

# GSO talk: Stability of matter

Gyu Eun Lee

*Abstract.* The Coulomb force law of classical electromagnetism says that the electrostatic force on an electron at distance  $r$  from a positively charged atomic nucleus is attractive and has magnitude  $\sim r^{-2}$ . This results in a very large force of attraction for small values of  $r$ . This begs the question: what stops the electron from irreversibly crashing into the nucleus? In this talk, we will discuss the history of this problem and how it prompted the development of quantum mechanics in the early 20th century. We will provide an elementary proof of the stability of the hydrogen atom, and highlight the role that uncertainty principles play in problems of this class. The talk will be entirely self-contained; any relevant physics will be presented at the start.

## 1 The classical model of the atom and the stability problem

By the early 20th century, physicists had begun to experimentally uncover the structure of the smallest known constituents of matter, the atom. The following picture began to emerge: the atom consists of a relatively massive core of positive electric charge called the nucleus, and negatively charged particle of relatively small mass moving in the space around the nucleus called the electron.

Consider an atomic configuration consisting of a nucleus with charge  $+Ze$  sitting at rest at the origin and a single electron of charge  $-e$  at position  $\mathbf{r} \in \mathbb{R}^3$ ,  $r = \|\mathbf{r}\|$ . The classical Coulomb force law tells us that the force on the electron due to the electrostatic interaction between the electron and the nucleus is given by

$$\mathbf{F} = -Ze^2 \frac{\mathbf{r}}{r^3},$$

with appropriately chosen units. Solving for the position  $\mathbf{r}$  of the electron as a function of time, we can show that much like in the case of Newtonian gravitation

(another inverse-square force law), the bounded trajectories of the electron take one of two forms: Kepler-esque ellipses with the nucleus as one of the foci, or degenerate trajectories that crash into the nucleus.

This model of the atom as a solar-system-esque configuration (i.e. the Rutherford model) turns out to have a major deficiency: it contradicts the observation that most atoms appear to have a consistent radius on the order of  $10^{-8}$  cm. Because the atomic nucleus is much smaller compared to this radius (around  $10^{-13}$  cm), this means that the vast majority of the volume of the atom is empty and defined by the electron orbital trajectories, which must mean that the electron spends a significant portion of its time at a distance of about  $10^{-8}$  cm from the nucleus.

However, no classical explanation exists as to what mechanism forces the electron to stay away from the nucleus in this manner. It appears that there is no obstruction to the trajectory of the electron being an arbitrarily small ellipse; nor does there appear to be something preventing a degenerate trajectory, which would result in the electron being irreversibly stuck to the nucleus by an attractive force of infinite order. In fact, from purely a classical physics perspective, it appears that an atom with bound electrons should be allowed to have arbitrarily small radius down to the radius of the nucleus, in disagreement with experiment.

This discrepancy is the essence of the problem called stability of the atom, which asks: *What mechanism prevents an atomic electron-nucleus configuration from collapsing to a point?*

## 2 The energy reformulation of stability

Let us examine the stability problem from a deeper perspective and with the aid of some physical concepts. First let us give a name to the single-electron nucleus configuration.

**Definition.** A *hydrogenic atom* is an atom consisting of a nucleus with positive charge  $+Ze$  and a single bound electron of negative charge  $-e$ .

The hydrogen atom is described by the case  $Z = 1$ . The Coulomb force in a hydrogenic atom configuration is a conservative vector field, given by  $e$  times as the negative gradient of the Coulomb potential generated by the nucleus,

$$V_C(\mathbf{r}) = -\frac{Ze}{r}.$$

The potential energy of the hydrogenic atom is the quantity

$$U(\mathbf{r}) = eV_C(\mathbf{r}) = -\frac{Ze^2}{r}.$$

We may also define the momentum of the electron as the vector  $\mathbf{p} = m_e\dot{\mathbf{r}}$ , where  $m_e$  is the mass of the electron, and the kinetic energy by

$$T(\mathbf{p}) = \frac{1}{2m_e} \|\mathbf{p}\|^2.$$

Then the total energy of the system is described by the Hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = T(\mathbf{p}) + U(\mathbf{r}) = \frac{1}{2m_e} \|\mathbf{p}\|^2 - \frac{Ze^2}{r}.$$

If the energy (that is,  $H$ ) is bounded from below by a finite (necessarily negative) constant, then small values of  $r$  must be accompanied by a corresponding sufficiently large value of  $\|\mathbf{p}\|$ . This suggests that if energy is bounded below, then the electron cannot spend too much time too close to the nucleus: its higher momentum will instead tend to propel it away, which would help to explain the value of the atomic radius. Therefore we may recast the stability problem for hydrogenic atoms in the following way: *Is there a lower bound for the energy  $H(\mathbf{r}, \mathbf{p})$ , and what mechanism provides the lower bound?*

However, classical physics fails us once again. A closer analysis of the possible bounded trajectories does not rule out trajectories with arbitrarily small momentum and radius. (Consider, in particular, degenerate trajectories.) Consequently, classical physics is incapable of providing a lower bound to the energy. This is, in fact, one of the major defects in the planetary model of the atom. Consider an atomic gas and fix a distinguished atom. If no lower bound to the energy of the atomic configuration exists, then one could imagine a case where this distinguished atom repeatedly loses energy to inter-atomic collisions, until its energy is extremely negative; this would then force the electron to be bound to an extremely tight orbit around the nucleus. Moreover, there is in fact a much more serious problem: even without inter-atomic interactions, an accelerating point charge radiates energy in the form of electromagnetic waves (with the change in energy described quantitatively by the Larmor formula). An electron orbiting around a nucleus is by definition in acceleration, and therefore spontaneously radiates away energy, and so there are actually no possible stable electron-nucleus orbits according to classical physics.

The physicists of the early 20th century, working just before the dawn of the quantum revolution, were also familiar with this defect with the planetary model. The defect was so severe that it prompted physicists to completely rethink their understanding of atomic physics. In fact, the stability problem was one of the major factors that led Niels Bohr to develop the Bohr hydrogen model, a model of the hydrogen atom that allows electrons to orbit only in certain pre-specified trajectories corresponding to different discrete energies and explicitly encodes a lowest energy level. Bohr's hydrogen model then went on to be refined into the first truly quantum mechanical models of the atom, by Werner Heisenberg (matrix mechanics) and independently by Erwin Schrödinger (Schrödinger equation). Today, we understand the stability of the atom to be a phenomenon that is truly quantum-mechanical in nature.

### 3 Crash course in quantum mechanics

To laymen quantum mechanics sounds like a highly technical and scary subject. This is true. However, its mathematical foundations are actually quite simple, and we can treat much of nonrelativistic quantum mechanics in an axiomatic setting.

We consider again the situation of the hydrogenic atom. We choose natural units so that  $e = 1$ . We will not need the strength of all of the axioms of nonrelativistic quantum mechanics. Here we use the Schrödinger formulation, through Schrödinger's equation will not be necessary. All relevant integrals will be assumed to exist (and, if necessary, be finite). Here are the axioms/facts/definitions relevant to us:

1. We denote by  $L^2(\mathbb{R}^3)$  the space of complex-valued Lebesgue measurable functions on  $\mathbb{R}^3$  that are square-integrable, i.e.

$$\|f\|_2^2 = \int_{\mathbb{R}^3} |f|^2 dx < \infty.$$

$L^2(\mathbb{R}^3)$  is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^3} \bar{f}g dx.$$

One can equivalently define  $L^2(\mathbb{R}^3)$  as the completion of the space of compactly supported smooth functions under the  $L^2$ -norm.

2. Contrary to classical physics, the electron no longer has a well-defined position and momentum. The *state* of the electron, which encodes all the information about the electron, is an element  $\psi \in L^2(\mathbb{R}^3)$  with unit norm, i.e.  $\|\psi\|_2 = 1$ .
3. *Born statistical interpretation*:  $|\psi|^2 = \rho$  is the probability density function of the position of the electron:

$$\int_E |\psi|^2 = \mathbb{P}(\text{the electron's position is in } E).$$

4. The *kinetic energy* of the state  $\psi$  is given by

$$T_\psi = \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 dx.$$

The *potential energy* of the state  $\psi$  is given by

$$U_\psi = \int_{\mathbb{R}^3} |\psi(x)|^2 V_C(x) dx = -Z \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} dx,$$

where  $V_C(x) = -Ze^2|x|^{-1}$  is the Coulomb potential.

5. The *total energy* of the state  $\psi$  is given by

$$E_\psi = T_\psi + U_\psi = \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 dx - Z \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} dx.$$

Equivalently we can express the total energy by the inner product

$$E_\psi = \langle \psi, H\psi \rangle$$

where  $H$  is the Hamiltonian operator

$$H = -\Delta + V_C(x) = -\Delta - \frac{Z}{|x|}.$$

6. The *ground state energy* is the real number

$$E_0 = \inf_{\|\psi\|_2=1} E_\psi.$$

If the infimum is achieved by some state  $\psi_0$ , we say that  $\psi_0$  is a *ground state* of the Coulomb potential.

## 4 Stability and uncertainty

We now reformulate the problem of stability of the hydrogenic atom in this language, in the following theorem:

**Theorem 1** (Stability of the first kind, hydrogenic atom). *There exists a constant  $E_0 > -\infty$ , independent of  $\psi$ , such that*

$$\langle \psi, H\psi \rangle \geq E_0 \text{ for all } \psi \in L^2(\mathbb{R}^3).$$

*In fact, we can take  $E_0 = -Z^2/4$ , and this lower bound is sharp: it is uniquely achieved by the ground state*

$$\psi_0(x) = \frac{Z^{3/2}}{\sqrt{8\pi}} e^{-Z|x|/2}.$$

The first proof of this result, without the sharp lower bound, was achieved by Freeman Dyson and Andrew Lenard in 1967. Elliott Lieb and Walter Thirring greatly simplified the argument in a 1976 paper. Here we present an elementary proof largely in the vein of Lieb's proof, but taken from lecture notes of Michael Loss.

The main idea behind the proof of Theorem (1) is the concept of quantum mechanical uncertainty. Qualitatively, uncertainty says the following: the position of a quantum particle and its momentum cannot be simultaneously restricted. One may think of the hydrogenic atom as a balloon filled with a compressible fluid, with the property that while the balloon can be squeezed as tight as you wish, it can only be done at the cost of the fluid inside gaining a large amount of momentum, and correspondingly kinetic energy. This is in stark contrast to the classical situation, in which the position and momentum of a particle can be measured exactly and simultaneously.

The most famous of all uncertainty principles is Heisenberg's:

**Theorem 2** (Heisenberg uncertainty principle). *Given a state  $\psi$ , let  $\sigma_x(\psi)^2$  denote the variance in position of the electron, and let  $\sigma_p(\psi)^2$  denote the variance in momentum. Then there is a positive constant  $C$  independent of  $\psi$  such that*

$$\sigma_x(\psi)^2 \sigma_p(\psi)^2 \geq C.$$

However, Heisenberg uncertainty is *not enough* to imply stability, despite claims to the contrary.<sup>1</sup> The uncertainty principle that we will use is the following:

**Theorem 3** (Coulomb uncertainty principle). *For  $\psi \in H^1(\mathbb{R}^3)$ ,*

$$\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} dx \leq \|\nabla\psi\|_2 \|\psi\|_2.$$

*Equality holds only if*

$$\psi(x) \sim e^{-c|x|}$$

*for some constant  $c > 0$ .*

Theorem (3) is an uncertainty principle in the following sense. Suppose we were to localize  $\psi$  around the origin; that is, we increase the probability that the electron's position is close to the nucleus. The normalization  $\|\psi\|_2 = 1$  forces  $|\psi|$  to take large values near the origin to compensate for the localization. This peaking behavior of  $\psi$  is then exaggerated by the factor  $|x|^{-1}$ , which forces  $\|\nabla\psi\|_2$  to be large since  $\|\psi\|_2 = 1$ .

From here we can immediately deduce the stability of the hydrogenic atom. We will only show the existence of the lower bound. The proof that the lower bound coincides with the ground state, and that of the expression for the ground state is left to the reader.

*Proof of Theorem (1).* Recall that the energy of an electron in state  $\psi$  is given by

$$E_\psi = \langle \psi, H\psi \rangle = \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 dx - Z \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} dx.$$

Applying Theorem 3,

$$E_\psi \geq \|\nabla\psi\|_2^2 - Z\|\nabla\psi\|_2.$$

The RHS is quadratic in  $\|\nabla\psi\|_2$  with a global minimum at  $\|\nabla\psi\|_2 = \frac{Z}{2}$ . Therefore

$$E_\psi \geq \frac{Z^2}{4} - \frac{Z^2}{2} = -\frac{Z^2}{4} > -\infty.$$

□

---

<sup>1</sup>The argument usually goes that if  $\psi$  is localized around the origin, then  $\sigma_x(\psi)$  is close to 0, so by Heisenberg uncertainty  $\sigma_x(\psi) = T_\psi$  is highly positive, which compensates for the fact that  $U_\psi$  would be highly negative for such a state  $\psi$ . What this argument misses is that nothing prevents  $\psi$  being so that half its mass is localized at the origin, and half its mass on the moon. Then  $\sigma_x(\psi)$  is very large, so there is effectively no useful lower bound on the kinetic energy, while  $U_\psi$  can still be arbitrarily negative for such a state. Therefore Heisenberg uncertainty cannot be used to obtain an actual lower bound on the energy.

*Proof of Theorem (3).* We recall that  $C_c^\infty(\mathbb{R}^3)$  is dense in  $H^1(\mathbb{R}^3)$ . We observe the following:

$$\frac{1}{|x|} = \partial_{x_i} \left( \frac{x_i}{|x|} \right) - \frac{x_i^2}{|x|^3}.$$

Summing over  $i = 1, 2, 3$ , we obtain

$$\frac{1}{|x|} = \frac{1}{2} \sum_{i=1}^3 \partial_{x_i} \left( \frac{x_i}{|x|} \right).$$

Assume  $\psi \in C_c^\infty(\mathbb{R}^3)$ . Integration by parts now yields the identity

$$\begin{aligned} 2\langle \psi, |x|^{-1} \psi \rangle &= \sum_{i=1}^3 \int_{\mathbb{R}^3} \bar{\psi} \partial_{x_i} \left( \frac{x_i}{|x|} \right) \psi \, dx \\ &= - \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_{x_i} (\bar{\psi} \psi) \frac{x_i}{|x|} \, dx \\ &= - \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_{x_i} \bar{\psi}) \psi + \bar{\psi} (\partial_{x_i} \psi)] \frac{x_i}{|x|} \, dx \\ &= -2\operatorname{Re} \sum_{i=1}^3 \langle \partial_{x_i} \psi, \frac{x_i}{|x|} \psi \rangle. \end{aligned}$$

Applying Cauchy-Schwarz for the inner product  $\langle \cdot, \cdot \rangle$ ,

$$\langle \psi, |x|^{-1} \psi \rangle = -\operatorname{Re} \sum_{i=1}^3 \langle \partial_{x_i} \psi, \frac{x_i}{|x|} \psi \rangle \leq \sum_{i=1}^3 \|\partial_{x_i} \psi\|_2 \|\frac{x_i}{|x|} \psi\|_2.$$

Now applying Cauchy-Schwarz for sums,

$$\sum_{i=1}^3 \|\partial_{x_i} \psi\|_2 \|\frac{x_i}{|x|} \psi\|_2 \leq \left( \sum_{i=1}^3 \|\partial_{x_i} \psi\|_2^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^3 \|\frac{x_i}{|x|} \psi\|_2^2 \right)^{\frac{1}{2}} = \|\nabla \psi\|_2 \|\psi\|_2,$$

and thus we have established

$$\langle \psi, \frac{1}{|x|} \psi \rangle \leq \|\nabla \psi\|_2 \|\psi\|_2,$$

for  $\psi \in C_c^\infty(\mathbb{R}^3)$ . For more general  $\psi \in H^1(\mathbb{R}^3)$ , approximate  $|\psi|$  from below by functions in  $C_c^\infty(\mathbb{R}^3)$ , and apply monotone convergence in the LHS.

We now treat the case of equality. If equality holds, then in particular it must hold in our first use of Cauchy-Schwarz:

$$-\operatorname{Re}\langle \partial_{x_i} \psi, \frac{x_i}{|x|} \psi \rangle \leq |\langle \partial_{x_i} \psi, \frac{x_i}{|x|} \psi \rangle| \leq \|\partial_{x_i} \psi\|_2 \|\frac{x_i}{|x|} \psi\|_2.$$

Equality in the second inequality tells us that

$$\partial_{x_i} \psi = c_i \frac{x_i}{|x|} \psi,$$

and submitting this identity with equality in the first tells us that

$$-\operatorname{Re}\langle c_i \frac{x_i}{|x|} \psi, \frac{x_i}{|x|} \psi \rangle = -\|\frac{x_i}{|x|} \psi\|_2^2 \operatorname{Re} c_i = \|\frac{x_i}{|x|} \psi\|_2^2 |c_i|,$$

and therefore  $c_i = -|c_i|$ . Equality must also hold in our second use of Cauchy-Schwarz:

$$\sum_{i=1}^3 \|\partial_{x_i} \psi\|_2 \|\frac{x_i}{|x|} \psi\|_2 \leq \left( \sum_{i=1}^3 \|\partial_{x_i} \psi\|_2^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^3 \|\frac{x_i}{|x|} \psi\|_2^2 \right)^{\frac{1}{2}}.$$

We thus see that

$$\|\partial_{x_i} \psi\|_2 = c \|\frac{x_i}{|x|} \psi\|_2, \quad c \geq 0,$$

and therefore  $c_i = -c$  for  $i = 1, 2, 3$ . Then

$$\nabla \psi = -c \frac{x}{|x|} \psi.$$

The general solution to this first-order equation is

$$\psi(x) = ae^{-c|x|},$$

and since  $\psi$  is square-integrable we must necessarily have  $c > 0$ . □

## 5 More stability problems, and the role of uncertainty

First, we remark that any intermediate-level undergraduate quantum mechanics class should include a proof of Theorem 1 by explicitly solving the Schrödinger

equation in the presence of the Coulomb potential. So what was the point of our proof? The moral of the proof is to specifically highlight the aspects of quantum mechanics that result in the stability of hydrogen. We saw in particular that uncertainty, and *uncertainty alone*, is enough to demonstrate stability. We operated on only a subset of the full axiomatic system of quantum mechanics: we did not even need to refer to the Schrödinger equation, or matrix mechanics for that matter, at all.

Moreover, the method is easily generalized to other Hamiltonians, i.e. non-Coulomb potentials, as we discuss below. This leads us to some very entertaining applications of analysis (especially harmonic analysis) to mathematical physics. This is not the case for explicitly solving the Schrödinger equation: while for the Coulomb potential this is relatively straightforward, for a general potential this is a difficult problem.

With the proof of Theorem 3 we have established stability for all isolated hydrogenic atoms. However, this is not the only question to be answered in the study of stability. Theorem 3 is explicitly well-suited for the study of hydrogenic atoms, and even gives the physically correct value of the ground state. But it is woefully inadequate for any scenario where the Hamiltonian is not the Coulomb Hamiltonian  $-\Delta - Z|x|^{-1}$ . A way around this obstacle is to employ alternative uncertainty principles, such as the following:

**Theorem 4** (Hardy's inequality).

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 dx \gtrsim_n \int_{\mathbb{R}^n} \frac{|\psi|^2}{|x|^2} dx; \quad \int_{\mathbb{R}^n} |\nabla \psi|^p dx \gtrsim_{n,p} \int_{\mathbb{R}^n} \frac{|\psi|^p}{|x|^p} dx.$$

(Sharp constants are known, with equality iff  $\psi \equiv 0$ .)

**Theorem 5** (Sobolev uncertainty principle).

$$\int_{\mathbb{R}^3} |\nabla \psi|^2 dx \geq \frac{3}{4} (4\pi^2)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} \rho_\psi^3 dx \right)^{\frac{1}{3}},$$

where  $\rho_\psi = |\psi|^2$ . The constant is optimal.

**Corollary 6.**

$$\int_{\mathbb{R}^3} |\nabla \psi|^2 dx \geq \frac{3}{4} (4\pi^2)^{\frac{2}{3}} \int_{\mathbb{R}^3} \rho_\psi^{\frac{5}{3}} dx.$$

This constant is not optimal.

Hardy's inequality can be used to prove the Sobolev uncertainty principle. Hardy's inequality also implies the Coulomb uncertainty principle by Hölder's inequality. Corollary 6 follows from the Sobolev uncertainty principle by Hölder's inequality. Notice that we have essentially reversed bounds from the Coulomb uncertainty principle: the behavior of  $\psi$  bounds that of  $\rho\psi$  from below. 5 and 6 can both be used to prove stability of the hydrogenic atom. For example, applying 6 to with  $H$  the hydrogenic atom Hamiltonian gives us

$$\langle \psi, H\psi \rangle = \int |\nabla\psi|^2 dx - Z \int \frac{|\psi|^2}{|x|} dx \geq \int \rho\psi^{\frac{5}{3}} dx - Z \int \frac{\rho\psi}{|x|} dx.$$

Demonstrating a finite lower bound can then be accomplished by minimizing the functional

$$J(\rho) = \int \rho^{\frac{5}{3}} dx - Z \int \frac{\rho}{|x|} dx$$

subject to the constraints

$$\int \rho dx = 1, \rho \geq 0.$$

This is a constrained variational problem that is actually quite feasible to solve by hand or numerically. The utility of 6 is that it eliminates the gradient in the variational problem, and variational problems are easier if they do not mix functions and their gradients. The lower bound we obtain from this variational approach is no longer sharp, but for the purpose of showing stability this is irrelevant. Moreover, this approach is better suited to non-hydrogenic potentials than the Coulomb uncertainty principle, and still demonstrates the same idea: stability is a consequence of the constraints due to uncertainty on the localization of position and momentum. One example of an interesting case we can handle with this new approach is a configuration of more than two charged particles, in which case the Hamiltonian is given by

$$H = -\Delta - W(x),$$

where  $W(x)$  is the electrostatic potential energy function of a many-particle system. This would describe several scenarios: for instance, a non-hydrogenic atom with multiple electrons, or a gas of several (possibly non-hydrogenic) atoms. Adding even one extra charged particle drastically changes the nature of the potential.

In many-particle systems we can also distinguish between two types of stability. Suppose a many-particle system of  $N$  electrons and  $K$  nuclei is described by the

Hamiltonian  $H$ . Let

$$E_0 = \inf\{\langle \psi, H\psi \rangle : \|\psi\|_2 = 1, \psi \in \mathcal{H}\},$$

where  $\mathcal{H} = \bigwedge^N L^2(\mathbb{R}^3; \mathbb{C}^2)$  is the Hilbert space of  $N$ -electron wavefunctions. We say such a system is *stable of the first kind* if

$$E_0 > -\infty.$$

We say the system is *stable of the second kind* if

$$E_0 \geq C(N + K),$$

for some constant  $C$  independent of  $N$  and  $K$ .

In many-particle systems, the property of quantum spin becomes important, because spin determines the statistical distribution of energies in a many-particle system. Particles with half-integer spin are called *fermions*, and in a many-particle system of fermions the average particle energies are governed by *Fermi-Dirac statistics*. Particles with integer spin are called *bosons*, and their energies obey *Bose-Einstein statistics*. The appropriate Hilbert spaces for the wavefunctions of multi-particle systems depends on the fermionic or bosonic nature of the particles: fermionic systems are described by alternating tensor powers of  $L^2$ , and bosonic systems by symmetric tensor powers, and we formulate the stability problems as above in each case. These considerations become increasingly important as the interparticle distance becomes comparable to the de Broglie wavelength of the particles.

In the case of fermionic many-particle systems, there is an analogous uncertainty principle to 6, which can be used to prove that *fermionic matter is stable of the second kind*. However, *bosonic matter is not stable of the second kind*. The proof relies strongly on the fact that fermionic systems must obey the Pauli exclusion principle. In particular, the stability of fermionic matter explains the fact that most everyday bulk matter has a well-defined volume, and that solid matter does not give way and compress to a point. A fuller discussion of these results leads one into a discussion of Thomas-Fermi theory.

There are further extensions of stability theory to more general situations. These include relativistic scenarios, magnetic field interactions, and radiation field interactions. Once again, uncertainty principles play a key role when proofs of stability can be obtained.

## References

- [1] Lieb, E.: The stability of matter: from atoms to stars, *Bulletin of the American Mathematical Society*, **22**, 1-49 (1990).
- [2] Lieb, E.: The stability of matter, *Reviews of Modern Physics*, **48**, 553-569 (1976).
- [3] Loss, M.: Stability of matter, <http://people.math.gatech.edu/~loss/papertalks/Nov14review.pdf> (Updated November 14, 2005) (Accessed November 3, 2016).
- [4] Loss, M.: Stability of matter, <http://people.math.gatech.edu/~loss/MUNICH/QUANTUMCOULOMB/PDFFILES/qcoulomb.pdf> (Updated August 4, 2005) (Accessed November 3, 2016).