 2.2.6, we derive that

$$\begin{aligned}\|xu(t)\|_{L^2}^2 &= \|xu_0\|_{L^2}^2 + 4t\int_{\mathbb{R}^n} \overline{u_0}x \cdot \nabla u_0 dx \\ &\quad + 4\int_0^t \int_0^s \left[2\|\nabla u(\tau)\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|u(\tau)\|_{L^{p+1}}^{p+1}\right] d\tau ds \\ &= \|xu_0\|_{L^2}^2 + 4t\int_{\mathbb{R}^n} \overline{u_0}x \cdot \nabla u_0 dx \\ &\quad + 4\int_0^t \int_0^s T(u(\tau))\, d\tau ds,\end{aligned} \tag{3.57}$$

for $t \in [0, \infty)$. Moreover, from the assumption $u_0 \in A_d$ and Lemma 3.2.2, it follows that

$$u(t) \in A_d, \quad \forall\, t \in [0, \infty),$$

which means $u(t)$ satisfies the hypothesis of Lemma 3.2.1 for all $t \in (0, \infty]$. It is then followed by

$$T(u(t)) \leqslant S_\eta(u(t)) - c_\eta \leqslant d - c_\eta.$$

Hence, the equation (3.57) can be converted to

$$\begin{aligned}\|xu(t)\|_{L^2}^2 \leqslant \|xu_0\|_{L^2}^2 + 4t\int_{\mathbb{R}^n} \overline{v_0}x \cdot \nabla u_0 dx \\ + 4t^2(d - c_\eta).\end{aligned}$$

Since $d < c_\eta$, the right-hand side of the inequality is negative when t is sufficiently large. It is obviously a contradiction since the square on the left-hand side can not be negative. Therefore, the solution $u(t)$ can not be extended to $t \in (0, \infty]$.

More specifically, there exists $T > 0$ such that

$$\lim_{t \to T^-} \|\nabla u(t)\|_{L^2} = \infty.$$

Step 2: Let w be a solution to the minimization problem (1.9) (i.e. $w \in K$ and $S_\eta(w) = c_\eta$), then by Theorem 1(ii), w is a weak solution to (1.5). Therefore, w satisfies

$$\|\nabla w\|_{L^2}^2 + \eta\|w\|_{L^2}^2 - \|w\|_{L^{p+1}}^{p+1} = 0. \tag{3.58}$$

Now, define

$$w_\lambda(x) := \lambda w(x),$$

where $\lambda > 0$. Substituting it into (1.7) yields

$$S_\eta(w_\lambda) = \lambda^2\|\nabla w\|_{L^2}^2 + \eta\lambda^2\|w\|_{L^2}^2 - \frac{2}{p+1}\lambda^{p+1}\|w\|_{L^{p+1}}^{p+1}. \tag{3.59}$$

Then differentiating (3.59) in λ, we have

$$\frac{d}{d\lambda}S_\eta(w_\lambda) = 2\lambda\left(\|\nabla w\|_{L^2}^2 + \eta\|w\|_{L^2}^2 - \lambda^{p-1}\|w\|_{L^{p+1}}^{p+1}\right).$$

It follows from (3.58) that for all $\lambda > 1$,

$$\begin{aligned}\frac{d}{d\lambda}S_\eta(w_\lambda) &< 2\lambda\left(\|\nabla w\|_{L^2}^2 + \eta\|w\|_{L^2}^2 - \|w\|_{L^{p+1}}^{p+1}\right)\\ &= 0.\end{aligned}$$

It is equivalent to

$$\frac{S_\eta(w_\lambda) - S_\eta(w)}{\lambda - 1} < 0.$$

Thus for $\lambda > 1$, we deduce $S_\eta(w_\lambda) < S_\eta(w) = c_\eta$.

Also, we have $T(w) = 0$ from $w \in K$. Then,

$$\begin{aligned}T(w_\lambda) &= \lambda^2\left[2\|w\|_{L^2}^2 - \lambda^{p-1}\frac{n(p-1)}{p+1}\|w\|_{L^{p+1}}^{p+1}\right]\\ &< \lambda^2\left[2\|w\|_{L^2}^2 - \frac{n(p-1)}{p+1}\|w\|_{L^{p+1}}^{p+1}\right]\\ &= \lambda^2 T(w) = 0.\end{aligned}$$

Moreover, by definition of w_λ, it is clear that the following holds as $\lambda \to 1$:

$$\begin{aligned}\|w_\lambda\|_{L^2} &\to \|w\|_{L^2},\\ \|w_\lambda\|_{L^{p+1}} &\to \|w\|_{L^{p+1}},\\ \|\nabla w_\lambda\|_{L^2} &\to \|\nabla w\|_{L^2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}w_\lambda &\to w \quad \text{in } H^1,\\ S_\eta(w_\lambda) &\to S_\eta(w),\end{aligned}$$

as $\lambda \to 1$. Now, let $d_\lambda := S_\eta(w_\lambda)$, then $d_\lambda \to c_\eta$ as $\lambda \to 1$. Hence, given any $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\|w_\lambda - w\|_{H^1} < \epsilon, \quad w_\lambda \in A_{d_\lambda}, \quad xw_\lambda \in L^2.$$

It follows from Step 1 that the solution u to (1.3) with initial condition $u(0) = w_\lambda$ blows up in finite time. This implies that a ground state solution v to (1.3) is orbitally unstable. $\square$

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